

K 517.1 P624 v

Carnegie Institute of Technology
Library
PITTSBURGH, PA.

Rules for Lending Books:

LECTURES
ON
THE THEORY OF FUNCTION
OF
REAL VARIABLES

VOLUME II

BY

JAMES PIERPONT, LL.D.

PROFESSOR OF MATHEMATICS IN YALE UNIVERSITY

K517.
P62
V. 2

COPYRIGHT, BY
JAMES PIERCE

ALL RIGHTS RESERVED

1913

PREFACE

THE present volume has been written to animate the first. The author has not written a treatise or a manual; he has aimed rather at university lectures with necessary modifications, freedom in the choice of subjects and freedom of action allowable in a lecture room may permit to a larger audience.

A distinctive feature of these Lectures is the theory of functions with reference to their definition. The first functions to be considered are combinations of the elementary functions. In a paper of 1854, "Ueber die Darstellbarkeit einer trigonometrischen Reihe," was the first step towards functions whose singularities ceased to be isolated. The searches of later mathematicians have led to the discovery of such functions, whose existence so long as unknown. The notion of a function was revolutionized the older notion of a function. The creation of finer tools of research. It was paid to the singular character of the functions. None was accorded to the domain over which the function was defined. After the epoch-making discovery by G. Cantor in the theory of point sets, it was necessary to consider a function of one variable on an interval, a function of two variables as a

prevails in the theory of determinants. One may find only two and three rowed determinants, but the ground of complaint if another prefers to state the demonstrations for general n . On the other hand the case may present unexpected and serious problems. Jordan has introduced the notion of functions of n having limited variation. How is this notion to be extended to two or more variables? An answer is far from simple, given by the author in Volume I; its serviceability has been shown by B. Camp. Another has been essayed by The reader must be warned, however, against extending the development always extended to the general case in the first place; would be quite impracticable in increasing the size of the present work. Secondly, it would be quite beyond the author's ability.

Another feature of the present work to which I call attention is the novel theory of integration in Chapter XVI of Volume I and Chapters I and II of Volume II. It rests on the notion of a cell and the division of a set into any set, into unmixed partial sets. The definition of multiple integrals leads to results more general than yet obtained with Riemann integrals.

Still another feature is a new presentation of the theory of measure. The demonstrations which the author has given are much to be desired in the way of completeness, and in attempting to find a general and rigorous treatment, at last led to adopt the form given in Chapter XI of Volume I.

The author also claims as original the theory of multiple integrals developed in Chapter XII. Lebesgue has shown that functions such that the points e at which $a \leq f(x)$ form a measurable set. His method leads to the

be divided into a finite number of metric sets then

$$\int_{\mathfrak{M}} f = \text{Max } \Sigma m_i \delta_i, \quad \int_{\mathfrak{M}} f = \text{Min}$$

where m_i , M_i are the minimum and maximum then is more natural than to ask what will $\delta_1, \delta_2 \dots$ are infinite instead of finite in number. This apparently trivial question results a theory of integration which contains the Lebesgue integrals as a special case. Furthermore, has the great advantage that not only the definition of the new integrals to the ordinary or Riemann integrals is perfectly obvious, but also the form of reasonings. Riemann's theory may be taken over to develop the theory of the new integrals.

Finally the author would call attention to the application of the new integrals to the area of a curved surface given at the end of the book. Though the above are the main features of the new theory, the author hopes that the experienced reader will discover some other features that are lacking in originality, but not of sufficient importance to be emphasized here.

It is now the author's pleasant duty to acknowledge the valuable assistance derived from his colleague Dr. W. A. Wilson. He has read the entire manuscript, proof with great care, corrected many errors, improved the demonstrations, besides contributing the suggestions on pages 373, 401-406, 414-424.

Unstinted praise is also due to the house of printing, who have met the author's wishes with unflinching fidelity and have given the utmost care to the press work.

CONTENTS

CHAPTER I

POINT SETS AND PROPER INTEGRALS

ARTICLES

- 1-10. Miscellaneous Theorems
- 11-15. Iterable Fields
- 16-25. Union and Divisor of Point Sets

CHAPTER II

IMPROPER MULTIPLE INTEGRALS

- 26-28. Classical Definition
- 29. Definition of de la Vallée-Poussin
- 30. Author's Definition
- 31-61. General Theory
- 62-69. Relation between Three Types
- 70-78. Iterated Integrals

CHAPTER III

SERIES

- 79-80. Preliminary Definitions and Theorems

ARTICLES

- 116-125. General Theory . . .
126-133. Iterated Series . . .

- 134-145. General Theory. Uniform
146. The Moore-Osgood Theorem
147-149. Continuity of a Series .
150-152. Termwise Integration .
153-156. Termwise Differentiation .

- 157-158. Termwise Differentiation and
159. Development of $\log(1+x)$
160. Equality of two Power Series
161-162. Development of a Power Series
163. Multiplication and Division
164-165. Undetermined Coefficients
166-167. Development of a Series with
168. Inversion of a Power Series
169-171. Taylor's Development .
172. Forms of the Remainder .
173. Development of $(1+x)^{\mu}$.
174. Development of $\log(1+x)$
175-181. Criticism of Current Errors

CONTENTS

ARTICLES

| | | |
|----------|--------------------------------|--|
| 219. | Bernouillian Numbers | |
| 220-228. | B and F Functions | |

CHAPTER VIII

AGGREGATES

| | | |
|----------|--|--|
| 229-230. | Equivalence | |
| 231. | Cardinal Numbers | |
| 232-241. | Enumerable Sets | |
| 242. | Some Space Transformations | |
| 243-250. | The Cardinal c | |
| 251-261. | Arithmetic Operations with Cardinals | |
| 262-264. | Numbers of Liouville | |

CHAPTER IX

ORDINAL NUMBERS

| | | |
|----------|-------------------------------|--|
| 265-267. | Ordered Sets | |
| 268-270. | Eutactic Sets | |
| 271-279. | Sections | |
| 280-284. | Ordinal Numbers | |
| 285-288. | Limitary Numbers | |
| 289-300. | Classes of Ordinals | |

CHAPTER X

POINT SETS

| | | |
|----------|-----------------------------------|--|
| 301-312. | Pantaxis | |
| 313-320. | Transfinite Derivatives | |
| 321-333. | Complete Sets | |

ARTICLES

- 407-409. Measurable Functions . . .
 410. Quasi and Semi Divisors . . .
 411-413. Limit Functions . . .
 414-424. Iterated Integrals . . .

IMPROPE

- 425-428. Upper and Lower Integrals
 429-431. I -Integrals . . .
 432-435. Iterated Integrals . . .

CHAI

FOURI

- 436-437. Preliminary Remarks . . .
 438. Summation of Fourier's Ser
 439-442. Validity of Fourier's Develo
 443-446. Limited Variation . . .
 447-448. Other Criteria . . .
 449-456. Uniqueness of Fourier's Dev

CHAI

DISCONTINU

- 457-462. Properties of Continuous Fu
 463-464. Pointwise and Total Discont
 465-473. Examples of Discontinuous
 474-489. Functions of Class I . . .
 490-497. Semicontinuous Functions

CHA

DERIVATES, EX

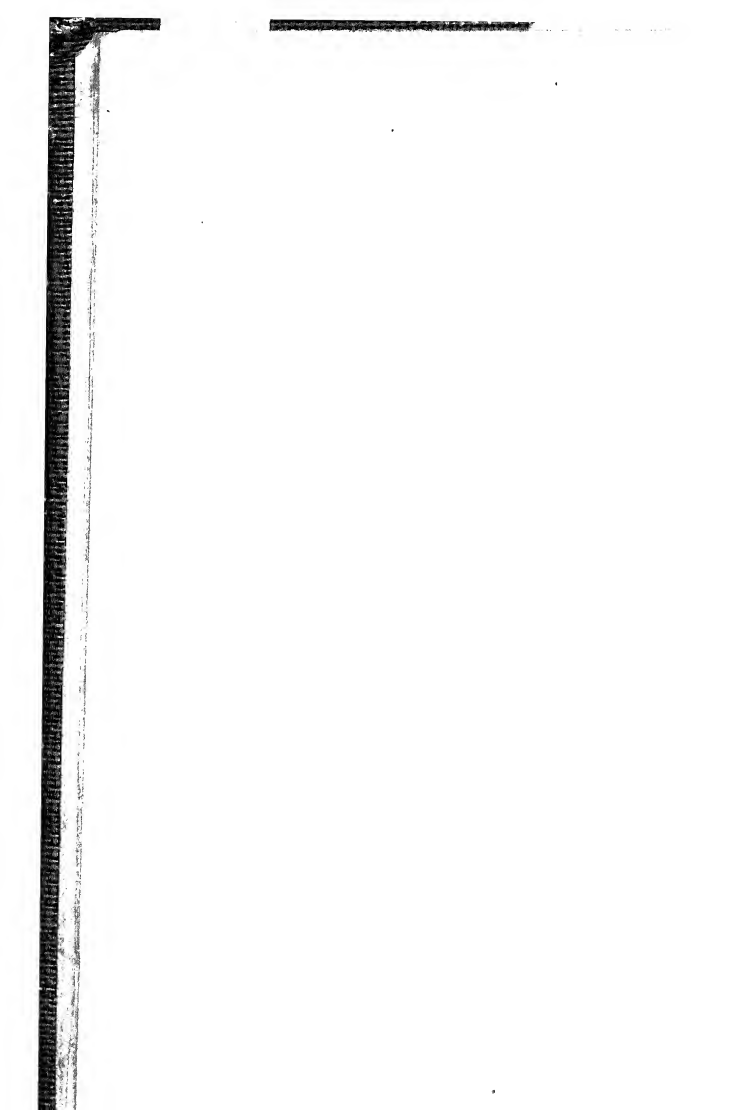
CONTENTS

CHAPTER XVII

GEOMETRIC NOTIONS

ARTICLES

| | |
|------------------|--|
| 562-563. | Properties of Intuitional Plane Curves |
| 564. | Motion |
| 565. | Curve as Intersection of Two Surfaces |
| 566. | Continuity of a Curve |
| 567. | Tangents |
| 568-572. | Length |
| 573. | Space-filling Curves |
| 574. | Hilbert's Curve |
| 575. | Equations of a Curve |
| 576-580. | Closed Curves |
| 581. | Area |
| 582. | Osgood's Curve |
| 583. | Résumé |
| 584-585. | Detached and Connected Sets |
| 586-591. | Images |
| 592-597. | Side Lights on Jordan Curves |
| 598-600. | Brouwer's Proof of Jordan's Theorem |
| 601. | Dimensional Invariance |
| 602. | Schönfliess' Theorem |
| 603-608. | Area of Curved Surfaces |
| Index | |
| List of Symbols. | |



FUNCTION THEORY OF VARIABLES

CHAPTER I

POINT SETS AND PROPER INTEGRALS

1. In this short chapter we wish to complete our proper multiple integrals and give a few *theorems* which we shall either need now or in the next chapter. We take up the important subject of improper multiple integrals.

In Volume I, 702, we have said that a limited set whose upper and lower contents are the same is measurable. It is best to reserve this term for another notion which has come of great prominence of late. We shall therefore in this chapter call sets whose upper and lower contents are equal, *metric*. A set \mathfrak{A} is metric, either symbol

$$\mathfrak{A} \text{ or } \mathfrak{A}$$

expresses its content. In the following it will be

given requires each cell to be metrizable; it is not necessary; it suffices that the division of the given set \mathfrak{A} . Such divisions of norm δ . [I, 711.] Under now theorems analogous to I, 714, 722.

2. Let \mathfrak{B} contain the limited point set \mathfrak{A} . Let \mathfrak{B} be a mixed division of \mathfrak{B} of norm δ . Let \mathfrak{A}_δ be the set of points of \mathfrak{A} contained in the cells of \mathfrak{B} of norm δ . Then

$$\lim_{\delta \rightarrow 0} \mathfrak{A}_\delta = \mathfrak{A}$$

The proof is entirely analogous to the proof of I, 714.

3. Let \mathfrak{B} contain the limited point set \mathfrak{A} . Let Δ be an unmixed division of \mathfrak{A} into cells $\delta_1, \delta_2, \dots$. Let $\mathfrak{M}_\delta, m_\delta$ be respectively the maximum and minimum of f in δ . Then

$$\lim_{\delta \rightarrow 0} S_\Delta = \lim_{\delta \rightarrow 0} \sum \mathfrak{M}_\delta \delta$$

$$\lim_{\delta \rightarrow 0} S_\Delta = \lim_{\delta \rightarrow 0} \sum m_\delta \delta$$

Let us prove 1); the relation 2) may be proved in a similar manner. In the first place we show that the relation is analogous to I, 722, that

$$S_\Delta < \int_{\mathfrak{A}} f d\mathfrak{A} + \epsilon.$$

PROPER INTEGRALS

The cells of \mathcal{E} containing points of \mathfrak{A} fall into
 1° the cells e_α containing points of the cell δ_i but not
 of Δ ; 2° the cells e'_i containing points of two or more
 Thus we have

$$S_E = \sum M_\alpha e_\alpha + \sum M'_i e'_i,$$

where M_α, M'_i are the maxima of f in e_α, e'_i . The
 have

$$S_E = \sum \mathfrak{M}_i e_\alpha + \frac{\epsilon}{1}, \quad \epsilon = e_0,$$

if e_0 is taken sufficiently small.

On the other hand, we have

$$S_A = \sum \mathfrak{M}_i e_\alpha = F \sum_i \delta_i = \sum e_\alpha.$$

Now we may suppose δ_0, e_0 are taken so small that

$$\sum \delta_i, \sum e_\alpha$$

differ from \mathfrak{A} by as little as we choose. We have
 properly chosen δ_0, e_0 .

$$S_A = \sum \mathfrak{M}_i e_\alpha = \frac{\epsilon}{1}.$$

This with 6) gives

$$S_A = S_E = \frac{\epsilon}{2},$$

which with 5) proves 4).

4. Let $f(x_1 \dots x_m)$ be limited in the limited cell \mathfrak{A}
 an unmixed division of \mathfrak{A} of norm δ , into cells $\delta_1, \delta_2, \dots$

has both signs in \mathfrak{A} . For example, let \mathfrak{A} be the class of all squares whose center call C . Let us effect a division of \mathfrak{A} into equal squares and let \mathfrak{A} be formed by the union of \mathfrak{A} and of C . Let us define f as follows:

$$f = 1 \text{ within } \mathfrak{A} \\ = -100 \text{ at } C$$

For the division E ,

$$S_E = -1 + 100$$

Hence,

$$\text{Min } S_D < S_E$$

On the other hand,

$$\lim_{d \rightarrow 0} \bar{S}_D = 100$$

The theorems I, 723, and its analogs show that for unmixed divisions of space, the division \mathfrak{A} must be unmixed divisions of the field \mathfrak{A} . This result, so, is shown by the example just given.

6. In certain cases the field \mathfrak{A} is not unmixed. In such a case we define

$$\int_{\mathfrak{A}} f = \sum_{\mathfrak{A}} f$$

7. From 4 we have at once :

Let Δ be an unmixed division of

$$\mathfrak{A} = \text{Min}$$

with respect to the class of all divisions

PROPER INTEGRALS

with respect to the class of rectangular division of D . But the class \mathcal{H} is a subclass of the class D .

Thus

$$\text{Min}_{\Delta} \sum \delta_i = \text{Min}_{\mathcal{H}} \sum d_i = \text{Min}_K \sum e_i.$$

Here the two end terms have the value \mathfrak{A} .

3. Let $f(x_1 \cdots x_m)$, $g(x_1 \cdots x_m)$ be limited in the \mathfrak{A} . We have then the following theorems:

1. *Let $f = g$ in \mathfrak{A} except possibly at the points of a measure zero set \mathfrak{S} . Then,*

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}} g.$$

For let $|f|$, $|g| \leq M$. Let D be a cubical division. Let M_i , N_i denote the maximum of f , g in the cell d_i . Let \mathfrak{A}_i denote the cells containing points of \mathfrak{S} , while \mathfrak{A} is the union of the other cells of \mathfrak{A}_D .

Then,

$$\sum_{\mathfrak{A}} M_i d_i = \sum_{\mathfrak{A}_i} M_i d_i + \sum_{\mathfrak{A}} M_i d_i;$$

$$\sum_{\mathfrak{A}} N_i d_i = \sum_{\mathfrak{A}_i} N_i d_i + \sum_{\mathfrak{A}} N_i d_i.$$

Hence, $\left| \sum_{\mathfrak{A}} M_i d_i - \sum_{\mathfrak{A}} N_i d_i \right| \leq \sum_{\mathfrak{A}_i} |M_i - N_i| d_i \leq 2M \sum_{\mathfrak{A}_i} d_i$

and the term on the right $\rightarrow 0$ as $d \rightarrow 0$.

2. *Let $f \neq g$ in \mathfrak{A} except possibly at the points of a measure zero set \mathfrak{S} . Then,*

$$3. \text{ If } c > 0, \quad \int_{\mathfrak{A}} cf = c \int_{\mathfrak{A}} f;$$

$$\text{but if } c < 0, \quad \int_{\mathfrak{A}} cf = c \int_{\mathfrak{A}} f; \quad \int_{\mathfrak{A}} cf = c \int_{\mathfrak{A}} f.$$

For in any cell d_i

$$\text{Max} \cdot cf = c \text{Max} f; \quad \text{Min} \cdot c$$

when $c > 0$; while

$$\text{Max} \cdot cf = c \text{Min} f; \quad \text{Min} \cdot c$$

when $c < 0$.

4. If g is integrable in \mathfrak{A} ,

$$\int_{\mathfrak{A}} (f + g) = \int_{\mathfrak{A}} f + \int_{\mathfrak{A}} g$$

For from

$$\text{Max} f + \text{Min} g \leq \text{Max} (f + g) \leq \text{Min}$$

we have

$$\int_{\mathfrak{A}} f + \int_{\mathfrak{A}} g = \int_{\mathfrak{A}} (f + g) = \int_{\mathfrak{A}} f$$

But g being integrable,

$$\int_{\mathfrak{A}} g = \int_{\mathfrak{A}} g.$$

Hence 2) gives

$$C \cdot c = C \cdot c = C \cdot c = C \cdot c$$

PROPER INTEGRALS

6. Let $f = g + h$, $|h| \leq H$ a constant, in \mathfrak{A} . Then

$$\left| \int_{\mathfrak{A}} f - \int_{\mathfrak{A}} g \right| \leq H\mathfrak{A}.$$

For $\quad \quad \quad = H + g \leq f \leq g + H.$

Then by 2 and 4

$$= \int_{\mathfrak{A}} H + \int_{\mathfrak{A}} g \leq \int_{\mathfrak{A}} f \leq \int_{\mathfrak{A}} g + \int_{\mathfrak{A}} H,$$

or

$$= H\mathfrak{A} + \int_{\mathfrak{A}} g \leq \int_{\mathfrak{A}} f \leq \int_{\mathfrak{A}} g + H\mathfrak{A}.$$

4. Let $f(x_1 \cdots x_m)$ be limited in limited \mathfrak{A} . Then,

$$\left| \int_{\mathfrak{A}} f - \int_{\mathfrak{A}} f' \right|$$

$$\leq \int_{\mathfrak{A}} f - \int_{\mathfrak{A}} f',$$

$$\int_{\mathfrak{A}} f - \int_{\mathfrak{A}} f' = \int_{\mathfrak{A}} f - \int_{\mathfrak{A}} f,$$

$$\int_{\mathfrak{A}} f - \int_{\mathfrak{A}} f' = \int_{\mathfrak{A}} f.$$

If $|f| \leq M$, we have also,

$$\int_{\mathfrak{A}} f \leq M\mathfrak{A}.$$

To prove 3), we use the relation

$$-|f|$$

Hence

$$\int_{\mathfrak{A}} -|f|$$

from which 3) follows on using

The demonstration of 4) is s

To prove 5), we observe that

$$\sum M_n d_n$$

5. 1. Let $f \geq 0$ be limited in
the aggregate formed of the points

$$\int_{\mathfrak{A}} f$$

This is obvious since the sum

$$\sum M_n d_n$$

may have terms in common.
twice on the right of 1) and on
to the limit.

Remark. The relation 1) may

Example. Let $\mathfrak{A} = (0, 1)$, \mathfrak{A} contains
rational points in \mathfrak{A} . Let $f = 1$ if

PROPER INTEGRALS

For

$$\int_{\mathfrak{Q}} g = \int_{\mathfrak{A}} g + \int_{\mathfrak{C}} g, \quad \text{by 1.}$$

But

$$\int_{\mathfrak{A}} g = \int_{\mathfrak{A}} f, \quad \text{by 3, 4,}$$

and obviously

$$\int_{\mathfrak{C}} g = 0.$$

3. The reader should note that the above theorem is true if \mathfrak{A} is not an unmixed part of \mathfrak{Q} .

Example. Let \mathfrak{A} denote the rational points in \mathfrak{Q} .

Let $f = g = 1$ in \mathfrak{A} .

Then

$$\int_{\mathfrak{A}} f = 1, \quad \int_{\mathfrak{A}} g = 1$$

4. Let \mathfrak{A} be a part of the limited field \mathfrak{Q} . Let $f \in \mathfrak{A}$. Let $g \in f$ in \mathfrak{A} and $= 0$ in $\mathfrak{C} = \mathfrak{Q} - \mathfrak{A}$. Then

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}} g;$$

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{Q}} g.$$

For let M_i, N_i be the maxima of f, g in the cell

$$\Sigma N_i \Delta_i = \Sigma N_i \Delta_i + \Sigma N_i \Delta_i.$$

For let $|f| \leq M$ in \mathfrak{A} . Let \mathfrak{Q}

$$\int_{\mathfrak{A}} f =$$

But

$$\left| \int_{\mathfrak{Q}_u} f \right|$$

Hence passing to the limit $u \rightarrow 0$

2. We note that 1 may be improved. For let \mathfrak{A} be the unit square. Let \mathfrak{Q}_u be the concentric square whose side is $1-u$. Then $\mathfrak{Q}_u \subset \mathfrak{A}$ and $\mathfrak{Q}_u \rightarrow \mathfrak{A}$ as $u \rightarrow 0$. For the other

$$\int_{\mathfrak{A}} f = 2,$$

7. In I, 716 we have given a theorem when each $\mathfrak{B}_u \subset \mathfrak{A}$. A similar theorem holds viz.:

Let $\mathfrak{B}_u \subset \mathfrak{B}_{u'}$ if $u < u'$. Let $\mathfrak{B}_u \rightarrow \mathfrak{A}$ as $u \rightarrow 0$. Then for each $\epsilon > 0$,

$$\mathfrak{B}_{u_0, \epsilon} = \mathfrak{A}_\epsilon \subset \mathfrak{B}_{u_0, \epsilon} \subset \mathfrak{A}$$

For $\mathfrak{B}_{u_0, \epsilon} \subset \mathfrak{A} + \frac{\epsilon}{2}$, u_0

Also for any division D of \mathfrak{A} into

PROPER INTEGRALS

belonging to a given point b of \mathfrak{B} , we denote by \mathfrak{C}_b the component of \mathfrak{A} containing b . We write

$$\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C},$$

and call \mathfrak{B} , \mathfrak{C} *components* of \mathfrak{A} .

We note that the fundamental relations of I

$$\int_{\mathfrak{A}} f \leq \int_{\mathfrak{B}} \int_{\mathfrak{C}} f \leq \int_{\mathfrak{A}} f$$

hold not only for the components \mathfrak{x} , \mathfrak{y} , etc., also for the general components \mathfrak{A} , \mathfrak{B} .

In what follows we shall often give a proof for the sake of clearness, but in such cases the reader may admit an easy generalization. In such cases the x -projection or component of \mathfrak{A} .

2. If $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ is limited and \mathfrak{B} is discrete,

For let \mathfrak{A} lie within a cube of edge $\frac{1}{2} C > d$ space. Then for any $d < \text{some } d_0$,

$$\overline{\mathfrak{B}}_d < \frac{\epsilon}{C^s}.$$

Then

$$\overline{\mathfrak{A}}_d < C^s \overline{\mathfrak{B}}_d < \epsilon.$$

3. That the converse of 2 is not necessarily true is shown by the two following examples, which we shall use

Example 1. Let \mathfrak{A} denote the points x

For

$$x = \frac{m}{n}, \quad m, n \text{ relative}$$

let

$$0 \leq y \leq \frac{1}{n}.$$

Then, \mathfrak{B} denoting the projection of \mathfrak{A}

$$\bar{\mathfrak{A}} = 0, \quad \mathfrak{B} = 1.$$

9. 1. Let $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ be a limited point

$$\mathfrak{A} \cdot \int_{\mathfrak{A}} \mathfrak{C} \cdot \mathfrak{A}.$$

For let $f = 1$ in \mathfrak{A} . Let $g = 1$ at each other points of a cube $A = B \cdot C$ containi

$$\mathfrak{A} = \int_A g, \quad \mathfrak{A} = \int$$

By I, 733,

$$\int_A g = \int_n \int_v g = \int_t$$

But by 5, 4,

$$\int_n \int_v g = \int_n \int_{\mathfrak{C}} f.$$

$$\int_n \int_v g = \int_n \int_{\mathfrak{C}} f.$$

Thus

$$\mathfrak{A} = \int g \leq \int \int f = \int \int f = \int$$

PROPER INTEGRALS

3. In this connection we should note, however, that of 2 is not always true, *i.e.* if \mathfrak{S} is integrable, then 2) and 2, 2) holds. This is shown by the following:

Example. In the unit square we define the points x ,

$$\text{For rational } x, \quad 0 \leq y \leq \frac{1}{2},$$

$$\text{For irrational } x, \quad \frac{1}{2} \leq y \leq 1.$$

Then $\mathfrak{S} = \frac{1}{2}$ for every x in \mathfrak{A} . Hence

$$\int_{\mathfrak{A}} \mathfrak{S} = \frac{1}{2}.$$

$$\text{But} \quad \mathfrak{A} = 0, \quad \mathfrak{A} = 1.$$

10. 1. Let $f(x_1 \cdots x_m)$ be limited in the limited field

$$\text{If } f \geq 0, \quad \int_{\mathfrak{A}} \int_{\mathfrak{S}} f \leq \int_{\mathfrak{A}} f.$$

$$\text{If } f \leq 0, \quad \int_{\mathfrak{A}} f \leq \int_{\mathfrak{A}} \int_{\mathfrak{S}} f.$$

Let us first prove 1). Let \mathfrak{A} , \mathfrak{B} , \mathfrak{S} lie in the space $r + s = m$. Then any cubical division D divides the cubical cells d , d' , d'' of volumes d , d' , d'' respectively $d = d' + d''$. D also divides \mathfrak{A} and each \mathfrak{S} into \mathfrak{A}' , \mathfrak{A}'' , \mathfrak{M} , \mathfrak{M}' , \mathfrak{M}'' and one of the cells d which

2. To illustrate the necessity of \mathfrak{A} to be the Pringsheim set of I . Then

$$\int_{\mathfrak{A}}.$$

On the other hand

$$\int_{\mathfrak{C}}.$$

Hence

$$\int_{\mathfrak{A}} \int_{\mathfrak{C}}.$$

and the relation 1) does not hold.

Iterable

11. 1. There is a large class of sets which have content and yet

\mathfrak{A}

Any limited point set satisfying condition 1) or more specifically *iterable with respect to*

Example 1. Let \mathfrak{A} consist of the points of a square. Obviously

$$\mathfrak{A} \int_{\mathfrak{A}} \mathfrak{C}$$

so that \mathfrak{A} is iterable both with respect to

Example 2. Let \mathfrak{A} consist of the points of a

ITERABLE FIELDS

Example 3. Let \mathfrak{A} consist of the points in the unit square defined thus:

For rational x let $0 < y < \frac{1}{2}$.

For irrational x let $\frac{1}{2} < y < 1$.

Here $\mathfrak{A} = 1$, while

$$\int_{\mathfrak{A}} \mathfrak{C} = \frac{1}{2}; \quad \int_{\mathfrak{C}} \mathfrak{A} = 1.$$

Hence \mathfrak{A} is iterable with respect to \mathfrak{C} but not with respect to \mathfrak{A} .

Example 4. Let \mathfrak{A} consist of the sides of the unit square and the rational points within the square.

Here $\mathfrak{A} = 1$, while

$$\int_{\mathfrak{A}} \mathfrak{C} = 0, \quad \int_{\mathfrak{C}} \mathfrak{A} = 1,$$

and similar relations for \mathfrak{C} . Thus \mathfrak{A} is not iterable with respect to either \mathfrak{A} or \mathfrak{C} .

Example 5. Let \mathfrak{A} be the Pringsheim set of 1, §4.

Here $\mathfrak{A} = 1$, while

$$\int_{\mathfrak{A}} \mathfrak{C} = 0, \quad \int_{\mathfrak{C}} \mathfrak{A} = 0.$$

Hence \mathfrak{A} is not iterable with respect to either \mathfrak{A} or \mathfrak{C} .

2. *Every limited metric point set is iterable with respect to its projections.*

This follows at once from the definition and §3, 2.

13. 1. Let \mathfrak{X} be a limited point space \mathfrak{R}_m . Let \mathfrak{B} , \mathfrak{C} be compact sets. A cubical division D of norm δ divides \mathfrak{X} into cells of volume d and \mathfrak{R}_r and \mathfrak{R}_s into cells of volume b and c . Let a be any point of \mathfrak{B} , lying in a cell d_r of all the cells d_s , containing a . Let $\sum_{d_r} d_s$ denote the sum of all d_s whose projection falls in d_r , not

We have now the following theorem.

If \mathfrak{X} is iterable with respect to \mathfrak{B} , then

$$\lim_{\delta \rightarrow 0} \sum_{\mathfrak{B}} d_r \left(\sum_{d_r} d_s \right) = \sum_{\mathfrak{B}} d_r$$

For

$$\mathfrak{C}_b = \sum_{\mathfrak{B}} d_r$$

Hence

$$\sum_{\mathfrak{B}} d_r \mathfrak{C}_b = \sum_{\mathfrak{B}} d_r$$

Let now $\delta \doteq 0$. The first and second limits are equal since \mathfrak{X} is iterable. Thus, the first limit is equal to the second, the same limit, and this gives 1.

2. *If \mathfrak{X} is iterable with respect to \mathfrak{C} , then*

$$\lim_{\delta \rightarrow 0} \sum_{\mathfrak{C}} d_r \left(\sum_{d_r} d_s \right) = \sum_{\mathfrak{C}} d_r$$

This follows at once from 1).

ITERABLE FIELDS

Let S denote, as in 13, 1, the cells of $\sum_r d_r$ which contain \mathfrak{Q} and H the cells containing points of both \mathfrak{Q} and \mathfrak{Q}' which fall in d_r . Then from

$$C'_{b_1} + C''_{b_1} \leq C'_{b_1} + S \leq C'_{b_2} + S \leq \sum_{d_r} d_r$$

we have

$$\mathfrak{G}_{b_1} \leq C'_{b_1} + C''_{b_1} \leq \text{Min } C'_b + \beta' + S \leq \text{Max } C'_r + \beta'$$

Multiplying by d_r and summing over \mathfrak{B} we find

$$\sum d_r \mathfrak{G}_{b_1} \leq \sum d_r \text{Min } C'_b + \sum \beta' d_r + \mathfrak{G}'_b \leq \sum d_r \text{Max } C'_r + \mathfrak{A}_b + \sum d_r$$

Passing to the limit, we have

$$\mathfrak{A} \leq \int_{\mathfrak{B}} C' + \eta' + \mathfrak{G}' \leq \int_{\mathfrak{B}} C' + \eta''$$

the limit of the last term vanishing since \mathfrak{Q} , \mathfrak{Q}' of \mathfrak{A} . Here η' , η'' are as small as we please on taking \mathfrak{B} small. From 2) we now have

$$\int_{\mathfrak{B}} C' \leq \mathfrak{A} - \mathfrak{G}' - \mathfrak{G}.$$

4. Let $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ be iterable with respect to \mathfrak{B} and A all those points of \mathfrak{A} whose projection on A is iterable with respect to B .

For let D be a cubical division of space of no

$$\mathfrak{A} = \lim \mathfrak{B} = \lim \int_{\mathfrak{B}} C' + \eta' + \mathfrak{G}' \leq \lim \int_{\mathfrak{B}} C' + \eta''$$

As each of the braces is ≥ 0 w

A

14. We can now generalize (733) as follows :

Let $f(x_1 \dots x_m)$ be limited in \mathfrak{A} with respect to \mathfrak{B} . Then

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}}$$

For let us choose the positive c

$$f + A > 0, \quad f$$

Let us effect a cubical division cells d . As in 13, this divides as well as their contents, by d . As usual let m, M denote the m cell d containing a point of \mathfrak{A} . extremes of f when we consider projection is b . Let $|f| \leq F$ in

Then for any b , we have by 1

$$\Sigma (m' - B)d_a = \int_{\mathfrak{A}}$$

or

$$= B(\Sigma d, \quad \mathfrak{C}$$

since $m \geq m'$.

In a similar manner

2. If \mathfrak{A} is not iterable with respect to the discrete set \mathfrak{D} . Let B, C be components of \mathfrak{A} . Then 1 gives

$$\int_{\mathfrak{A}} f < \int_{\mathfrak{B}} f + \int_{\mathfrak{C}} f$$

since

3. The reader should guard against assuming that only \mathfrak{A} is iterable on removal of \mathfrak{D} . The following :

Example. Let the points of \mathfrak{A} be the points of \mathfrak{A}_1 consist of all the points of \mathfrak{A} . Let \mathfrak{D} lie on the rational ordinals.

$$x = \frac{m}{n},$$

$$0 \leq y < 1$$

Let us define f over \mathfrak{A} thus :

$$f = 1$$

$$f = 0$$

The relation 1) is false in this case.

while

$$\int_{\mathfrak{A}} f = 1$$

15. 1. Let $f(x_1 \cdots x_m)$ be a function of m variables. Let D denote the rectangle $[0, 1] \times \cdots \times [0, 1]$ of \mathfrak{A} , except possibly those

To show that 1) exist we need only show that there exists a d_0 such that for any $d < d_0$ the norms $< d_0$

$$\Delta = \left| \int_{\mathfrak{A}_{D'}} - \int_{\mathfrak{A}_D} \right|$$

To this end, we denote by E the diagonal of D' on D . Then E is a rectangle.

Let $\mathfrak{A}_E - \mathfrak{A}_{D'} = A'$, $\mathfrak{A}_E - \mathfrak{A}_D = A''$.

If d_0 is sufficiently small, A' , A'' are

an arbitrarily small positive number.

$$\Delta = \left| \left(\int_{\mathfrak{A}_D} - \int_{\mathfrak{A}_E} \right) - \left(\int_{\mathfrak{A}_{D'}} - \int_{\mathfrak{A}_E} \right) \right|$$

if η is taken small enough.

2. The integrals

$$\int_{\mathfrak{A}} f,$$

heretofore considered may be called *iterated integrals*, in contradistinction.

3. Let f be limited in the limited D . Then the inner and outer lower (upper) integrals are

For \mathfrak{A}_D is an unmixed part of \mathfrak{A} so

$$\text{Cont } \mathfrak{A}_D \doteq \mathfrak{A},$$

Then by 6, 1,

$$\lim_{d \rightarrow 0} \int_{\mathfrak{A}_D} f =$$

For each $\mathfrak{A}_D = 0$, and

16. Let $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$ b

For let D be a cubic
of $\overline{\mathfrak{A}}_D$ fall into three cla
these form \mathfrak{B}_D . 2°, c
3°, cells containing fro
or 2°. Call these \mathfrak{f}_D .

Let now $d \doteq 0$. A
Front \mathfrak{A} and this is dis

17. 1. Let

be point sets, limited
The aggregate formed
sets 1) is called their

or more shortly by

If \mathfrak{A} is a general sy
may also be denoted b
or even more briefly b

or by

$$Dv\{\mathfrak{A}\},$$

if \mathfrak{A} is a general symbol as before.

2. Examples.

Let \mathfrak{A} be the interval $(0, 2)$; \mathfrak{B} the interval $(1, \infty)$

$$U(\mathfrak{A}, \mathfrak{B}) = (0, \infty), \quad Dv(\mathfrak{A}, \mathfrak{B}) = (1, 2).$$

Let

$$\mathfrak{A}_1 = (0, 1), \quad \mathfrak{A}_2 = (1, 2) \dots$$

Then

$$U(\mathfrak{A}_1, \mathfrak{A}_2 \dots) = (0, \infty),$$

$$Dv(\mathfrak{A}_1, \mathfrak{A}_2 \dots) = 0.$$

Let

$$\mathfrak{A}_1 = (\frac{1}{2}, 1), \quad \mathfrak{A}_2 = (\frac{1}{3}, \frac{1}{2}), \quad \mathfrak{A}_3 = (\frac{1}{4}, \frac{1}{3})$$

Then

$$U(\mathfrak{A}_1, \mathfrak{A}_2 \dots) = (0^*, 1),$$

$$Dv(\mathfrak{A}_1, \mathfrak{A}_2 \dots) = 0.$$

Let

$$\mathfrak{A}_1 = (\frac{1}{2}, 1\frac{1}{2}), \quad \mathfrak{A}_2 = (\frac{1}{4}, 1\frac{3}{4}), \quad \mathfrak{A}_3 = (\frac{1}{8}, 1\frac{7}{8})$$

Then

$$U(\mathfrak{A}_1, \mathfrak{A}_2 \dots) = (0^*, 2^*),$$

$$Dv(\mathfrak{A}_1, \mathfrak{A}_2 \dots) = \mathfrak{A}_1.$$

3. Let

$$\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{A}_3 + \dots$$

Let

$$\mathfrak{C} = Dv(\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}_2, \dots).$$

Let

$$\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{C}_1, \quad \mathfrak{A}_1 = \mathfrak{A}_2 + \mathfrak{C}_2, \dots$$

Then

$$\mathfrak{A} = \mathfrak{C} + \mathfrak{C}_1 + \mathfrak{C}_2 + \dots$$

Let us first exclude the $=$ sign in 1). Then every

Example. Let $\mathfrak{A} =$

$$\mathfrak{B} =$$

$$\mathfrak{C} =$$

Then

$$U$$

and

$$(U$$

are different.

Thus if we write $+$ for

$$\mathfrak{A} + (\mathfrak{B} -$$

18. 1. Let $\mathfrak{A}_1 \geq \mathfrak{A}_2 \geq \mathfrak{A}_3$
aggregates. Then

$$\mathfrak{B} = D$$

Moreover \mathfrak{B} is complete.

Let a_n be a point of \mathfrak{A}_n , n

Any limiting point α of
ing point of

$$a_m,$$

But all these points lie in \mathfrak{A}
 \mathfrak{A}_m , and therefore in every
 $\mathfrak{B} > 0$.

\mathfrak{B} is complete. For let β

$$b_1, b$$

As each b_m is in each \mathfrak{A}_n , and
is in \mathfrak{B} .

POINT SETS

For each point b of \mathfrak{B} lies in some \mathfrak{A}_n , or it lies in \mathfrak{A} , hence in every A_n . In the first case b lies in \mathfrak{A} , in the second in A . Moreover it cannot lie in both A and \mathfrak{A} .

20. 1. Let $\eta_1, \dots, \eta_n, \eta_{n+1}, \dots$

be an infinite sequence of point sets whose union can in fact may be more briefly indicated by the notation

$$\mathfrak{M}_{\text{ind}} \models I'(\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3, \dots).$$

Obviously when \mathfrak{A} is limited,

$$y = \lim y_n.$$

That the inequality may hold as well as the equal shown by the following examples.

Example 1. Let \mathfrak{A}_n be the segment $\left(\frac{1}{n}, 1\right)$.

Then $\|u_n - I^* u_n\|_{L^2(\Omega)} \rightarrow 0$ as $n \rightarrow \infty$.

$$M_1 = M_2 = \dots = M_n = M$$

Example 2. Let a_n denote the points in the unit interval whose abscissae are given by

$$x = \frac{m}{n}, m = n = 1, 2, 3, \dots, m, n \text{ relatively prime}$$

$$1, \dots, n$$

There are

is the totality of rational numbers in $(0^*, 1^*)$.

444 JOURNAL OF DOCUMENTATION

Example 1. Let $\mathfrak{B}_n =$ the segment $\left(0, \frac{1}{n}\right)$.

Then $\mathfrak{B} = \text{Dev} \{\mathfrak{B}_n\} = (0)$, the origin.

Here $\mathfrak{B} = 0, \quad \lim \mathfrak{B}_n = \lim \frac{1}{n} = 0,$

and $\mathfrak{B} = \lim \mathfrak{B}_n.$

Example 2. Let \mathfrak{A}_n be as in 1, Example 2. Let

Let $\mathfrak{B}_n = (1, 2) + b_n.$

Here $\mathfrak{B} =$ the segment $(1, 2)$ and $\mathfrak{B}_n \rightarrow 2.$

Hence $\mathfrak{B} < \lim \mathfrak{B}_n.$

3. Let $\mathfrak{B}_1 < \mathfrak{B}_2 < \dots$ be unmixed parts of \mathfrak{A} . Let $\mathfrak{B} = U \{\mathfrak{B}_n\}$. Then $\mathfrak{C} = \mathfrak{A} - \mathfrak{B}$ is discrete.

For let $\mathfrak{A} = \mathfrak{B}_n + \mathfrak{C}_n$; then \mathfrak{C}_n is an unmixed part of

$$\mathfrak{A} - \mathfrak{B}_n + \mathfrak{C}_n.$$

Passing to the limit $n = \infty$, this gives

$$\lim \mathfrak{C}_n = 0.$$

Hence \mathfrak{C} is discrete by 2.

4. We may obviously apply the terms monotone increasing, monotone decreasing sequences, etc. [Cf. I, 108, 211, of the type 1), 3).

21. Let $\mathfrak{C} = \mathfrak{A} + \mathfrak{B}$. If $\mathfrak{A}, \mathfrak{B}$ are complete,

2. 1. If \mathfrak{A} , \mathfrak{B} are complete, so are also

$$\mathfrak{C} = (\mathfrak{A}, \mathfrak{B}), \quad \mathfrak{E} = Dr(\mathfrak{A}, \mathfrak{B}).$$

Let us first show that \mathfrak{C} is complete. Let c be a limiting point of \mathfrak{C} . Let c_1, c_2, \dots be points of \mathfrak{C} which $\rightarrow c$. Let us separate the c_n into two classes, according as they belong to \mathfrak{A} , or do not. One of these classes must embrace an infinite number of points which $\rightarrow c$. As both \mathfrak{A} and \mathfrak{B} are complete, c lies in either \mathfrak{A} or \mathfrak{B} . Hence it lies in \mathfrak{C} .

To show that \mathfrak{E} is complete. Let d_1, d_2, \dots be points of \mathfrak{E} which $\rightarrow d$. As each d_n is in both \mathfrak{A} and \mathfrak{B} , their limiting point d is in both \mathfrak{A} and \mathfrak{B} , since these are complete. Hence d is in \mathfrak{E} .

2. If \mathfrak{A} , \mathfrak{B} are metric so are

$$\mathfrak{C} = (\mathfrak{A}, \mathfrak{B}) \quad \mathfrak{E} = Dr(\mathfrak{A}, \mathfrak{B}).$$

For the points of Front \mathfrak{C} lie either in Front \mathfrak{A} or in Front \mathfrak{B} . Hence the points of Front $\mathfrak{E} = \text{Front } \mathfrak{A}$ and also $= \text{Front } \mathfrak{B}$. Front \mathfrak{A} and Front \mathfrak{B} are discrete since \mathfrak{A} , \mathfrak{B} are metric.

3. Let the complete set \mathfrak{A} have a complete part \mathfrak{B} . Then for every small $\epsilon > 0$ is taken, there exists a complete set \mathfrak{C} in \mathfrak{A} , having no point in common with \mathfrak{B} such that

$$\mathfrak{C} \subset \mathfrak{A} - \mathfrak{B} \subset \epsilon.$$

Moreover there exists no complete set \mathfrak{C} , having no point in common with \mathfrak{B} such that

$$\mathfrak{C} \subset \mathfrak{A} - \mathfrak{B}.$$

PROOF. — Let ϵ be any small positive number. The theorem follows from (1). To prove

But the cells of $\overline{\mathfrak{C}}_D$ may be subdivided, forming a new \mathfrak{C}_D which does not change the cells of \mathfrak{B}_D , so that $\mathfrak{B}_D = \mathfrak{B}_A$.

$$\overline{\mathfrak{C}}_A = \overline{\mathfrak{C}} + \epsilon''', \quad 0 < \epsilon''' < \epsilon$$

Thus 2), 3), 4) give

$$\mathfrak{A} + \epsilon' = \mathfrak{B} + \epsilon'' + \overline{\mathfrak{C}} + \epsilon''',$$

or

$$\overline{\mathfrak{C}} = \mathfrak{A} - \mathfrak{B} - (\epsilon'' + \epsilon''' - \epsilon'),$$

$$\overline{\mathfrak{C}} \geq \mathfrak{A} - \mathfrak{B} - \epsilon.$$

24. Let $\mathfrak{A}, \mathfrak{B}$ be complete. Let

$$\mathfrak{U} = (\mathfrak{A}, \mathfrak{B}), \quad \mathfrak{F} = \text{Int}(\mathfrak{A}, \mathfrak{B}).$$

Then

$$\mathfrak{A} + \mathfrak{B} = \mathfrak{U} + \mathfrak{F}.$$

For let

$$\mathfrak{U} = \mathfrak{A} + A.$$

Then A contains complete sets C' , such that

$$C' \cap \mathfrak{U} = \mathfrak{A} - \epsilon,$$

but no complete set such that

$$C' \cap \mathfrak{U} = \mathfrak{A},$$

by 23. On the other hand,

$$\mathfrak{B} = A + \mathfrak{F}.$$

Hence A contains complete sets C' , such that

$$C' \cap \mathfrak{B} = \mathfrak{F} - \epsilon,$$

but no complete set such that

For suppose

$$I = k + \mathfrak{L} \neq 0.$$

Let

$$I = \epsilon + \eta, \quad \epsilon, \eta \neq 0.$$

Then by 23 there exists in \mathfrak{A}_1 a complete set \mathfrak{C}_1 having no common with \mathfrak{L} such that

$$\mathfrak{C}_1 + \mathfrak{A}_1 = \mathfrak{L} + \epsilon,$$

as $\mathfrak{A}_1 \neq k$, such that

$$\mathfrak{C}_1 \neq \eta.$$

Let

$$\mathfrak{C}_2 = \text{Dec } (\mathfrak{A}_2, \mathfrak{C}_1), \quad \Pi = \text{c}(\mathfrak{A}_2, \mathfrak{C}_1).$$

Then by 24,

$$\mathfrak{A}_2 + \mathfrak{C}_1 = \Pi + \mathfrak{C}_2.$$

Thus

$$\mathfrak{C}_2 = \mathfrak{A}_2 + \mathfrak{C}_1 - \Pi$$

$$= \mathfrak{A}_2 + \mathfrak{C}_1 - \mathfrak{A}_1$$

$$= (\mathfrak{A}_2 + \text{c}(\mathfrak{A}_1, \mathfrak{L} + \epsilon)) - \mathfrak{A}_1 = \mathfrak{A}_2 - \mathfrak{A}_1 + \mathfrak{L} + \epsilon$$

$$= k + \mathfrak{L} + \epsilon$$

$$= \eta.$$

Thus \mathfrak{A}_2 contains the non-vanishing complete set \mathfrak{C}_2 having no common with \mathfrak{L} . In this way we may continue, $\mathfrak{A}_{2^i} = \text{c}(\mathfrak{A}_2, \mathfrak{C}_i)$ contain a non-vanishing complete component not in k is absurd.

Corollary. Let $\mathfrak{A} = \text{c}(\mathfrak{A}_1, \mathfrak{A}_2, \dots)$ be complete. Then \mathfrak{A}_n

This follows easily from 23, 24.

CHAPTER II

IMPROPER MULTIPLE INTEGRALS

26. Up to the present we have considered only proper multiple integrals. We take up now the case when the integrand is not limited. Such integrals are called improper. In Vol. I, Chapter IV, we get the integrals treated in Vol. I, Chapter IV. In the application of the theory we are now to develop, the order of integration in iterated improper integrals. The treatment of this question given in Vol. I may be generalized by making use of the properties of improper integrals.

27. Let \mathfrak{A} be a limited point set in m -way space. At a point of \mathfrak{A} let $f(x_1 \cdots x_m)$ have a definite value. The points of infinite discontinuity of f which we denote by \mathfrak{S} . In general \mathfrak{S} is discrete, and this is the most important case. But it is not necessary. We call \mathfrak{S} *singular points*.

Example. Let \mathfrak{A} be the unit square. At the points of \mathfrak{A} let $f = \frac{r}{s}$, these fractions being irreducible, let $r = ns$. At the points of \mathfrak{A} let $f = 1$. Here every point of \mathfrak{A} is a point of discontinuity and hence $\mathfrak{S} = \mathfrak{A}$.

Several types of definition of improper integrals have been proposed. We shall mention only three.

GENERAL THEORY

all possible complete divisions Δ of norm δ_n are called *lower* and *upper integrals* of f in \mathfrak{A} , and are denoted by

$$\int_{\mathfrak{A}} f d\mathfrak{A}, \quad \int_{\mathfrak{A}} f d\mathfrak{A},$$

or more shortly by

$$\int_{\mathfrak{A}} f, \quad \int_{\mathfrak{A}} f.$$

When the limits (1) are finite, the corresponding integrals are *convergent*. We also say *f admits a lower or an upper integral in \mathfrak{A}* . When the two integrals (2) are equal, we say *f is integrable in \mathfrak{A}* and denote their common value by

$$\int_{\mathfrak{A}} f d\mathfrak{A} \quad \text{or by} \quad \int_{\mathfrak{A}} f.$$

We call (3) the *improper integral* of f in \mathfrak{A} ; we also say *f admits an improper integral in \mathfrak{A}* and that the integral is *improper*.

The definition of an improper integral just given is an extension of that given in Vol. I, Chapter IV. It is the natural development of the idea of an improper integral which goes back to the beginning of the calculus.

It is convenient to speak of the symbols (2) as upper and lower integrals, even when the limits (1) do not exist. A definition which applies to the symbol (3).

Let us replace f by $g + f$ in one of the symbols (2) or (3). The resulting symbol is called the *integral of the function $g + f$* in \mathfrak{A} . We write

$$\int_{\mathfrak{A}} (g + f) = \mathfrak{M} \int_{\mathfrak{A}} g + \int_{\mathfrak{A}} f$$

We define now the lower integral as

$$\int_{\mathfrak{R}} f = \lim_{\alpha, \beta \rightarrow 0} \int_{\mathfrak{R}} f_{\alpha\beta}.$$

A similar definition holds for the upper integral. The terms introduced in 28 apply here without change.

This definition of an improper integral is due to *L. Poussin*. It has been employed by him and *R. G. D.* with great success.

30. Type III. Let $\alpha, \beta > 0$. Let $\mathfrak{R}_{\alpha\beta}$ denote the region at which

$$\alpha < f(x_1, \dots, x_m) < \beta.$$

We define now

$$\int_{\mathfrak{R}} f = \lim_{\alpha, \beta \rightarrow 0} \int_{\mathfrak{R}_{\alpha\beta}} f \quad ; \quad \int_{\mathfrak{R}} f = \lim_{\alpha, \beta \rightarrow 0} \int_{\mathfrak{R}_{\alpha\beta}} f.$$

The other terms introduced in 28 apply here without change. This type of definition originated with the author and was developed in his lectures.

31. When the points of infinite discontinuity \mathfrak{R} and the upper integrals are absolutely convergent, all the definitions lead to the same result, as we shall show.

When this condition is not satisfied, the results may be different.

Example. Let \mathfrak{R} be the unit square. Let $\mathfrak{R}_1, \mathfrak{R}_2$ denote respectively the upper and lower halves. At the rational* $x = \frac{m}{n}, y = \frac{r}{s}$, in \mathfrak{R}_1 , let $f = ns$. At the other points of \mathfrak{R}_1 let $f = 0$. In \mathfrak{R}_2 let $f = 0$.

*¹⁰ Definition. Here $\mathfrak{R} = \mathfrak{R}_1 \cup \mathfrak{R}_2$.

31. *Definition.* Here $\mathfrak{A}_{\alpha\beta}$ embraces all the points of finite number of points of \mathfrak{A} for $\alpha > 2$, β arbitrarily large.

$$\int_{\mathfrak{A}} f = 1, \quad \int_{\mathfrak{A}} f = 1,$$

and thus

$$\int_{\mathfrak{A}} f = 1.$$

32. In the following we shall adopt the third type as it seems to lead to more general results when treating the important subject of inversion of the order of integration of integrals.

We note that if f is limited in \mathfrak{A} ,

$$\lim_{\alpha, \beta \rightarrow \infty} \int_{\mathfrak{A}} f = \text{the proper integral } \int_{\mathfrak{A}} f.$$

For α, β being sufficiently large, $\mathfrak{A}_{\alpha\beta} = \mathfrak{A}$.

Also, if \mathfrak{A} is discrete,

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}} f = 0.$$

For $\mathfrak{A}_{\alpha\beta}$ is discrete, and hence

$$\int_{\mathfrak{A}_{\alpha\beta}} f = 0.$$

Hence the limit of these integrals is 0.

33. Let $m = \text{Min}_f$, $M = \text{Max}_f$ in \mathfrak{A} .

Then

$$\lim_{\alpha, \beta \rightarrow \infty} \int_{\mathfrak{A}_{\alpha\beta}} f = \lim_{\alpha \rightarrow \infty} \int_{\mathfrak{A}_{\alpha, \infty}} f = 0 \text{ if } f \text{ is } f.$$

Thus in these cases we may simplify our notation by writing

$$\mathfrak{A}_{\alpha, M} = \mathfrak{A}_{\text{imp}}$$

by

$$\mathfrak{A}_{\alpha} = \mathfrak{A}_{\beta},$$

respectively.

2. Thus we have:

$$\int_{\mathfrak{A}} f = \lim_{\beta \rightarrow \beta_0} \int_{\mathfrak{A}_{\beta}} f, \quad \text{when } \text{Min } f \text{ is finite.}$$

$$\int_{\mathfrak{A}} f = \lim_{\alpha \rightarrow \alpha_0} \int_{\mathfrak{A}_{\alpha}} f, \quad \text{when } \text{Max } f \text{ is finite.}$$

3. Sometimes we have to deal with several functions. In this case the notation $\mathfrak{A}_{\text{imp}}$ is ambiguous. To make it clear let $\mathfrak{A}_{f, \alpha, \beta}$ denote the points of \mathfrak{A} where

$$a = \alpha, \quad f = \beta.$$

Similarly, $\mathfrak{A}_{g, \alpha, \beta}$ denotes the points where

$$a = \alpha, \quad g = \beta, \text{ etc.}$$

34. $\int_{\mathfrak{A}_{\alpha\beta}} f$ is a monotone decreasing function of α for each

$\int_{\mathfrak{A}_{\alpha\beta}} f$ is a monotone increasing function of β for each

If $\text{Max } f$ is finite

$\int_{\mathfrak{A}_{\alpha}} f$ are monotone decreasing functions of α .

If $\text{Min } f$ is finite

$\int_{\mathfrak{A}_{\beta}} f$ are monotone increasing functions of β .

But each cell d_i of $\mathfrak{A}_{\alpha\beta}$ lies among the cells can break up the sum 2), getting

$$\sum_{\mathfrak{A}_{\alpha'\beta}} m'_i d'_i = \sum_{\mathfrak{A}_{\alpha\beta}} m'_i d'_i + \sum m''_i d''_i.$$

Here the second term on the right is sum not containing points of $\mathfrak{A}_{\alpha\beta}$. It is thus \leq on the right $m'_i < m_i$. It is thus less than the

$$\sum_{\mathfrak{A}_{\alpha'\beta}} m'_i d'_i < \sum_{\mathfrak{A}_{\alpha\beta}} m_i d_i.$$

Thus

$$\int_{\mathfrak{A}_{\alpha'\beta}} f < \int_{\mathfrak{A}_{\alpha\beta}} f, \quad \alpha' > \alpha.$$

In a similar manner we may prove the same for us turn to the third.

We need only to show that

$$\int_{\mathfrak{A}_{-\alpha}} f \text{ is monotone decreasing.}$$

Let $\alpha' > \alpha$. Then

$$\int_{\mathfrak{A}_{-\alpha}} f = \lim_{d \rightarrow 0} \sum_{\mathfrak{A}_{-\alpha}} M_i d_i.$$

$$\int_{\mathfrak{A}_{-\alpha'}} f = \lim_{d \rightarrow 0} \sum_{\mathfrak{A}_{-\alpha'}} M'_i d'_i.$$

As before

$$\sum_{\mathfrak{A}_{-\alpha'}} M'_i d'_i = \sum_{\mathfrak{A}_{-\alpha}} M'_i d'_i + \sum M''_i d''_i$$

But in the cells d_i , $M'_i = M_i$. Hence the same as Σ in 3). The second term of 5) follows now as before.

For by 34

$$\int_{\mathfrak{A}_n} f \leq \int_{\mathfrak{A}_{n+1}} f$$

are limited monotone functions. Their limits exist.

36. If $M = \text{Max } f$ is finite, and $\int_{\mathfrak{A}} f$ is convergent, then the corresponding upper integral is convergent and

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}} f = M \lim_{n \rightarrow \infty} \mathfrak{A}_n,$$

where $f \rightarrow \infty$ in \mathfrak{A}_n .

Similarly, if $m = \text{Min } f$ is finite and $\int_{\mathfrak{A}} f$ is convergent, then the corresponding lower integral is convergent and

$$m \lim_{n \rightarrow \infty} \mathfrak{A}_n = \int_{\mathfrak{A}} f = \int_{\mathfrak{A}} f, \quad f \rightarrow -\infty \text{ in } \mathfrak{A}_n.$$

Let us prove the first half of the theorem.

We have

$$\int_{\mathfrak{A}} f = \lim_{n \rightarrow \infty} \int_{\mathfrak{A}_n} f$$

Now

$$\int_{\mathfrak{A}} f \leq \int_{\mathfrak{A}_n} f = \int_{\mathfrak{A}_n} f \leq M \mathfrak{A}_n.$$

We have now only to pass to the limit.

37. If $\int_{\mathfrak{A}} f$ is convergent, and $\mathfrak{B} = \mathfrak{A}$,

$$\int_{\mathfrak{A}} f$$

Let

$$f = 1 \quad \text{when } x \text{ is irr}$$

$$= \frac{1}{y} \quad \text{when } x \text{ is rational}$$

Then

$$\int_{\mathfrak{U}_B} f = 1 \quad ; \quad \text{hence } \int_{\mathfrak{U}'} f$$

On the other hand,

$$\int_{\mathbb{Q}_p} = \int_0^1 dx \int_1^{\frac{1}{x}} \frac{dy}{y} = \log \mathcal{E}$$

Hence

$$\int_{\mathbb{Q}^3} \lim_{\beta \rightarrow \infty} \int_{\mathbb{Q}^3 \beta} = \lim \log \beta$$

is divergent.

38. 1. In the future it will be convenient to call the points of \mathfrak{A} where $f = 0$, and \mathfrak{B} the points where $f > 0$, and call them the *positive* and *negative components* of \mathfrak{A} .

2. If $\int_{\mathfrak{A}} f$ converges, so do $\int_{\mathfrak{B}} f$.

If $\int_{\mathfrak{A}} f$ converges, so do $\int_{\mathfrak{B}} f$.

For let us effect a cubical division of s so that $\beta' \geq \beta$. Let e denote those cells containing those cells containing a point of $\mathfrak{B}_{\beta'}$ but not those cells containing a point of $\mathfrak{A}_{\alpha\beta}$ but none of $\mathfrak{A}_{\beta'}$.

There

We find similarly

$$\bar{\int}_{\mathfrak{B}_{\beta'}} - \bar{\int}_{\mathfrak{B}_{\beta}} = \lim_{d \rightarrow 0} \{ \Sigma (M'_e - M_e)e + \Sigma M'_e e' \}.$$

Now

$$\left| \bar{\int}_{\mathfrak{A}_{\alpha\beta'}} - \bar{\int}_{\mathfrak{A}_{\alpha\beta}} \right| < \epsilon$$

for a sufficiently large α , and for any $\beta, \beta' \geq \beta_0$.

Hence the same is true of the left side of (1).

As corollaries we have:

3. *If the upper integral of f is convergent in \mathfrak{A} , then*

$$\bar{\int}_P f \leq \bar{\int}_{\mathfrak{B}} f \quad P \in \mathfrak{B}.$$

If the lower integral of f is convergent in \mathfrak{A} ,

$$\underline{\int}_N f > \underline{\int}_{\mathfrak{A}} f \quad N \in \mathfrak{A}.$$

For

$$\bar{\int}_P f \leq \bar{\int}_{\mathfrak{B}_{\beta}} f \leq \bar{\int}_{\mathfrak{B}} f \quad \text{etc.}$$

4. *If $f \geq 0$ and $\bar{\int}_{\mathfrak{A}} f$ is convergent, so is*

$$\bar{\int}_{\mathfrak{B}} f, \quad \mathfrak{B} \in \mathfrak{A}.$$

Moreover the second integral is \leq the first.

This follows at once from 3, as $\mathfrak{A} = \mathfrak{B}$.

Now any cell d_i of $\mathfrak{A}_{\alpha\beta}$ is a cell of $\mathfrak{A}_{\alpha'\beta'}$, and hence $A_{\alpha'\beta'} \geq A_{\alpha\beta}$. Thus $A_{\alpha\beta}$ is a monotone inc. of α, β . On the other hand

$$\lim A_{\alpha\beta} = 0,$$

by hypothesis. Hence $A_{\alpha\beta} = 0$ and thus f is integr.

Next let f be limited in \mathfrak{B} , then $|f| \leq \text{some } M$ in $\mathfrak{B} \subset \mathfrak{A}_{\gamma, \gamma'}$. But f being integrable in $\mathfrak{A}_{\gamma, \gamma'}$, it is in \mathfrak{B} .

Let us now consider the general case. Since f

$$\int_{\mathfrak{B}} f, \quad \int_{\mathfrak{A}} f,$$

both converge by 38. Let now P, N be the points in \mathfrak{B} . Then

$$\int_P f = \int_{\mathfrak{B}} f, \quad \left| \int_P f \right| = \left| \int_{\mathfrak{B}} f \right|.$$

Thus

$$\int_P f = \int_N f$$

both converge. Hence by 39,

$$\int_{\mathfrak{B}} f$$

both converge. But if $\mathfrak{A}_{a,b}$ denote the points $a < f < b$,

$$\int_{\mathfrak{B}} f = \lim_{a, b \rightarrow 0} \int_{\mathfrak{A}_{a,b}} f,$$

by definition.

But as just seen,

$$\int_{\mathfrak{A}_{a,b}} = \int_{\mathfrak{A}_{a,b}}$$

Hence

$$C = c = C = c$$

For let

$$\int_{\mathfrak{A}} f = \lim_{\alpha, \beta \rightarrow \epsilon} \int_{\mathfrak{A}_{\alpha\beta}} f$$

exist. Since

$$\int_{\mathfrak{A}_{\alpha\beta}} = \int_{\mathfrak{A}_{\alpha\beta}},$$

necessarily

$$\int_{\mathfrak{A}} f = \lim_{\alpha, \beta \rightarrow \epsilon} \int_{\mathfrak{A}_{\alpha\beta}} f$$

exists and 1), 2) are equal.

42. 1. In studying the function f it is so introduce two auxiliary functions defined as

$$\begin{aligned} g &= f & \text{where } f > 0 \\ &= 0 & \text{where } f \leq 0 \\ h &= -f & \text{where } f < 0 \\ &= 0 & \text{where } f \geq 0 \end{aligned}$$

Thus g, h are both ≥ 0 and

$$f = g - h,$$

$$|f| = g + h.$$

We call them the *associated non-negative f*

2. As usual let $\mathfrak{A}_{\alpha\beta}$ denote the points of \mathfrak{A} . Let \mathfrak{A}_α denote the points where $g \leq \beta$, and \mathfrak{A}_α . Then

$$\int_{\mathfrak{A}} g = \lim_{\alpha, \beta \rightarrow \epsilon} \int_{\mathfrak{A}_{\alpha\beta}} g,$$

$$\int_{\mathfrak{A}} h = \lim_{\alpha, \beta \rightarrow \epsilon} \int_{\mathfrak{A}_{\alpha\beta}} h.$$

For

Let $f = 1$ at the irrational points in $\mathfrak{A} = (0, 1)$,
 $= -n$, for $x = \frac{m}{n}$ in \mathfrak{A} .

Then

$$\int_{\mathfrak{A}_n} g = 0 \quad , \quad \int_{\mathfrak{A}_{n\beta}} g = 1.$$

Again let $f = -1$ for the irrational points in \mathfrak{A} ,
 $= n$ for the rational points $x = \frac{m}{n}$.

Then

$$\int_{\mathfrak{A}_n} h = 0, \quad \int_{\mathfrak{A}_{n\beta}} h = 1.$$

43. 1.

$$1) \quad \int_{\mathfrak{A}} g = \int_{\mathfrak{B}} f; \quad \int_{\mathfrak{A}} g \leq \int_{\mathfrak{B}} f,$$

$$3) \quad \int_{\mathfrak{A}} h = - \int_{\mathfrak{B}} f, \quad \int_{\mathfrak{A}} h \geq - \int_{\mathfrak{B}} f,$$

*provided the integral on either side of the equations converges,
provided the integrals on the right side of the inequalities converge.*

Let us prove 1); the others are similarly established.
a cubical division of space of norm d , we have for a fixed

$$\begin{aligned} \int_{\mathfrak{A}_n} g &= \lim_{d \rightarrow 0} \left\{ \sum_{\mathfrak{B}_n} M_i d_i + \sum 0 \cdot d_i \right\} \\ &= \lim_{d \rightarrow 0} \sum_{\mathfrak{B}_n} M_i d_i = \int_{\mathfrak{B}_n} f. \end{aligned}$$

3. If $\int_{\mathfrak{A}} f$ is convergent, we cannot say the same for $\int_{\mathfrak{B}} f$. A similar remark holds for the limit.

For let $f = 1$ at the rational points of \mathfrak{A} and $f = -\frac{1}{x}$ at the irrational points of \mathfrak{A} .

Then

$$\int_{\mathfrak{A}} f = 1 \quad , \quad \int_{\mathfrak{B}} f = -\infty$$

4. That the inequality sign in 2) or 4) is not shown thus :

Let

$$f = \frac{1}{\sqrt{x}} \text{ for rational } x \text{ in } \mathfrak{A} = \mathfrak{A}_1 \\ = -\frac{1}{\sqrt{x}} \text{ for irrational } x.$$

Then

$$\int_{\mathfrak{A}} f = 0 \quad , \quad \int_{\mathfrak{B}} f = 2.$$

44. 1.

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}} g = \lim_{\alpha, \beta \rightarrow \epsilon} \int_{\mathfrak{A}_{\alpha\beta}} h$$

$$\int_{\mathfrak{A}} f = \lim_{\alpha, \beta \rightarrow \epsilon} \int_{\mathfrak{A}_{\alpha\beta}} g = \int_{\mathfrak{A}} h$$

provided, if the integral on the left exists, or if the limit on the right exist.

For let us effect a cubical division of \mathfrak{A} containing points of \mathfrak{A} fall into two classes :

Let now $\alpha, \beta \doteq \infty$. If the integral on the left of 1) is convergent, the integral on the right of 1) is convergent by 43, 1, and the limit on the right of 1) exists. Using now 42, 2, v

Let us now look at the 2^o hypothesis. By 42, 2,

$$\lim_{\alpha, \beta \rightarrow \infty} \int_{\mathfrak{A}_{\alpha\beta}} g = \int_{\mathfrak{A}} g.$$

Thus passing to the limit in 4), we get 1).

2. A relation of the type

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}} g + \int_{\mathfrak{A}} h$$

does not always hold as the following shows.

Example. Let $f = n$ at the points $x = \frac{m}{2^{2n}}$

$$= -n \text{ for } x = \frac{m}{2^{2n+1}}$$

$= -1$ at the other points of $\mathfrak{A} = (0, 1)$.

Then
$$\int_{\mathfrak{A}} f = -1 \quad \int_{\mathfrak{A}} g = 0 \quad \int_{\mathfrak{A}} h = 0.$$

45. If $\int_{\mathfrak{A}} f$ is convergent, it is in any unmixed part \mathfrak{A}'

Let us consider the upper integral first. By 43, 2,

$$\int_{\mathfrak{A}} g$$

As the limit of the right side exists, that
From this fact, and because 1) exists,

$$\int_{\mathfrak{A}} f$$

exists by 44, 1.

A similar demonstration holds for the low

46. If $\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_m$, form an unmixed divis

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}_1} f + \dots + \int_{\mathfrak{A}_m} f$$

provided the integral on the left exists or a
right exist.

For if $\mathfrak{A}_{m, \alpha \beta}$ denote the points of $\mathfrak{A}_{\alpha \beta}$ in \mathfrak{A}_m

$$\int_{\mathfrak{A}_{\alpha \beta}} f = \int_{\mathfrak{A}_{1 \alpha \beta}} f + \dots + \int_{\mathfrak{A}_{m \alpha \beta}} f$$

Now if the integral on the left of 1) is con
on the right of 1) all converge by 45. Pas
gives 1). On the other hypothesis, the int
1) existing, a passage to the limit in 2) sh
this case also.

47. If $\int_{\mathfrak{A}} f$ and $\int_{\mathfrak{A}} f$ converge, so does $\int_{\mathfrak{A}} \frac{1}{f}$

$$\begin{aligned} \int_{\mathfrak{A}} f &= \int_{\mathfrak{A}} f - \int_{\mathfrak{A}} f \\ &= \int_{\mathfrak{A}} g + \int_{\mathfrak{A}} h. \end{aligned}$$

For let A_n denote the points of \mathfrak{A} where

$$0 < f < \beta.$$

48. 1. If $\int_{\mathfrak{A}} |f|$ converges, both $\int_{\mathfrak{A}} f$ converge.

For as usual let \mathfrak{B} denote the points of \mathfrak{A} where $f \geq 0$. Then

$$\int_{\mathfrak{B}} f = \int_{\mathfrak{B}} |f|$$

is convergent by 38, 3, since $\int_{\mathfrak{A}} |f|$ is convergent.

Similarly,

$$\int_{\mathfrak{A}} (-f) = - \int_{\mathfrak{A}} f$$

is convergent. The theorem follows now by 39.

2. If $\int_{\mathfrak{A}} |f|$ converges, so do

$$\int_{\mathfrak{A}} g, \quad \int_{\mathfrak{A}} h.$$

For by 1,

$$\int_{\mathfrak{A}} f$$

both converge. The theorem now follows by 43, 2.

3. For

$$\int_{\mathfrak{A}} f$$

both to converge it is necessary and sufficient that

$$\int_{\mathfrak{A}} |f|$$

is convergent.

For if 3) converges, the integrals 2) both converge.

On the other hand if both the integrals 2) converge

But by 41, 1, f is integrable in A_μ . Hence A_μ by 1, 720. Thus

$$\int_{\mathfrak{A}} |f| = \lim_{\mu \rightarrow \infty} \int_{A_\mu} |f|.$$

49. From the above it follows that if both

$$\int_{\mathfrak{A}} f$$

converge, they converge absolutely. Thus, if

$$\int_{\mathfrak{A}} f$$

converges, it is absolutely convergent.

We must, however, guard the reader against supposing that *only absolutely* convergent upper exist.

Example. At the rational points of $\mathfrak{A} = (0$

$$f(x) = \frac{1}{2\sqrt{x}}.$$

At the irrational points let

$$f(x) = \frac{1}{x}.$$

Here

$$\int_{\mathfrak{A}} f = 1 \quad \int_{\mathfrak{A}} f = \infty.$$

Thus, f admits an upper, but not a lower. On the other hand the upper integral of f does not

For obviously $\int_{\mathfrak{A}} f = +\infty$.

Let us consider, for example,

$$J = \int_0^1 \int_{\frac{1}{x}}^{\frac{1}{\sin x}} dx = \int_{\frac{1}{2}}^1 f dx.$$

If we set $x = \frac{1}{u}$, we get

$$J = \int_1^2 \frac{\sin u}{u} du,$$

which converges by I, 667, but is not absolutely convergent by I, 646.

This apparent discrepancy at once disappears when we note that according to the definition laid down in Vol. I,

$$J = R \lim_{a \rightarrow 0} \int_a^1 f dx,$$

while in the present chapter

$$J = \lim_{a, n \rightarrow \infty} \int_{a, n} f dx,$$

Now it is easy to see that, taking α large at pleasure

$$\int_{a, n} f dx \leq \epsilon \quad \text{as } \beta \geq \epsilon,$$

so that J does not converge according to our present definition.

In the theory of integration as ordinarily developed on the calculus a similar phenomenon occurs, viz. convergent integrals exist when $m > 1$.

51. 1. If $\int |f|$ is convergent,

2. If $\int_{\mathfrak{A}} |f|$ is convergent, $\int_{\mathfrak{A}} f$ are convergent for all \mathfrak{A} .

For $\int_{\mathfrak{A}} f$ is convergent by 38, 4.

Hence

$$\int_{\mathfrak{A}} f$$

converge by 48, 3.

3. If $1 + \int_{\mathfrak{A}} f$ is convergent and $\text{Min } f$ is finite, or convergent and $\text{Max } f$ is finite, then

$$\int_{\mathfrak{A}} |f|$$

is convergent.

This follows by 36 and 48, 3.

52. Let $f \rightarrow 0$ in \mathfrak{A} . Let the integral

$$\int_{\mathfrak{A}} f$$

converge. If

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}_0} f + \alpha,$$

then for any unimixed part $\mathfrak{A}_0 \subset \mathfrak{A}$

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}_0} f + \alpha',$$

where

$$0 \leq \alpha' \leq \alpha.$$

For let $\mathfrak{A} = \mathfrak{A}_0 + \mathfrak{C}$. Then $\mathfrak{A}_0 = \mathfrak{A}_0 + \mathfrak{C}_0$ is an unimixed part. Also

Hence

$$\int_{\mathfrak{A}} f + \int_{\mathfrak{C}} f = \int_{\mathfrak{A}_0} f + \int_{\mathfrak{C}_0} f + \alpha.$$

From 2)

$$\begin{aligned} \alpha' &= \int_{\mathfrak{A}} f - \int_{\mathfrak{A}_0} f \\ &= \alpha - \left(\int_{\mathfrak{C}} f - \int_{\mathfrak{C}_0} f \right) \\ &= \alpha, \end{aligned}$$

which establishes 3).

53. If the integral

$$\int_{\mathfrak{A}} f$$

converges, then

$$\epsilon > 0, \quad \sigma > 0, \quad \int_{\mathfrak{A}} f = \epsilon$$

for any $\mathfrak{A} \supset \mathfrak{A}$ such that

$$\mathfrak{A} \subset \sigma.$$

Let us suppose first that $f \geq 0$. If the theorem exists, however small $\sigma > 0$ is taken, a \mathfrak{A} sat that

$$\int_{\mathfrak{A}} f = \epsilon.$$

Then there exists a cubical division of space points of \mathfrak{A} , call them \mathfrak{C} , which lie in cells containing \mathfrak{A} , are such that $\mathfrak{C} \subset \sigma$ also. Moreover \mathfrak{C} is an un Then from 4) follows, as $f \geq 0$, that

$$\int_{\mathfrak{C}} f = \epsilon$$

also.

Let us now take β so that

Let now $\beta\sigma < \epsilon$, then

$$\int_{\mathfrak{G}_1} f > \epsilon,$$

which contradicts 5).

Let us now make no restrictions on the sign

$$\left| \int_{\mathfrak{G}'} f \right| < \int_{\mathfrak{G}} |f|.$$

But since 1) converges, the present case is proceeding.

54. 1. Let $\int_{\mathfrak{G}} f$ converge.

Let as usual $\mathfrak{A}_{\alpha\beta}$ denote the points of \mathfrak{A} at which $A_{\alpha\beta}$ be such that each $\mathfrak{A}_{\alpha\beta}$ lies in some $A_{\beta\gamma}$ in which $\int_{A_{\beta\gamma}} f = 0$. Let $\mathfrak{L}_{\alpha\beta} = A_{\alpha\beta} - \mathfrak{A}_{\alpha\beta}$ and let $\alpha, \beta \rightarrow \infty$ with α, β .

$$\lim_{\alpha, \beta \rightarrow \infty} \mathfrak{L}_{\alpha\beta} = 0.$$

For if not, let

$$\lim_{\alpha, \beta \rightarrow \infty} \mathfrak{L}_{\alpha\beta} = l, \quad l > 0.$$

Then for any $0 < \lambda < l$, there exists a monotone such that

$$\mathfrak{L}_{\alpha_n\beta_n} > \lambda \quad \text{for } n > \text{some } n.$$

Let $\mu_n = \text{Min}(\alpha_n, \beta_n)$, then $\int_{\mathfrak{L}_{\alpha_n\beta_n}} f > \mu_n$ in Hence

$$\int_{\mathfrak{L}_{\alpha_n\beta_n}} |f| > \mu_n \lambda > \epsilon.$$

But $\mathfrak{L}_{\alpha_n\beta_n}$ being a part of \mathfrak{A}

If, 1°,

$$\lim_{\beta \rightarrow \infty} \int_{\mathfrak{A}_\beta} |f|,$$

or if, 2°,

$$\lim_{b \rightarrow \infty} \int_{I_b} |f|$$

exist, then,

$$\lim_{a, b \rightarrow \infty} \int_{A_{a, b}} f = \int_{\mathfrak{A}} f.$$

For, if 2° holds, 1° holds also, since

$$\int_{\mathfrak{A}_\beta} |f| \leq \int_{I_b} |f|, \quad \text{as } \mathfrak{A}_\beta \subset I_b.$$

Thus case 2° is reduced to 1°. Let then the 1° hold. We have

$$\int_{\mathfrak{A}_{a\beta}} f = \int_{\mathfrak{A}_{a\beta}} g + \int_{\mathfrak{A}_{a\beta}} h,$$

as 4) in 44, 1 shows. Let now

$$\mathfrak{D}_{a\beta} = A_{ab} \cap \mathfrak{A}_{a\beta}.$$

Then,

$$\int_{\mathfrak{A}_{a\beta}} g = \int_{I_{ab}} g + \int_{\mathfrak{A}_{a\beta}} g + \int_{\mathfrak{D}_{a\beta}} g.$$

But $\mathfrak{D}_{a\beta} \doteq 0$, as $a, \beta \doteq \infty$, by 54. Let us now pass $a, \beta \doteq \infty$ in 3). Since the limit of the last term is 0, we get

$$\lim_{a, \beta \rightarrow \infty} \int_{\mathfrak{A}_{a\beta}} g = \lim_{a, b \rightarrow \infty} \int_{I_{a, b}} g.$$

Similarly,

$$\lim_{a, b \rightarrow \infty} \int_{A_{a, b}} h = \lim_{a, b \rightarrow \infty} \int_{I_{a, b}} h.$$

2. The following example is instructive as showing that the conditions imposed in 1 are not fulfilled, the result does not hold.

Example. Since

$$\int_0^1 \frac{dx}{x^2} = +\infty,$$

there exists, for any $b_n > 0$, a $0 < b_{n+1} < b_n$, such that

$$\int_{b_{n+1}}^{b_n} \frac{dx}{x^2} = G_{n+1},$$

then

$$G_1 + G_2 + \cdots = +\infty,$$

as $b_n \rightarrow 0$. Let now

$$f(x) = \begin{cases} 1 & \text{for the rational points in } \mathfrak{R} \\ \frac{1}{x^2} & \text{for the irrational.} \end{cases}$$

Then

$$\int_{b_1}^{b_2} f(x) dx = 1 + \frac{1}{b_2^2}.$$

Let

$$\int_{b_1}^{b_2} \frac{dx}{x^2} = G_1, \quad 0 < b_1 < 1.$$

Let A_n denote the points of \mathfrak{R} in $(b_n, 1)$ and the irrationals in (b_{n+1}, b_n) .

Then

$$\int_{b_n}^{b_{n+1}} f(x) dx = +\infty.$$

But obviously the set A_n is conjugate to \mathfrak{R}_n . On the other hand

$$\int_{b_1}^{b_2} f(x) dx = 1,$$

while

$$\lim_{n \rightarrow \infty} \int_{b_n}^{b_{n+1}} f(x) dx = +\infty.$$

Let us establish the theorem for the upper integral; similar reasoning may be used for the lower. Since 1) is convergent,

$$\lambda = \lim_{a, b \rightarrow +\infty} \int_{\mathfrak{A}_{a|b}} g$$

exists by 44, 1. Since 3) exists, we have by 53,

$$0 < \int_{\mathfrak{A}} g < \frac{\epsilon}{4},$$

for any $\mathfrak{B} \in \mathfrak{A}$ such that $\mathfrak{B} \subset \sigma$ for some σ' .

Since 4) exists, there exists a pair of values a, b such that

$$\lambda < \int_{\mathfrak{A}_{a|b}} h + \eta, \quad 0 < \eta < \frac{\epsilon}{4},$$

since the integral on the right side of 4) is a monotone increasing function of a, b .

Since $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$ is an unmixed division of \mathfrak{A} ,

$$\int_{\mathfrak{A}_{a|b}} h = \int_{\mathfrak{B}_{a|b}} h + \int_{\mathfrak{C}_{a|b}} h.$$

Since $h \geq 0$, and the limit 4) exists, the above shows that

$$\mu = \lim_{a, b \rightarrow +\infty} \int_{\mathfrak{B}_{a|b}} h, \quad \nu = \lim_{a, b \rightarrow +\infty} \int_{\mathfrak{C}_{a|b}} h$$

exist and that

$$\lambda = \mu + \nu.$$

Then a, b being the same as in 6),

Thus,

$$\mu \leq \frac{\epsilon}{4}.$$

But

$$\int_{\mathfrak{A}} f - \int_{\mathfrak{A}} g = \mu$$

by 44, 1. Thus 2) follows on using 5), 11) and

57. *If the integral $\int_{\mathfrak{A}} f$ converges and \mathfrak{A}_n is \mathfrak{A} such that $\mathfrak{A}_n \rightarrow \mathfrak{A}$ as $n \rightarrow \infty$, then*

$$\lim_{n \rightarrow \infty} \int_{\mathfrak{A}_n} f = \int_{\mathfrak{A}} f.$$

For if we set $\mathfrak{A} = \mathfrak{A}_n + \mathfrak{C}_n$, the first set is an \mathfrak{A} and $\mathfrak{C}_n \rightarrow 0$. Now

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}_n} f + \int_{\mathfrak{C}_n} f.$$

Passing to the limit, we get 1) on using 56.

58. 1. *Let* $\mathfrak{F}_{\alpha\beta} = D(\mathfrak{A}_{\alpha\beta}, \mathfrak{A}_{\alpha\beta})$

If, 1, the upper contents of

$$\mathfrak{f}_{\alpha\beta} = \mathfrak{A}_{\alpha\beta} - \mathfrak{F}_{\alpha\beta}, \quad \mathfrak{g}_{\alpha\beta} = \mathfrak{A}_{\alpha\beta} - \mathfrak{F}_{\alpha\beta}, \quad \mathfrak{h}_{\alpha\beta} = \mathfrak{f}_{\alpha\beta} + \mathfrak{g}_{\alpha\beta}$$

$$0 \text{ as } \alpha, \beta \rightarrow \infty,$$

and if, 2, the upper integrals of $f, g, f + g$ are

$$\int_{\mathfrak{A}} (f + g) \leq \int_{\mathfrak{A}} f + \int_{\mathfrak{A}} g.$$

Then $D_{\alpha\beta}$ may be chosen so that $\mathfrak{F}_{\alpha\beta} \doteq 0$.

Now

$$\int_{\mathfrak{A}_{f, \alpha\beta}} f = \int_{\mathfrak{E}_{\alpha\beta}} f + \int_{\mathfrak{F}_{\alpha\beta}} f,$$

since the fields are unmixed. By 56, the second integral on the right $\doteq 0$ as $\alpha, \beta \doteq \infty$. Hence

$$\lim_{\alpha, \beta \rightarrow \infty} \int_{\mathfrak{A}_{f, \alpha\beta}} f = \lim_{\alpha, \beta \rightarrow \infty} \int_{\mathfrak{E}_{\alpha\beta}} f.$$

Similar reasoning applies to g and $f + g$.

Again,

$$\int_{\mathfrak{E}_{\alpha\beta}} (f + g) = \int_{\mathfrak{E}_{\alpha\beta}} f + \int_{\mathfrak{E}_{\alpha\beta}} g.$$

Thus, letting $\alpha, \beta \doteq \infty$ we get 2).

2. *When the singular points of f, g are discrete, the conclusion holds.*

3. *If g is integrable and the conditions 1', 2', 3' are satisfied,*

$$\int_{\mathfrak{A}} (f + g) = \int_{\mathfrak{A}} f + \int_{\mathfrak{A}} g.$$

4. *If f, g are integrable and condition 1' is satisfied, $f + g$ is integrable and*

$$\int_{\mathfrak{A}} (f + g) = \int_{\mathfrak{A}} f + \int_{\mathfrak{A}} g.$$

$$5. \quad \int_{\mathfrak{A}} (f + C) = \int_{\mathfrak{A}} f + C \lim_{\alpha, \beta \rightarrow \infty} \mathfrak{A}_{\alpha\beta}$$

Example. Let \mathfrak{A} consist of the rational points

Let

$$f = 1 + n \quad , \quad g = 1 - n$$

at the point $x = \frac{m}{n}$. Then

$$f + g = 2 \quad \text{in } \mathfrak{A}.$$

Now

$$\mathfrak{A}_{f, \alpha\beta} \quad , \quad \mathfrak{A}_{g, \alpha\beta}$$

embrace only a finite number of points for a given α, β . On the other hand,

$$\mathfrak{A}_{f+g, \alpha\beta} = \mathfrak{A} \quad \text{for } \beta > 2.$$

Thus the upper content of the last set in \mathfrak{A} is $\alpha, \beta = \infty$ and condition 1° is not fulfilled. A similar argument does not hold in this case. For

$$\int_{\mathfrak{A}} (f + g) = 2 \quad , \quad \int_{\mathfrak{A}} f = 0 \quad , \quad \int_{\mathfrak{A}} g = 0$$

59.

$$\text{If } c \neq 0, \text{ then } \int_{\mathfrak{A}} cf = c \int_{\mathfrak{A}} f,$$

$$\text{if } c = 0, \text{ then } \int_{\mathfrak{A}} cf = c \int_{\mathfrak{A}} f,$$

provided the integral on either side is convergent.

For

$$\begin{aligned} \int_{\mathfrak{A}_{cf, \alpha\beta}} cf &= c \int_{\mathfrak{A}_{cf, \alpha\beta}} f \\ &= c \int_{\mathfrak{A}_{f, \alpha\beta}} f \end{aligned}$$

Then $D_{\alpha\beta}$ may be chosen so that $\lambda_{\alpha\beta} = 0$.

Now

$$\int_{\mathfrak{M}_{\alpha\beta, \text{all}}} f = \int_{\mathfrak{M}_{\alpha\beta}} f + \int_{\lambda_{\alpha\beta}}$$

since the fields are unmixed. By 56, the second integral on the right $\doteq 0$ as $\alpha, \beta \doteq \infty$. Hence

$$\lim_{\alpha, \beta \rightarrow \infty} \int_{\mathfrak{M}_{\alpha\beta, \text{all}}} f = \lim_{\alpha, \beta \rightarrow \infty} \int_{\mathfrak{M}_{\alpha\beta}} f.$$

Similar reasoning applies to g and $f + g$.

Again,

$$\int_{\mathfrak{M}_{\alpha\beta}} (f + g) = \int_{\mathfrak{M}_{\alpha\beta}} f + \int_{\mathfrak{M}_{\alpha\beta}} g.$$

Thus, letting $\alpha, \beta \doteq \infty$ we get 24.

2. *When the singular points of f, g are discrete, the above holds.*

3. *If g is integrable and the conditions 1, 2, 3, 4 are satisfied,*

$$\int_{\mathfrak{M}} (f + g) = \int_{\mathfrak{M}} f + \int_{\mathfrak{M}} g.$$

4. *If f, g are integrable and condition 1 is satisfied, f is integrable and*

$$\int_{\mathfrak{M}} (f + g) = \int_{\mathfrak{M}} f + \int_{\mathfrak{M}} g.$$

5.

$$\int_{\mathfrak{M}} (f + c) = \int_{\mathfrak{M}} f + c \lim_{\alpha, \beta \rightarrow \infty} \lambda_{\alpha\beta}.$$

GENERAL THEORY

Example. Let \mathfrak{A} consist of the rational points in $(0, 1)$.

Let $f = 1 + n^{-1}$, $g = 1 - n^{-1}$

at the point $x = \frac{n}{n+1}$. Then

$$f(x) + g(x) = 2 \quad (x \in \mathfrak{A}).$$

Now

$$\mathfrak{A}_{x_0+\varepsilon} = \mathfrak{A}_{x_0-\varepsilon}$$

embrace only a finite number of points for a given ε , on other hand,

$$\mathfrak{A}_{x_0+\varepsilon} \cap \mathfrak{A}_{x_0-\varepsilon} = \mathfrak{A} \quad (0 < \varepsilon < 1).$$

Thus the upper content of the set set in I does not depend on ε and condition 1 is not fulfilled. Also relation (1) does not hold in this case. For

$$\int_A f(x) + g(x) dx = 2, \quad \int_A f(x) dx = \int_A g(x) dx = 0.$$

$$\begin{aligned} 59. \quad P &= 0, \quad P^2 = \int_A f^2 = \int_A g^2 \\ &= 0, \quad P^2 = \int_A f^2 + \int_A g^2 \end{aligned}$$

provided the set \mathfrak{A} is countable and $\mathfrak{A} \cap \mathfrak{A} = \mathfrak{A}$.

For

$$\begin{aligned} \int_{\mathfrak{A} \cap \mathfrak{A}} f^2 &= \int_{\mathfrak{A} \cap \mathfrak{A}} g^2 = 0 \\ &= \int_{\mathfrak{A}} f^2 = \int_{\mathfrak{A}} g^2 = 0. \end{aligned}$$

Similarly $\mathfrak{M}_{(f, \alpha\beta)} = \mathfrak{M}_{(f, \frac{\beta}{c}, \frac{\alpha}{c})}$ when $c > 0$.

Thus 3), 4) give

$$\begin{aligned} \int_{\mathfrak{M}_{(f, \alpha\beta)}} f &= c \int_{\mathfrak{M}_{(f, \frac{\beta}{c}, \frac{\alpha}{c})}} f = c \int_{\mathfrak{M}_{(f, \frac{\beta}{c}, \frac{\alpha}{c})}} f \\ &= c \int_{\mathfrak{M}_{(f, \frac{\beta}{c}, \frac{\alpha}{c})}} f = c \int_{\mathfrak{M}_{(f, \alpha\beta)}} f. \end{aligned}$$

We now need only to pass to the limit $\alpha, \beta \rightarrow \infty$.

60. *Let one of the integrals*

$$\int_{\mathfrak{M}} f, \quad \int_{\mathfrak{M}} g$$

converge. If $f = g$, except at a discrete set \mathfrak{D} converge and are equal. A similar theorem holds for multiple integrals.

For let us suppose the first integral in 1) converges.

$$\mathfrak{M} = A + \mathfrak{D};$$

then

$$\int_{\mathfrak{M}} f = \int_A f + \int_{\mathfrak{D}} f = \int_A f.$$

Now

$$\begin{aligned} \int_{\mathfrak{M}} g &= \lim_{\alpha, \beta \rightarrow \infty} \int_{\mathfrak{M}_{(\alpha, \beta)}} g = \lim_{\alpha, \beta \rightarrow \infty} \int_{\mathfrak{M}_{(\alpha, \beta)}} f \\ &= \lim_{\alpha, \beta \rightarrow \infty} \int_{A_{(\alpha, \beta)}} f = \int_A f. \end{aligned}$$

Thus the second integral in 1) converges, and the integrals in 1) are equal.

RELATION BETWEEN THE INTEGRALS OF TYPES I.

For let \mathcal{Q}_1 be defined as in (8, 1). Then

$$\int_{\mathcal{Q}_1} p = \int_{\mathcal{Q}_1} q.$$

Let $\alpha, \beta \in \mathcal{Q}_1$, we get (2) by the same style of reasoning.

2. In the integrals (1) $\alpha \in \mathcal{Q}_1$, and their singular points, the relation (1) holds.

This follows from (2), 2.

3. If the conditions of 1 do not hold, the relation (1) is true.

Example. Let \mathcal{H} denote the rational points on \mathbb{C}^* , 1

$$t \in \mathcal{H} \iff \exists^! t = \frac{1}{n} \text{ for } n \in \mathcal{H},$$

$$p = 1 \text{ for } t \in \mathcal{H}.$$

Then

$$t \in \mathcal{H} \iff t \in \mathcal{H}.$$

But

$$\int_{\mathcal{H}} p = 0 \neq \int_{\mathcal{H}} q = 1.$$

Relation between t, t_1, t_2, t_3 of type I, II

62. Let us denote t_1, t_2, t_3, t_4 as the singular points

$$t_1 = 1, \quad t_2 = 1, \quad t_3 = P_3$$

respectively. The singular points t_1, t_2, t_3 are the points of the path γ on \mathcal{H} and t_4 is the point P_3 . When t_1, t_2, t_3 are singular points, t_4 is not a singular point. The singular points

Hence

$$\begin{aligned} P_{\mathfrak{A}} &= \lim_{\alpha \rightarrow 0} C'_{\mathfrak{A}_\alpha} \\ &= C'_{\mathfrak{A}}, \quad \text{by definition} \end{aligned}$$

64. If \bar{C} is convergent, we cannot say the similar remark holds for the lower integrals.

Example. For the rational points in $\mathfrak{A} = (0, 1)$

$$f(x) = \frac{1}{2 \vee x};$$

for the irrational points let

$$f(x) = \frac{1}{x}.$$

Then

$$\bar{C}_{\mathfrak{A}} = \lim_{\alpha \rightarrow 0} \int_{\mathfrak{A}_\alpha} f(x) dx = \lim [x] = 1$$

On the other hand,

$$P_{\mathfrak{A}} = \lim_{\alpha, n \rightarrow \infty} \int_{\mathfrak{A}_{\alpha, n}} f$$

does not exist. For however large n is taken

$$\int_{\mathfrak{A}_{\alpha, n}} f > \frac{1}{2} - \epsilon \quad \text{for } \alpha > \frac{1}{2} - \epsilon$$

65. If C is absolutely convergent and \bar{X} is convergent and are equal to the corresponding C' and X' .

For let D be any complete division of \mathfrak{A} of order

$$\int_{\mathfrak{A}_1} + \int_{\mathfrak{A}_2} + \cdots + \int_{\mathfrak{A}_n} + \cdots$$

RELATION BETWEEN THE INTEGRALS OF TYPES I,

Hence (1), (2) give

$$\int_{A_{\delta}} f = C_A + c' \quad \text{for any } \delta > \delta_0.$$

On the other hand, if c_0 is sufficiently small,

$$C_A = C_A + c'' \quad \text{for } \delta = \delta_0.$$

Hence
$$\int_{A_0} f = C_A + c''' = c''' = c.$$

Passing to the limit as $\delta \rightarrow 0$, we get

$$P = C.$$

66 *If f is absolutely convergent, the singular part is discrete.*

For suppose $\lambda \neq 0$. Let \mathfrak{W} denote the points of $f = d$. Then $\mathfrak{W} \subset \lambda$ for any d . Hence

$$\int_{\mathfrak{W}} f = \int_{\mathfrak{W}} f = \lim_{d \rightarrow 0} \int_{\mathfrak{W}} f = \lim_{d \rightarrow 0} \int_{\mathfrak{W}} f = c,$$

and $f = c$ since $\lambda \neq 0$.

67 *If f is absolutely convergent, c is C .*

For let H be a subspace of space of norm d .

Then

$$c = \lim_{d \rightarrow 0} \int_H f.$$

For

$$V_{\mathfrak{A}} f = \int_{\mathfrak{A}} f_{\lambda} + \epsilon', \quad 0 < \epsilon' < \epsilon$$

for λ sufficiently large. Let λ be so taken, that

$$V_{\mathfrak{A}} f = \int_{\mathfrak{A}} f_{\lambda} + \epsilon'', \quad 0 < \epsilon'' < \epsilon$$

Also,

$$\int_{\mathfrak{A}} f_{\lambda} = \lambda \mathfrak{B} + \frac{\epsilon}{\lambda},$$

if σ is taken sufficiently small in 2).

From 3), 4) follows 1).

69. If $V_{\mathfrak{A}} f$ is absolutely convergent, both equal to the corresponding V integrals

For by 67, G is absolutely convergent. Hence

Thus

$$G_{\mathfrak{A}} f = \int_{\mathfrak{A}_1} f + \alpha, \quad \alpha = \frac{\epsilon}{\lambda} < \epsilon$$

Also

$$V_{\mathfrak{A}} f = \int_{\mathfrak{A}} f_{\lambda_n} + \beta, \quad \beta = \frac{\epsilon}{\lambda} < \epsilon$$

Hence

$$\eta = G_{\mathfrak{A}} f - V_{\mathfrak{A}} f = \int_{\mathfrak{A}_1} f - \int_{\mathfrak{A}} f_{\lambda_n}$$

Now

$$\int_{\mathfrak{A}} f_{\lambda_n} = \int_{\mathfrak{A}_1} f_{\lambda_n} + \int_{\mathfrak{A}_2} f_{\lambda_n}$$

But

$$\left| \int_{\mathfrak{A}_1} f_{\lambda_n} - \int_{\mathfrak{A}_1} f \right| < \epsilon_n$$

Iterated Integrals

70. 1. We consider now the relations which exist between integrals

$$\int_{\mathfrak{A}} f,$$

and

$$\int_{\mathfrak{M}} \int_{\mathfrak{Q}} f,$$

where $\mathfrak{M} = \mathfrak{M} \times \mathfrak{Q}$ lies in a space \mathfrak{R}_m , $m = p + q$, and \mathfrak{Q} is a projective \mathfrak{R} in the space \mathfrak{R}_p .

It is sometimes convenient to denote the last q coordinates of $x = (x_1, \dots, x_p, x_{p+1}, \dots, x_{p+q})$ by y_1, \dots, y_q . Thus the coordinates x_1, \dots, x_p refer to \mathfrak{M} and y_1, \dots, y_q to \mathfrak{Q} . The section of \mathfrak{M} corresponding to the point x in \mathfrak{Q} may be denoted by \mathfrak{S}_x when it is desired to indicate which of the sections \mathfrak{S} is meant.

2. Let us set

$$\phi(x_1, \dots, x_p) = \int_{\mathfrak{Q}} f;$$

then the integral (2) is

$$\int_{\mathfrak{M}} \phi.$$

It is important to note at once that although the integrand is defined for each point in \mathfrak{M} , the integrand ϕ in (4) may not be

Example. Let \mathfrak{M} consist of the points (x, y) in the unit square

$$x = \frac{m}{n}, \quad 0 \leq y \leq \frac{1}{n}$$

(Fig. 1). Let \mathfrak{Q} be the line $y = 0$. At the point (x, y) in \mathfrak{M} , let

To provide for the case that ϕ may not be defined at some of the points of \mathfrak{B} we give the symbol \mathfrak{B} the following meaning:

$$\int_{\mathfrak{B}} \int_{\mathfrak{C}} f = \lim_{\alpha, \beta \rightarrow 0} \int_{\mathfrak{B}} \int_{\mathfrak{C}} f$$

where $\Gamma = \mathfrak{C}$ when the integral \mathfrak{B} is convergent; in the contrary case Γ is such a part of \mathfrak{C} that

$$\int_{\Gamma} f < \alpha,$$

and such that the integral in \mathfrak{C} is numerically less than β permit.

Sometimes it is convenient to denote Γ by \mathfrak{C}' .

The points $\mathfrak{B}_{\alpha\beta}$ are the points of \mathfrak{B} at which ϕ is not defined. We notice that each $\mathfrak{B}_{\alpha\beta}$ in (5) contains all the points at which the integral \mathfrak{B} is not convergent. Then

$$\mathfrak{B} \cap \mathfrak{B}_{\alpha\beta} = \mathfrak{B}_{\alpha\beta}.$$

Hence when \mathfrak{B} is complete or metric,

$$\lim_{\alpha, \beta \rightarrow 0} \mathfrak{B}_{\alpha\beta} = \mathfrak{B}$$

Before going farther it will aid the reader to consider some examples.

71. *Example 1.* Let \mathfrak{A} be as in the example of § 69 at $x = \frac{m}{n}$. We see that

$$\int_{\mathfrak{A}} f = 0.$$

the discrete point set used in Ex. 1 by \mathfrak{E} . We define f as follows: f shall have in \mathfrak{E} the values assigned to it at t in Ex. 1. At the other points $A \in \mathfrak{Q} - \mathfrak{E}$, f shall have the value 0. Then

$$\int_a^b \int_1^2 \int_1^2 \int_1^2 f = 1.$$

On the other hand \mathfrak{Q}_1 consists of the irrational points and a finite number of other points. Thus

$$\int_a^b \int_1^2 f = 1.$$

Hence again the two (3), (4) exist and are equal.

Let us look at the results we get if we use integrals I and II. We will denote them by C and F as in 62.

We see at once that

$$C_A = F_A = P_A = 1.$$

Let us now calculate the iterated integrals

$$C_A C_A,$$

and

$$F_A F_A.$$

We observe that

$$C_A = 1 \quad \text{for } x \text{ irrational}$$

$$C_A = 0 \quad \text{for } x \text{ rational.}$$

Thus the integral C_A either is not defined at all, since \mathfrak{Q}_1 does not exist, or if we interpret the definition as possible, its value is 0. In neither case is

$$C_A = C_A C_A.$$

Let us now look at the integral F_A . We see at once

$$F_A F_A$$

does not exist, as $F_A = 1$ for rational x , and $F_A = 0$ for

Example 3. Let \mathfrak{A} be the unit square.

Let

$$f = n \quad \text{for } x = \frac{m}{n} \quad (n \leq m \leq n+1)$$

$$z = n \quad \text{for } x = \frac{m}{n}$$

At the other points of \mathfrak{A} let $f = 1$.

Then

$$\int_{\mathfrak{A}} f = \int_0^1 \int_0^1 f = 1.$$

Here every point of \mathfrak{A} is a point of multi- \mathfrak{A} , thus $\mathfrak{Z} = \mathfrak{A}$.

Here $\mathcal{C}_{\mathfrak{A}}$ is not defined, as \mathfrak{A} , does not satisfy the definition its most liberal interpretation,

$$\mathcal{C}_{\mathfrak{A}} = 0.$$

The same remarks hold for $\mathcal{C}'_{\mathfrak{A}}, \mathcal{C}_{\mathfrak{A}}'$.

On the other hand

$$I_{\mathfrak{A}} = 1, \quad \mathfrak{A} \in \mathfrak{A},$$

while

$$I'_{\mathfrak{A}} I_{\mathfrak{A}}$$

does not exist, since

$$I'_{\mathfrak{A}} = \frac{1}{n} \quad \text{for } n \leq m \leq n+1$$

$$= 1 \quad \text{for } m > n+1$$

Moreover

$$I'_{\mathfrak{A}} I_{\mathfrak{A}} = \frac{1}{n} \quad \text{for } n \leq m \leq n+1$$

Example 4. Let \mathfrak{A} denote the unit square

We see at once that

$$C_A = F_A = 1.$$

Again the integral $C_A C'_Q$ does not exist, or on a liberal interpretation it has the value 0. Also in this example

$$C'_A C'_Q \text{ and } C_A C'_Q$$

do not exist or on a liberal interpretation, they are 0.

Turning to the F integrals we see that

$$F_A F_Q = +\infty, \quad F_A F'_Q = +\infty,$$

while $F'_A F_Q$ does not exist finite or infinite.

Example 5. Let our field of integration \mathfrak{A} consist of square considered in Ex. 4, let us call it \mathfrak{Q} , and another square \mathfrak{A} , lying to its right. Let f be defined over \mathfrak{Q} as defined in Ex. 4, and let $f = 1$ in \mathfrak{A} .

Then

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}} \int_{\mathfrak{Q}} = 2.$$

Also

$$C_A = F_A = 2.$$

Then

$$C'_A C'_Q = 1,$$

while

$$F'_A F'_Q \text{ does not exist,}$$

and

$$F_A F_Q = +\infty, \quad F_A F'_Q = +\infty.$$

72-1 In the following sections we shall restrict our notions

1. \mathfrak{A} shall be limited and iterable with respect to \mathfrak{A} .
2. \mathfrak{A} shall be complete or metric.
3. The singular points \mathfrak{A} of the integrand f shall be dis-

Let $\mathfrak{A}_n = \mathfrak{B}_n \cdot \mathfrak{C}_n$ denote the points of \mathfrak{A} which contain no point of \mathfrak{B} . We observe *take without loss of generality*

$$\mathfrak{B}_n \subset \mathfrak{B}.$$

For let us adjoin to \mathfrak{A} a discrete set \mathfrak{E} disjoint from \mathfrak{A} such that the projection of \mathfrak{E} on \mathfrak{B}_n

$$\text{Let } A = \mathfrak{A} + \mathfrak{E} = \mathfrak{B} + \mathfrak{C} + \mathfrak{E}, \quad \mathfrak{C} \cap \mathfrak{E} = \emptyset.$$

$$\text{We now set } \phi = f \text{ on } \mathfrak{A} \\ 0 \text{ on } \mathfrak{E}.$$

$$\text{Then } \int_1 \phi = \int_{\mathfrak{A}} \phi + \int_2 \phi + \int_{\mathfrak{E}} \phi \\ = \int_{\mathfrak{A}} \phi.$$

$$\text{Similarly } \int_2 \phi = \int_{\mathfrak{B}} \phi + \int_1 \phi + \int_{\mathfrak{E}} \phi$$

$$\text{Hence } \int_{\mathfrak{A}} \int_{\mathfrak{C}} \phi = \int_{\mathfrak{B}} \int_{\mathfrak{E}} \phi$$

3. The set \mathfrak{C}_n being as in 2, we shall write

$$\mathfrak{C} = \mathfrak{C}_n + \mathfrak{C}_\infty$$

73. Let $B_{\sigma, n}$ denote the points $x \in \mathfrak{B}$ at which σ is iterable, with respect to \mathfrak{B} .

$$\lim_{n \rightarrow \infty} B_{\sigma, n} = \mathfrak{B}.$$

For since \mathfrak{B} is iterable,

$$\mathfrak{B} = \int_{\mathfrak{B}} \mathfrak{C} \quad \text{by definition.}$$

Hence \mathfrak{C} , considered as a function of x , is

We have now

$$Q = Q_n + Q_{n+1} + \cdots + Q_m = 0$$

as Q_n, Q_m are unimodal. Hence Q_n is an integrable function. But

$$\begin{aligned} A_n &= A_n \left(\int_0^1 (Q_n + Q_m) \right) = \int_0^1 Q_n \\ &= \int_0^1 Q_m. \end{aligned}$$

As the left side $\rightarrow 0$ as $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} \int_0^1 Q_m = 0.$$

But

$$\int_0^1 Q_m = \int_0^1 Q_n = a B_{n,m}.$$

As the left side $\rightarrow 0$ we have for a given a

$$\lim_{n \rightarrow \infty} B_{n,m} = 0,$$

which is (1).

74 Let $A = \mathcal{B} = \mathcal{C}$ be isomorphic. Let the integral

$$\int_0^1 f(x) dx = 0$$

be convergent in the usual sense. Let \mathcal{B} . Let Q_n denote \mathcal{B} at n levels.

$$\int_0^1 Q_n = 0$$

Then

$$\lim_{n \rightarrow \infty} Q_n = \mathcal{B}$$

For let

0

Let $\mathfrak{A}_n = \mathfrak{B}_n \cdot \mathfrak{C}_n$ denote the points which contain no point of \mathfrak{A} . We take without loss of generality

$$\mathfrak{A}_n \subset \mathfrak{A}_{n-1}$$

For let us adjoin to \mathfrak{A} a discrete set \mathfrak{D} from \mathfrak{A} such that the projection of \mathfrak{A} on \mathfrak{D} is dense.

Let $A = \mathfrak{A} + \mathfrak{D} = \mathfrak{B} + \mathfrak{C} + \mathfrak{D}$.

We now set $\phi = f$ on \mathfrak{B} , $\phi = 0$ on $\mathfrak{C} + \mathfrak{D}$.

Then

$$\int_1 \phi = \int_{\mathfrak{B}} \phi + \int_2 \phi \\ = \int_{\mathfrak{B}} f$$

Similarly

$$\int_2 \phi = \int_3 \phi + \int_4 \phi$$

Hence

$$\int_{\mathfrak{B}} \int_1 \phi = \int_{\mathfrak{B}} \int_2 \phi$$

3. The set \mathfrak{C}_n being as in 2, we shall

$$\mathfrak{C}_n = \mathfrak{C}_{n-1} + \mathfrak{D}_n$$

73. Let $B_{a,n}$ denote the points of \mathfrak{B} at which f is iterable, with respect to \mathfrak{B} ,

$$\lim_{n \rightarrow \infty} B_{a,n} = \mathfrak{B}$$

We have now

$$\mathfrak{C} = \mathfrak{C}_n + \mathfrak{C}_n' \quad \text{with} \quad \mathfrak{C}_n' \rightarrow 0$$

as $\mathfrak{C}_n, \mathfrak{C}_n'$ are mixed. Hence \mathfrak{C} is an integrable But

$$\begin{aligned} \mathfrak{A} &= \mathfrak{A}_n + \int_0^a (\mathfrak{C}_n + \mathfrak{C}_n') = \int_0^a \mathfrak{C}_n \\ &\quad + \int_0^a \mathfrak{C}_n'. \end{aligned}$$

As the left side $\rightarrow 0$ as $a \rightarrow \infty$,

$$\lim \int_0^a \mathfrak{C}_n' = 0.$$

But

$$\int_0^a \mathfrak{C}_n' = \int_{\mathfrak{C}_n} \mathfrak{C}_n' = a B_{an}.$$

As the left side $\rightarrow 0$, we have for a given a

$$\lim B_{an} = 0,$$

which is (1).

74. Let $\mathfrak{A} = \mathfrak{W}$ or be its \mathfrak{A} . Let the integral

$$\int_0^a f(x) dx = 0$$

be convergent and finite for all complete \mathfrak{W} . Let $\mathfrak{C}_n \rightarrow 0$ as $n \rightarrow \infty$ such

$$\int_0^a f(x) dx = \mathfrak{C}_n$$

Then

$$\lim_{n \rightarrow \infty} \mathfrak{C}_n = 0$$

For let

For

We may now reason on \mathfrak{A}_1 as we did on \mathfrak{A} , i.e., p. We get a complete set $\mathfrak{A}_2 \sim \mathfrak{A}_1$ such that

$$\mathfrak{A}_2 \cap \mathfrak{A}_1 = \emptyset.$$

Continuing we get

$$\mathfrak{A}_n \cap \mathfrak{A}_{n-1} = \emptyset, \quad n = 1, 2, \dots$$

Thus

$$\overline{\mathfrak{A}_n} \cap \mathfrak{A} = \frac{1}{\sqrt{n}} + \frac{1}{1 + \sqrt{n}} + \dots + \frac{1}{(n-1)^{1/2}} \\ \leq \mathfrak{A} \cap \eta.$$

Let now

$$\mathfrak{b} = \bigcap_{n=1}^{\infty} \mathfrak{A}_n$$

Then

$$\mathfrak{b} \cap \mathfrak{A} \cap \eta$$

by 25.

Let \mathfrak{b}_n denote those points of \mathfrak{b} for which $x \leq \frac{1}{n}$. $\mathfrak{b} = \{\mathfrak{b}_n\}$. For let b be any point of \mathfrak{b} . Since there exists a σ_1 such that

$$\text{at } b, \quad \int_{\mathfrak{A}} f < \epsilon,$$

for any ϵ such that $\epsilon < \sigma_1$. Thus $b \in \mathfrak{b}_{\sigma_1} \subset \mathfrak{b}_n$. $\{\mathfrak{b}_n\}$. Thus $\mathfrak{b}_n \supset \mathfrak{b}$ as \mathfrak{b} is complete. But $\mathfrak{b}_n \subset \mathfrak{A}_n$.

Hence

$$\lim_{n \rightarrow \infty} \mathfrak{b}_n = \mathfrak{b},$$

which with 4), gives 3).

75. Let $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ be iterable. Let the integral

$$\int_{\mathfrak{A}} f = \lim_{n \rightarrow \infty} \int_{\mathfrak{A}_n} f$$

be convergent and limited in complete \mathfrak{B} .

Let d_i'' denote the other cells containing points of

$$S_i = \sum C_i + \sum d_i'' M,$$

where

$$0 \leq \int_{d_i''} f \leq M.$$

Hence

$$S_i \leq \epsilon \mathfrak{W}_n + M(\mathfrak{W}_n + \mathfrak{S}_{n,n}).$$

Letting $d \rightarrow 0$, we get

$$\int_n, \int_n f \leq \epsilon \mathfrak{W} + M(\mathfrak{W} + \mathfrak{S}_{n,n}).$$

Letting now $n \rightarrow \infty$ and using 3) of 74, we get what is planned.

76 Let $\mathfrak{A} = \mathfrak{W}$ be iterably with respect to \mathfrak{W} , where $\epsilon = \epsilon(\delta, \eta)$. Let the singular points X of f be di-

$$\mathfrak{A}_1 = \{f \in G \mid \int_{\mathfrak{A}_1} f \leq \int_n, \int_n f + \int_n f\}$$

$$\mathfrak{A}_2 = \{f \in G \mid \int_{\mathfrak{A}_2} f \leq \int_n, \int_n f + \int_n f\}$$

Here any one of the members in \mathfrak{A}_1 may be infinite that tells us the infinite. A similar remark applies

to \mathfrak{A}_2 that applies.

$$\epsilon = \epsilon(\delta, \eta) \mid \mathfrak{W} \text{ is complete} \implies \int_n, \int_n f \text{ is convex.}$$

We have by 44,

$$\int_{\mathfrak{A}_1} f \leq \int_n, \int_n f + \int_n f$$

By 76, the function

But for a fixed n

$$\int_{\mathfrak{B}_n} f = \int_{\mathfrak{B}} f - \int_{\mathfrak{B} - \mathfrak{B}_n} f.$$

Hence for G_n sufficiently large,

$$\int_{\mathfrak{B}_n} f = \int_{\mathfrak{B}} f, \quad \text{at each point } (x, y) \in \mathfrak{B}.$$

Then

$$\int_{\mathfrak{B}_n} \int_{\mathfrak{B}_n} f = \int_{\mathfrak{B}} \int_{\mathfrak{B}} f - \int_{\mathfrak{B} - \mathfrak{B}_n} \int_{\mathfrak{B}} f - \int_{\mathfrak{B}} \int_{\mathfrak{B} - \mathfrak{B}_n} f$$

where Γ_n, γ_n are points of \mathfrak{B} on $\mathfrak{B} - \mathfrak{B}_n$.

Hence

$$\int_{\mathfrak{B}_n} \int_{\mathfrak{B}} f = \int_{\mathfrak{B}} \int_{\mathfrak{B}_n} f + \int_{\mathfrak{B}} \int_{\mathfrak{B} - \mathfrak{B}_n} f.$$

Now \mathfrak{B}_n may not be complete, if $\mathfrak{B} \neq \mathfrak{B}_n \cup H$.
As \mathfrak{B} is complete,

$$\int_{\mathfrak{B}_n} \int_{\mathfrak{B}_n} f = \int_{\mathfrak{B}_n} \int_{\mathfrak{B}} f.$$

We may therefore write $\mathfrak{B}_n \cup H = \mathfrak{B}_n$ and

$$\epsilon + \int_{\mathfrak{B}_n} \int_{\mathfrak{B}} f = \int_{\mathfrak{B}_n} \int_{\mathfrak{B}_n} f + \int_{\mathfrak{B}_n} \int_{\mathfrak{B} - \mathfrak{B}_n} f + \int_{\mathfrak{B}} \int_{\mathfrak{B} - \mathfrak{B}_n} f.$$

By 75, the last term on the right $\rightarrow 0$ as $n \rightarrow \infty$,
to the limit,

$$\int_{\mathfrak{B}} \int_{\mathfrak{B}} f = \lim_{n \rightarrow \infty} \int_{\mathfrak{B}_n} \int_{\mathfrak{B}_n} f,$$

since $\epsilon > 0$ is small at pleasure.

On the other hand, passing to the limit $n \rightarrow \infty$,
 $n = \infty$, we get

Let us now suppose \mathfrak{N} is metric. We effect a cut of \mathfrak{R}_β of norm β , and denote by B_β those cells containing points of \mathfrak{N} . Then B_β is complete and

$$\lim_{\beta \rightarrow 0} B_\beta = \mathfrak{N}.$$

Let A_β denote those points of \mathfrak{N} whose projection on B_β is iterable with respect to B_β by 13, 3, and in the preceding case

$$\int_{A_\beta} = \int_{B_\beta} \int_{\mathfrak{Q}} = \int_{A_1}.$$

If the middle integral in 11) is divergent, $\int_{\mathfrak{N}}$ is divergent, also if the last integral in 11) is divergent, 1) pose then that the two last integrals in 11) are finite. Then by 6)

$$\lim_{\beta \rightarrow 0} \int_{A_\beta} \int_{\mathfrak{Q}} = \int_{\mathfrak{N}} \int_{\mathfrak{Q}},$$

$$\lim_{\beta \rightarrow 0} \int_{A_\beta} = \int_{\mathfrak{N}}.$$

Thus passing to the limit $\beta \rightarrow 0$ in 11) we get 1).

Let us now suppose $\mathfrak{N} = G_1 \cup G_2 \neq 0$.

Then

$$g = f + G = 0,$$

and we can apply 1) to the new function g .

Then

$$\int_{\mathfrak{N}} g = \int_{\mathfrak{N}} \int_{\mathfrak{Q}} = \int_{\mathfrak{N}} g.$$

Now

$$\int_{\mathfrak{N}} g = \int_{\mathfrak{N}} f + G \mathfrak{N},$$

by 18, 6, since \mathfrak{N} is denumerable. Also by the same theorem

$$\int_{\mathfrak{Q}} g = \int_{\mathfrak{Q}} f + G \lim_{\beta \rightarrow 0} \mathfrak{Q} = \int_{\mathfrak{Q}} f + G \Gamma,$$

But for a fixed n

$$\int_{\mathfrak{B}} \quad \text{is limited in } \mathfrak{B}.$$

Hence for \mathfrak{U}_n sufficiently large,

$$\int_{\mathfrak{U}_n} f \cdot \int_{\mathfrak{V}} f \quad \text{at each point of } \mathfrak{B}, \quad \mathfrak{U}_n \rightarrow \mathfrak{U}. \quad (6)$$

Then

$$\int_{\mathfrak{U}_n} \int_{\mathfrak{V}_n} f \cdot \int_{\mathfrak{U}_n} \int_{\mathfrak{V}_n} f = \int_{\mathfrak{U}_n} \left\{ \int_{\mathfrak{V}_n} f \right\} \cdot \int_{\mathfrak{V}_n} \left\{ \int_{\mathfrak{U}_n} f \right\}, \quad (7)$$

where Γ_n, γ_n are points of Γ in $\mathfrak{U}_n, \mathfrak{V}_n$.

Hence

$$\int_{\mathfrak{U}_n} \int_{\mathfrak{V}_n} f \cdot \int_{\mathfrak{U}_n} \int_{\mathfrak{V}_n} f + \int_{\mathfrak{U}_n} \int_{\mathfrak{V}_n} f \quad (8)$$

Now \mathfrak{B}_n may not be complete; if not let H_n be completed \mathfrak{B}_n . As \mathfrak{B} is complete,

$$\int_{\mathfrak{U}_n} \int_{\mathfrak{V}_n} f = \int_{H_n} \int_{\mathfrak{V}_n} f.$$

We may therefore write 8), using 5),

$$= \epsilon + \int_{\mathfrak{U}_n} \int_{\mathfrak{V}_n} f \cdot \int_{\mathfrak{U}_n} \int_{\mathfrak{V}_n} f + \int_{H_n} \int_{\mathfrak{V}_n} f \cdot \int_{\mathfrak{U}_n} \int_{\mathfrak{V}_n} f + \int_{H_n} \int_{\mathfrak{V}_n} f.$$

By 75, the last term on the right $\rightarrow 0$ as $n \rightarrow \infty$. Thus passing to the limit,

$$\int_{\mathfrak{U}} \int_{\mathfrak{V}} f \cdot \lim_{n \rightarrow \infty} \int_{\mathfrak{U}_n} \int_{\mathfrak{V}_n} f, \quad (9)$$

since $\epsilon > 0$ is small at pleasure.

On the other hand, passing to the limit $\mathfrak{U} = \mathfrak{U}$ in 74, and then $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} \int_{\mathfrak{U}_n} \int_{\mathfrak{V}_n} f = \int_{\mathfrak{U}} \int_{\mathfrak{V}} f. \quad (10)$$

Thus 8), 10), 9), and 4) give 1).

Let us now suppose that the middle term of 1) is divergent. We have as before

$$\int_{\mathfrak{U}_n} \int_{\mathfrak{V}_n} f \cdot \lim_{n \rightarrow \infty} \int_{\mathfrak{U}_n} \int_{\mathfrak{V}_n} f = \int_{\mathfrak{U}} f$$

Hence the integral on the right of 1) is divergent.

Let us now suppose \mathfrak{B} is metric. We effect a cubical division of \mathfrak{R}_p of norm d , and denote by B_d those cells containing only points of \mathfrak{B} . Then B_d is complete and

$$\lim_{d \rightarrow 0} B_d = \mathfrak{B}.$$

Let A_d denote those points of \mathfrak{A} whose projections fall on B_d . Then A_d is iterable with respect to B_d by 13, 3, and we have as in the preceding case

$$\int_{A_d} \cdot \int_{B_d} \int_{\mathfrak{G}} \cdot \int_{A_d} \cdot \quad (11)$$

If the middle integral in 11) is divergent, $\int_{\mathfrak{A}}$ is divergent and 1) holds, also if the last integral in 11) is divergent, 1) holds. Suppose then that the two last integrals in 11) are convergent. Then by 57

$$\lim_{d \rightarrow 0} \int_{A_d} \int_{B_d} \int_{\mathfrak{G}} \cdot = \int_{\mathfrak{A}} \int_{\mathfrak{G}} \cdot$$

$$\lim_{d \rightarrow 0} \int_{A_d} \cdot = \int_{\mathfrak{A}} \cdot$$

Thus passing to the limit $d \rightarrow 0$ in 11) we get 1).

Let us now suppose $f \in \mathcal{C}$, $\mathcal{C} \in \mathcal{O}$.

Then

$$g = f + \mathcal{C} \in \mathcal{O},$$

and we can apply 1) to the new function g .

Thus

$$\int_{\mathfrak{A}} g = \int_{\mathfrak{A}} \int_{\mathfrak{G}} g \leq \int_{\mathfrak{A}} g. \quad (12)$$

Now

$$\int_{\mathfrak{A}} g = \int_{\mathfrak{A}} f + \mathcal{C} \mathfrak{A}, \quad (13)$$

by 58, 6, since \mathfrak{A} is discrete. Also by the same theorem,

$$\int_{\mathfrak{G}} g = \int_{\mathfrak{G}} f + \mathcal{C} \lim_{\gamma \rightarrow 0} \mathfrak{G}_{\gamma} = \int_{\mathfrak{G}} f + \mathcal{C} \Gamma, \quad (14)$$

denoting by \mathfrak{G}_{γ} the points of \mathfrak{G} where

$$-G < f < \gamma$$

and setting

$$\Gamma = \lim_{\gamma \rightarrow 0} \mathfrak{G}_{\gamma}.$$

Now for any n

$$\int_{\mathfrak{A}_n} G = \int_{\mathfrak{A}} \int_{\mathfrak{A}_n} G = \int_{\mathfrak{A}_n} G$$

$$\text{Hence } G\mathfrak{A} = \lim_{n \rightarrow \infty} \int_{\mathfrak{A}} \int_{\mathfrak{A}_n} G = G \lim_{n \rightarrow \infty} \int_{\mathfrak{A}_n} G,$$

$$\text{or } \mathfrak{A} = \lim_{n \rightarrow \infty} \int_{\mathfrak{A}_n} G. \quad (15)$$

Now for a fixed n , γ may be taken so large that for all points of \mathfrak{B} ,

$$G_n \leq G_\gamma.$$

Hence

$$G \geq \lim_{n \rightarrow \infty} G_n \geq G_\gamma.$$

Hence

$$\mathfrak{A} = \int_{\mathfrak{A}} G \geq \int_{\mathfrak{A}} G_\gamma = \int_{\mathfrak{A}} G_\gamma = \mathfrak{A}$$

Hence

$$\mathfrak{A} = \int_{\mathfrak{A}} G. \quad (16)$$

and thus Γ is integrable in \mathfrak{A} .

This result in 14) gives, on using 6a, 3,

$$\int_{\mathfrak{A}} \int_{\mathfrak{A}} g = \int_{\mathfrak{A}} \int_{\mathfrak{A}} f + G\mathfrak{A} \quad (17)$$

From 12), 13), and 17) follows 14).

77. As corollaries of the last theorem we have, supposing \mathfrak{A} to be as in 76,

1. If f is integrable in \mathfrak{A} and $f \leq G$, then

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}} \int_{\mathfrak{A}} f.$$

If $f \leq G$, then

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}} \int_{\mathfrak{A}} f$$

2. If $f \geq -G$ and $\int_{\mathfrak{A}}$ is divergent, then

$$\int_{\mathfrak{A}} \int_{\mathfrak{A}} f$$

are divergent

3. If $f \in \mathcal{G}$ and one of the integrals $\int_{\mathfrak{A}} \int_{\mathfrak{G}} f$ is convergent, then

$$\int_{\mathfrak{A}} f$$

is convergent.

78. Let $\mathfrak{A} = \mathfrak{A} \cdot \mathfrak{G}$ be iterable with respect to \mathfrak{A} , which last is complete or metric. Let the singular points \mathfrak{A} be discrete. If

$$\int_{\mathfrak{A}} f, \quad (1)$$

$$\int_{\mathfrak{A}} \int_{\mathfrak{G}} f, \quad (2)$$

both converge, they are equal.

For let D_1, D_2, \dots be a sequence of superimposed cubical divisions as in 72, 2. We may suppose as before that each $\mathfrak{A}_n = \mathfrak{A}$.

Since 1) is convergent

$$\epsilon > 0, \quad n_0, \quad \left| \int_{\mathfrak{A}} f - \int_{\mathfrak{A}_n} f \right| < \frac{\epsilon}{2} \quad n > n_0. \quad (3)$$

Since f is limited in \mathfrak{A}_n , which latter is iterable,

$$\int_{\mathfrak{A}_n} f = \int_{\mathfrak{A}} \int_{\mathfrak{G}_n} f. \quad (4)$$

This shows that

$$\int_{\mathfrak{G}_n} f \quad (5)$$

is an integrable function in \mathfrak{A} , and hence in any part of \mathfrak{A} .

From 3), 4) we have

$$\left| \int_{\mathfrak{A}} f - \int_{\mathfrak{A}} \int_{\mathfrak{G}_n} f \right| < \frac{\epsilon}{2} \quad n > n_0. \quad (6)$$

We wish now to show that

$$\left| \int_{\mathfrak{A}} \int_{\mathfrak{G}} f - \int_{\mathfrak{A}} \int_{\mathfrak{G}_n} f \right| < \frac{\epsilon}{2} \quad n > n_0. \quad (7)$$

When this is done, 6) and 7) prove the theorem.

To establish 7) we begin by observing that

$$\int_{\mathfrak{A}} \int_{\mathfrak{G}} f = \lim_{n, n' \rightarrow \infty} \int_{\mathfrak{A}_{n, n'}} f.$$

Now for a fixed n , α, β may be taken so that Γ shall embrace all the points of \mathcal{Q}_n for every point of \mathcal{Q} . Let us set

$$\Gamma = \mathcal{Q}_n + \gamma_n.$$

Then

$$\int_{\mathcal{Q}_{\alpha\beta}} \int_{\gamma_n} f = \int_{\mathcal{Q}_{\alpha\beta}} \int_{\mathcal{Q}_n} f + \int_{\mathcal{Q}_{\alpha\beta}} \int_{\gamma_n} f. \quad (8)$$

As

$$\lim_{\alpha, \beta \rightarrow \infty} \int_{\mathcal{Q}_{\alpha\beta}} \int_{\mathcal{Q}_n} f = \int_{\mathcal{Q}} \int_{\mathcal{Q}_n} f \quad \text{by 1, 724.}$$

On the other hand,

$$\left| \int_{\mathcal{Q}_{\alpha\beta}} \int_{\gamma_n} f \right| = \int_{\mathcal{Q}_{\alpha\beta}} \int_{\gamma_n} |f| = \int_{\mathcal{Q}_{\alpha\beta}} \int_{\gamma_n} |f| = \int_{\mathcal{Q}_{\alpha\beta}} \int_{\gamma_n} |f|.$$

Thus 7) is established when we show that

$$\int_{\mathcal{Q}} \int_{\gamma_n} |f| = \frac{\epsilon}{2} \quad n > n_0. \quad (9)$$

To this end we note that $|f|$ is integrable in \mathcal{R} by 48, 1. Hence by 77, 1,

$$\int_{\mathcal{R}} |f| = \int_{\mathcal{R}} \int_{\mathcal{Q}} |f|. \quad (10)$$

Also by 1, 734,

$$\int_{\mathcal{R}} |f| = \int_{\mathcal{R}} \int_{\mathcal{Q}_n} |f|. \quad (11)$$

From 10), 11) we have for $n > n_0$,

$$\int_{\mathcal{R}} |f| = \int_{\mathcal{R}} \int_{\mathcal{Q}} |f| = \int_{\mathcal{R}} \int_{\mathcal{Q}_n} |f| + \int_{\mathcal{R}} \int_{\gamma_n} |f| = \frac{\epsilon}{2}, \quad (12)$$

since the left side $\neq 0$.

But as in 8)

$$\int_{\mathcal{Q}_{\alpha\beta}} \int_{\gamma_n} |f| = \int_{\mathcal{Q}_{\alpha\beta}} \int_{\mathcal{Q}_n} |f| + \int_{\mathcal{Q}_{\alpha\beta}} \int_{\gamma_n} |f|.$$

Passing to the limit $\alpha, \beta \rightarrow \infty$ gives

$$\int_{\mathcal{Q}} \int_{\gamma_n} |f| = \int_{\mathcal{Q}} \int_{\mathcal{Q}_n} |f| + \int_{\mathcal{Q}} \int_{\gamma_n} |f|.$$

This in 12) gives 9).

CHAPTER III

SERIES

Preliminary Definitions and Theorems

79. Let a_1, a_2, a_3, \dots be an infinite sequence of numbers.

The symbol $A = a_1 + a_2 + a_3 + \dots$ (1)

is called an *infinite series*. Let

$$A_n = a_1 + a_2 + \dots + a_n. \quad (2)$$

If $\lim_{n \rightarrow \infty} A_n$ (3)

is finite, we say the series (1) is *convergent*. If the limit (3) is infinite or does not exist, we say (1) is *divergent*. When (1) is convergent, the limit (3) is called the *sum* of the series. It is customary to represent a series and its sum by the same letter, when no confusion will arise. Whenever practicable we shall adopt the following uniform notation. The terms of a series will be designated by small *Roman* letters, the series and its sum will be denoted by the corresponding capital letter, when no confusion will arise. The *sum of the first n terms* of a series as A will be denoted by A_n . The infinite series formed by removing the first n terms, as for example,

$$a_{n+1} + a_{n+2} + a_{n+3} + \dots \quad (4)$$

will be denoted by A_n , and will be called the *remainder after n terms*.

The series formed by replacing each term of a series by its numerical value is called the *adjoint series*. We shall designate it by replacing the *Roman* letters by the corresponding Greek or German letters. Thus the adjoint of (1) would be denoted by

$$A = a_1 + a_2 + a_3 + \dots = \text{Adj } A \quad (5)$$

where

$$a_n = |a_n|.$$

If all the terms of a series are ≥ 0 , it is identical with its adjoint.

A sum of p consecutive terms as

$$a_{n+1} + a_{n+2} + \cdots + a_{n+p}$$

we denote by $A_{n,p}$.

$$\text{Let } B = a_1 + a_2 + a_3 + \cdots + a_1 + a_2$$

be the series obtained from A by omitting all its terms that vanish. Then A and B converge or diverge simultaneously, and when convergent they have the same sum.

$$\text{For } B_n = A_{n,n}.$$

Thus if the limit on either side exists, the limit of the other side exists and both are equal.

This shows that in an infinite series we may omit its zero terms without affecting its character or value. We shall suppose this done unless the contrary is stated.

A series whose terms are all ≥ 0 we shall call a *positive term series*; similarly if its terms are all ≤ 0 , we call it a *negative term series*. If $a_n > 0$, $n > m$ we shall say the series is *essentially a positive term series*. Similarly if $a_n < 0$, $n > m$ we call it an *essentially negative term series*.

If A is an essentially positive term series and divergent, $\lim A_n = +\infty$; if it is an essentially negative term series and divergent, $\lim A_n = -\infty$.

When $\lim A_n = \pm \infty$, we sometimes say A is $\pm \infty$.

80. 1. For A to converge, it is necessary and sufficient that

$$\epsilon > 0, m, |A_{n,p}| < \epsilon, n > m, p = 1, 2, \dots \quad (1)$$

For the necessary and sufficient condition that

$$\lim A_n$$

exists is

$$\epsilon > 0, m, |A_r - A_n| < \epsilon, r, n > m \quad (2)$$

But if $r = n + p$

$$A_r - A_n = A_{n,p} = a_{n+1} + a_{n+2} + \cdots + a_{n+p}.$$

Thus 2) is identical with 1).

2. *The two series A , A_1 converge and diverge simultaneously. When convergent,*

$$A = A_1 + A_2. \quad (3)$$

For obviously if either series satisfies theorem 1, the other must, since the first terms of a series do not enter the relation 1). On the other hand,

$$A_{n+p} = A_n + A_{n,p}.$$

Letting $p \rightarrow \infty$ we get 3).

3. *If A is convergent, $A_n \rightarrow 0$.*

For

$$\begin{aligned} \lim A_n &= \lim (A - A_n) \\ &= A - \lim A_n = A - A \\ &= 0. \end{aligned}$$

For A to converge it is necessary that $a_n \rightarrow 0$.

For in 1) take $p = 1$; it becomes

$$|a_{n+1}| < \epsilon \quad n > m$$

We cannot infer conversely because $a_n \rightarrow 0$, therefore A is convergent. For as we shall see in §1, 2,

$$1 + \frac{1}{2} + \frac{1}{4} + \dots$$

is divergent, yet $\lim a_n = 0$.

4. *The positive term series A is convergent if A_n is limited.*

For then $\lim A_n$ exists by I, 109.

5. *A series whose adjoint converges is convergent.*

For the adjoint A of A being convergent,

$$\epsilon > 0, \quad m, \quad |A_{n,p}| < \epsilon, \quad n > m, \quad p = 1, 2, 3, \dots$$

But

$$A_{n,p} = a_{n+1} + a_{n+2} + \dots + a_{n+p} = a_{n+1} + a_{n+2} + \dots + a_{n,p} = |A_{n,p}|.$$

Thus

$$|A_{n,p}| < \epsilon$$

and A is convergent.

Definition. A series whose adjoint is convergent is called *absolutely convergent*.

Series which do not converge absolutely may be called, when necessary to emphasize this fact, *simply convergent*.

6. Let $A = a_1 + a_2 + \dots$
be absolutely convergent.

Let $B = a_{i_1} + a_{i_2} + \dots$; $i_1 < i_2 < \dots$

be any series whose terms are taken from A , preserving their relative order. Then B is absolutely convergent and

$$|B| \leq |A|.$$

For $|B_m| = B_m = A_n = A,$ (1)

choosing n so large that A_n contains every term in B_m . Moreover for $m \geq \text{some } m'$, $A_n = B_m = \text{some term of } A$. Thus passing to the limit in (1), the theorem is proved.

7. Let $A = a_1 + a_2 + \dots$. The series $B = ka_1 + ka_2 + \dots$, $k \neq 0$, converges or diverges simultaneously with A . When convergent,

$$B = kA.$$

For $B_n = kA_n.$

We have now only to pass to the limit.

From this we see that a negative or an essentially negative term series can be converted into a positive or an essentially positive term series by multiplying its terms by $k = -1$.

8. If A is simply convergent, the series B formed of its positive terms taken in the order they occur in A , and the series C formed of the negative terms, also taken in the order they occur in A , are both divergent.

If B and C are convergent, so are B, C' . Now

$$A_n = B_{n_1} + C'_{n_2}, \quad n = n_1 + n_2.$$

Hence A would be convergent, which is contrary to hypothesis. If only one of the series B, C' is convergent, the relation

$$A_n = B_{n_1} + C'_{n_2}$$

shows that A would be divergent, which is contrary to hypothesis.

9. The following theorem often affords a convenient means of estimating the remainder of an absolutely convergent series.

Let $A = a_1 + a_2 + \dots$ be an absolutely convergent series. Let $B = b_1 + b_2 + \dots$ be a positive term convergent series whose sum is known either exactly or approximately. Then if $|a_n| \leq b_n$, $n \geq m$

$$|\bar{A}_n| \leq \bar{B}_n < B.$$

For

$$\begin{aligned} |A_{n,p}| &\leq a_{n+1} + \dots + a_{n+p} \\ &\leq b_{n+1} + \dots + b_{n+p} \\ &< B_n < B. \end{aligned}$$

Letting $p \doteq \infty$ gives the theorem.

EXAMPLES

81. 1. The geometric series is defined by

$$G = 1 + g + g^2 + g^3 + \dots \quad (1)$$

The geometric series is absolutely convergent when $|g| < 1$ and divergent when $|g| \geq 1$. When convergent,

$$G = \frac{1}{1-g}. \quad (2)$$

When $g \neq 1$,

$$\frac{1}{1-g} = 1 + g + g^2 + \dots + g^{n-1} + \frac{g^n}{1-g}.$$

Hence

$$G_n = \frac{1}{1-g} - \frac{g^n}{1-g}.$$

When $|g| < 1$, $\lim g^n = 0$, and then

$$\lim G_n = \frac{1}{1-g}.$$

When $|g| \geq 1$, $\lim g^n$ is not 0, and hence by 80, 3, G is not convergent.

2. The series $H = 1 + \frac{1}{2^\mu} + \frac{1}{3^\mu} + \frac{1}{4^\mu} + \dots$ (3)

is called the *general harmonic series of exponent μ* . When $\mu = 1$, it becomes

$$J = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \quad (4)$$

the *harmonic series*. We show now that

The general harmonic series is convergent when $\mu > 1$ and is divergent for $\mu \leq 1$.

Let $\mu > 1$. Then

$$\frac{1}{2^\mu} + \frac{1}{3^\mu} < \frac{1}{2^\mu} + \frac{1}{2^\mu} < \frac{2}{2^\mu} = \frac{1}{2^{\mu-1}} = g \quad ; \quad g < 1.$$

$$\frac{1}{4^\mu} + \frac{1}{5^\mu} + \frac{1}{6^\mu} + \frac{1}{7^\mu} < \frac{1}{4^\mu} + \frac{1}{4^\mu} + \frac{1}{4^\mu} + \frac{1}{4^\mu} = \frac{4}{4^\mu} = g^2.$$

$$\frac{1}{8^\mu} + \dots + \frac{1}{15^\mu} < \frac{1}{8^\mu} + \frac{1}{8^\mu} + \dots + \frac{1}{8^\mu} = \frac{8}{8^\mu} = g^3, \text{ etc.}$$

Let $n \leq 2^v$. Then

$$H_n < 1 + g + \dots + g^v < \frac{1}{1-g}.$$

Thus $\lim H_n$ exists, by I, 109, and

$$H < \frac{1}{1 - \frac{1}{2^{\mu-1}}}. \quad (5)$$

Let $\mu < 1$. Then

$$\frac{1}{n^\mu} > \frac{1}{n}.$$

Thus 3) is divergent for $\mu < 1$, if it is for $\mu = 1$.

But we saw, I, 141, that

$$\lim J_n = \infty,$$

hence J is divergent.

It is sometimes useful to know that

$$\lim \frac{J_n}{\log n} = 1. \quad (6)$$

In fact, by I, 180,

$$\begin{aligned} \lim \frac{J_n}{\log n} &= \lim \frac{J_n - J_{n-1}}{\log n - \log (n-1)} = \lim \frac{\frac{1}{n}}{\log \left(\frac{n}{n-1} \right)} \\ &= \lim \frac{1}{\log \left(1 - \frac{1}{n} \right)^{-n}} = 1. \end{aligned}$$

Since $*n = \log n = l_2 n$, we have

$$\lim_{n \rightarrow \infty} \frac{J_n}{n} = 0 \quad ; \quad \lim_{l_2 n \rightarrow \infty} \frac{J_n}{l_2 n} = r \quad , \quad r > 1. \quad (7)$$

Another useful relation is

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} = \log(n+1). \quad (8)$$

For $\log(1+m) = \log m = \log\left(1 + \frac{1}{m}\right) = \frac{1}{m}$.

Let $m = 1, 2, \dots, n$. If we add the resulting inequalities we get 8).

3. *Alternating Series.* This important class of series is defined as follows. Let $a_1 - a_2 + a_3 - \dots = 0$.

Then $A = a_1 - a_2 + a_3 - a_4 + \dots$ (9)

whose signs are alternately positive and negative, is such a series.

The alternating series (9) is convergent and

$$|A_n| \leq a_{n+1}. \quad (10)$$

For let $p > 3$. We have

$$A_{n,p} = (-1)^n(a_{n+1} - a_{n+2} + \dots + (-1)^{p-1}a_{n+p}) \\ = (-1)^n P.$$

If p is even,

$$P = (a_{n+1} - a_{n+2}) + \dots + (a_{n+p-1} - a_{n+p}).$$

If p is odd,

$$P = (a_{n+1} - a_{n+2}) + \dots + (a_{n+p-1} - a_{n+p-1}) + a_{n+p}.$$

Thus in both cases,

$$P = a_{n+1} - a_{n+2} > 0. \quad (11)$$

Again, if p is even,

$$P = a_{n+1} - (a_{n+2} - a_{n+3}) - \dots - (a_{n+p-2} - a_{n+p-1}) = a_{n+p}.$$

* In I, 401, the symbol "lim" in the first relation should be replaced by lim.

Example 3.
$$A = \sum_1^{\infty} \frac{1}{(x+n-1)(x+n)} \quad x \neq 0, -1, -2, \dots$$

$$= \sum \left\{ \frac{1}{x+n-1} - \frac{1}{x+n} \right\}$$

is telescopic.

$$A_n = \frac{1}{x} - \frac{1}{x+n} \doteq \frac{1}{x}.$$

82. *Dini's Series.* Let $A = a_1 + a_2 + \dots$ be a divergent positive term series. Then

$$D = \frac{a_1}{A_1} + \frac{a_2}{A_2} + \frac{a_3}{A_3} + \dots$$

is divergent.

For

$$\begin{aligned} D_{m,p} &= \frac{a_{m+1}}{A_{m+1}} + \dots + \frac{a_{m+p}}{A_{m+p}} \\ &> \frac{1}{A_{m+p}} (a_{m+1} + \dots + a_{m+p}) \\ &> \frac{A_{m,p}}{A_m + A_{m,p}} = 1 - \frac{A_m}{A_{m+p}}. \end{aligned}$$

Letting m remain fixed and $p \doteq \infty$, we have $\bar{D}_m \geq 1$, since $A_{m+p} \doteq \infty$. Hence D is divergent.

Let

$$A = 1 + 1 + 1 + \dots \quad \text{Then } A_n = n.$$

Hence

$$D = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots \text{ is divergent.}$$

Let

$$A = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

Then

$$D = \frac{1}{1} + \frac{1}{2(1+\frac{1}{2})} + \frac{1}{3(1+\frac{1}{2}+\frac{1}{3})} + \dots = \sum \frac{1}{nA_n}$$

is divergent, and hence, *a fortiori*,

$$\sum \frac{1}{nA_{n-1}}.$$

But

$$A_{n-1} > \log n.$$

Hence

$$\sum_2^{\infty} \frac{1}{n \log n} = \frac{1}{2 \log 2} + \frac{1}{3 \log 3} + \dots$$

is divergent, as *Abel* first showed.

83. 1. *Abel's Series.*

An important class of series have the form

$$B = a_1 t_1 + a_2 t_2 + a_3 t_3 + \dots \quad (1)$$

As Abel first showed how the convergence of certain types of these series could be established, they may be appropriately called in his honor. The reasoning depends on the simple identity (*Abel's identity*),

$$\begin{aligned} B_{n,p} &= t_{n+1} A_{n,1} + t_{n+2} (A_{n,2} - A_{n,1}) + \dots + t_{n+p} (A_{n,p} - A_{n,p-1}) \\ &= A_{n,1} (t_{n+1} - t_{n+2}) + \dots + A_{n,p-1} (t_{n+p-1} - t_{n+p}) + t_{n+p} A_{n,p}, \quad (2) \end{aligned}$$

where as usual $A_{n,m}$ is the sum of the first m terms of the remainder series A_n . From this identity we have at once the following cases in which the series 1) converges.

2. *Let the series $A = a_1 + a_2 + \dots$ and the series $\Sigma |t_{n+1} - t_n|$ converge. Let the t_n be limited. Then $B = a_1 t_1 + a_2 t_2 + \dots$ converges.*

For since A is convergent, there exists an m such that

$$|A_{n,p}| < \epsilon; \quad n > m, \quad p = 1, 2, 3 \dots$$

Hence

$$|B_{n,p}| < \epsilon \{ |t_{n+1} - t_{n+2}| + |t_{n+2} - t_{n+3}| + \dots + |t_{n+p}| \}.$$

3. *Let the series $A = a_1 + a_2 + \dots$ converge. Let $t_1, t_2, t_3 \dots$ be a limited monotone sequence. Then B is convergent.*

This is a corollary of 2.

4. *Let $A = a_1 + a_2 + \dots$ be such that $|A_n| < G, n = 1, 2, \dots$. Let $\Sigma |t_{n+1} - t_n|$ converge and $t_n \rightarrow 0$. Then B is convergent.*

For by hypothesis there exists an m such that

$$|t_{n+1} - t_{n+2}| + |t_{n+2} - t_{n+3}| + \dots + |t_{n+p}| < \epsilon$$

for any $n > m$.

5. *Let $|A_n| < G$ and $t_1 \geq t_2 \geq t_3 \geq \dots \rightarrow 0$. Then B is convergent.*

This is a special case of 4.

6. As an application of 5 we see the alternating series

$$B = t_1 - t_2 + t_3 - \dots$$

is convergent. For as the A series we may take $A = 1 - 1 + 1 - 1 + \dots$ as $|A_n| \leq 1$.

84. Trigonometric Series.

Series of this type are

$$C = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots \quad (1)$$

$$S = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \dots \quad (2)$$

As we see, they are special cases of Abel's series. Special cases of the series 1), 2) are

$$\Gamma = \frac{1}{2} + \cos x + \cos 2x + \cos 3x + \dots \quad (3)$$

$$\Sigma = \sin x + \sin 2x + \sin 3x + \dots \quad (4)$$

It is easy to find the sums Γ_n, Σ_n as follows. We have

$$2 \sin mx \sin \frac{1}{2}x = \cos \frac{2m-1}{2}x - \cos \frac{2m+1}{2}x.$$

Letting $m = 1, 2, \dots, n$ and adding, we get

$$2 \sin \frac{1}{2}x \cdot \Sigma_n = \cos \frac{1}{2}x - \cos \frac{2n+1}{2}x. \quad (5)$$

Keeping x fixed and letting $n \doteq \infty$, we see Σ_n oscillates between fixed limits when $x \neq 0, \pm 2\pi, \dots$

Thus Σ is divergent except when $x = 0, \pm \pi, \dots$

Similarly we find when $x \neq 2m\pi$,

$$\Gamma_n = \frac{\sin(n - \frac{1}{2})x}{2 \sin \frac{1}{2}x}. \quad (6)$$

Hence for such values Γ_n oscillates between fixed limits. For the values $x = 2m\pi$ the equation 3) shows that $\Gamma_n \doteq +\infty$.

From the theorems 4, 5 we have at once now

If $\Sigma |a_{n+1} - a_n|$ converges and $a_n \doteq 0$, and hence in particular if $a_1 \geq a_2 \geq \dots \doteq 0$, the series 1) converges for every x , and 2) converges for $x \neq 2m\pi$.

If in 3) we replace x by $x + \pi$, it goes over into

$$\Delta = \frac{1}{2} - \cos x + \cos 2x - \cos 3x + \dots \quad (7)$$

Thus Δ_n oscillates between fixed limits if $x \neq \pm (2m-1)\pi$, when $n \rightarrow \infty$. Thus

If $\sum |a_{n+1} + a_n|$ converges and $a_n \rightarrow 0$, and hence in particular if $a_1 \geq a_2 \geq \dots \rightarrow 0$, the series $a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots$ converges for $x \neq (2m-1)\pi$.

85. Power Series.

An extremely important class of series are those of the type

$$P = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \dots \quad (1)$$

called power series. Since P reduces to a_0 if we set $x = a$, we see that every power series converges for at least one point. On the other hand, there are power series which converge at but one point, e.g.

$$a_0 + 1!(x-a) + 2!(x-a)^2 + 3!(x-a)^3 + \dots \quad (2)$$

For if $x \neq a$, $\lim n! |x-a|^n = \infty$, and thus (2) is divergent.

1. If the power series P converges for $x = b$, it converges absolutely within

$$D_\lambda(a) \quad , \quad \lambda = |a-b|.$$

If P diverges for $x = b$, it diverges without $D_\lambda(a)$.

Let us suppose first that P converges at b . Let x be a point in D_λ , and set $|x-a| = \xi$. Then the adjoint of P becomes for this point

$$\begin{aligned} II &= a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3 + \dots \\ &= a_0 + a_1 \lambda \cdot \frac{\xi}{\lambda} + a_2 \lambda^2 \cdot \left(\frac{\xi}{\lambda}\right)^2 + a_3 \lambda^3 \cdot \left(\frac{\xi}{\lambda}\right)^3 + \dots \end{aligned}$$

But
$$\lim a_n \lambda^n = 0,$$

since series P is convergent for $x = b$.

Hence
$$a_n \lambda^n < M \quad n > 1, 2, \dots$$

Thus
$$II_n < M \left(1 + \frac{\xi}{\lambda} + \dots + \frac{\xi^n}{\lambda^n} \right) < \frac{M}{1 - \frac{\xi}{\lambda}}$$

and II is convergent.

If P diverges at $x = b$, it must diverge for all b' such that $|b' - a| > \lambda$. For if not, P would converge at b by what we have just proved, and this contradicts the hypothesis.

2. Thus we conclude that the set of points for which P converges form an interval $(a - \rho, a + \rho)$ about the point a , called the *interval of convergence*; ρ is called its norm. We say P is *developed about the point a* . When $a = 0$, the series 1) takes on the simpler form

$$a_0 + a_1x + a_2x^2 + \dots$$

which for many purposes is just as general as 1). We shall therefore employ it to simplify our equations.

We note that the geometric series is a simple case of a power series.

86. *Cauchy's Theorem on the Interval of Convergence.*

The norm ρ of the interval of convergence of the power series,

$$P = a_0 + a_1x + a_2x^2 + \dots$$

is given by

$$\frac{1}{\rho} = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} \quad a_n = |a_n|.$$

We show Π diverges if $\xi > \rho$. For let

$$\frac{1}{\rho} > \beta > \frac{1}{\xi}.$$

Then by I, 338, 1, there exist an infinity of indices $\iota_1, \iota_2 \dots$ for which

$$\sqrt[\iota_n]{|a_{\iota_n}|} > \beta.$$

Hence

$$|a_{\iota_n}| > \beta^{\iota_n},$$

and thus

$$|a_{\iota_n} \xi^{\iota_n}| > (\xi \beta)^{\iota_n} > 1,$$

since $\xi \beta > 1$. Hence

$$\sum_n |a_n \xi^n|$$

is divergent and therefore Π .

We show now that Π converges if $\xi < \rho$. For let

$$\xi < \frac{1}{\beta} < \rho.$$

Then there exist only a finite number of indices for which

$$\sqrt[n]{|a_n|} > \beta.$$

Let m be the greatest of these indices. Then

$$\sqrt[n]{|a_n|} < \beta \quad n > m.$$

Hence

$$\alpha_n < \beta^n,$$

and

$$\alpha_n \xi^n < (\beta \xi)^n.$$

Thus

$$\begin{aligned} \alpha_{m+1} \xi^{m+1} + \dots + \alpha_{m+n} \xi^{m+n} &< (\beta \xi)^{m+1} \{1 + (\beta \xi) + \dots + (\beta \xi)^{m+n-1}\} \\ &< \frac{(\beta \xi)^{m+1}}{1 - \beta \xi}, \end{aligned}$$

and Π is convergent.

Example 1.

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Here

$$\forall \alpha_n = \frac{1}{n!} \neq 0 \quad \text{by I, 185, 4.}$$

Hence $\rho = \infty$ and the series converges absolutely for every x .

Example 2.

$$\frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

Here

$$\forall \alpha_n = \frac{1}{n} \neq 1 \quad \text{by I, 185, 3.}$$

Hence $\rho = 1$, and the series converges absolutely for $|x| < 1$.

Tests of Convergence for Positive Term Series

87. To determine whether a given positive term series

$$A = a_1 + a_2 + \dots$$

is convergent or not, we may compare it with certain standard series whose convergence or divergence is known. Such comparisons enable us also to establish criteria of convergence of great usefulness.

We begin by noting the following theorem which sometimes proves useful.

1. *Let A, B be two series which differ only by a finite number of terms. Then they converge or diverge simultaneously.*

This follows at once from 80, 2. Hence if a series A whose convergence is under investigation has a certain property only

after the m th term, we may replace A by A_m , which has this property from the start.

2. The *fundamental theorem* of comparison is the following :

Let $A = a_1 + a_2 + \dots$, $B = b_1 + b_2 + \dots$ be two positive term series. Let $r > 0$ denote a constant. If $a_n \leq r b_n$, A converges if B does and $A \leq rB$. If $a_n \geq r b_n$, A diverges if B does.

For on the first hypothesis

$$A_n \leq r B_n.$$

On the second hypothesis

$$A_n \geq r B_n.$$

The theorem follows on passing to the limit.

3. From 2 we have at once :

Let $A = a_1 + a_2 + \dots$, $B = b_1 + b_2 + \dots$ be two positive term series. Let r, s be positive constants. If

$$r \leq \frac{a_n}{b_n} \leq s \quad n = 1, 2, \dots$$

or if

$$\lim \frac{a_n}{b_n}$$

exists and is $\neq 0$, A and B converge or diverge simultaneously. If B converges and $\frac{a_n}{b_n} \neq 0$, A also converges. If B diverges and $\frac{a_n}{b_n} \neq \infty$, A also diverges.

4. Let $A = a_1 + a_2 + \dots$, $B = b_1 + b_2 + \dots$ be positive term series. If B is convergent and

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n} \quad n = 1, 2, 3 \dots$$

A converges. If B is divergent and

$$\frac{a_{n+1}}{a_n} \geq \frac{b_{n+1}}{b_n},$$

A diverges.

For on the first hypothesis

$$\frac{a_{n+1}}{b_{n+1}} \leq \frac{a_n}{b_n} \leq \dots \leq \frac{a_1}{b_1}.$$

We may, therefore, apply 3. On the second hypothesis, we have

$$\frac{a_n}{b_n} < \frac{a_1}{b_1}$$

and we may again apply 3.

Example 1. $A = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots$

is convergent. For

$$a_n = \frac{1}{n \cdot n+1} < \frac{1}{n^2}$$

and $\sum \frac{1}{n^2}$ is convergent. The series A was considered in 81, 4, Ex. 1.

Example 2. $A = e^{-x} \cos x + e^{-2x} \cos 2x + \dots$
is absolutely convergent for $x > 0$.

For

$$|a_n| < \frac{1}{e^{nx}}$$

which is thus < the n th term in the convergent geometric series

$$\frac{1}{e^x} + \frac{1}{e^{2x}} + \frac{1}{e^{3x}} + \dots$$

Example 3. $A = \sum_n \frac{1}{n} \log \frac{n+1}{n}$
is convergent.

For

$$\log \left(1 + \frac{1}{n} \right) = \frac{1}{n} - \frac{\theta_n}{n^2}, \quad 0 < \theta_n < 1.$$

Hence

$$0 < a_n = \frac{1}{n^2} \left(1 - \frac{\theta_n}{n} \right) < \frac{1}{n^2}.$$

Thus A is comparable with the convergent series $\sum \frac{1}{n^2}$.

88. We proceed now to deduce various tests for convergence and divergence. One of the simplest is the following, obtained by comparison with the hyperharmonic series.

Let $A = a_1 + a_2 + \dots$ be a positive term series. It is convergent if

$$\lim a_n n^\mu < \infty, \quad \mu > 1,$$

and divergent if

$$\lim na_n > 0.$$

For on the first hypothesis there exists, by I, 338, a constant $G > 0$ such that

$$a_n < \frac{G}{n^\mu} \quad n = 1, 2, \dots$$

Thus each term of A is less than the corresponding term of the convergent series $G \sum \frac{1}{n^\mu}$.

On the second hypothesis there exists a constant c such that

$$a_n > \frac{c}{n} \quad n = 1, 2, \dots$$

and each term of A is greater than the corresponding term of the divergent series $c \sum \frac{1}{n}$.

Example 1.
$$A = \sum \frac{1}{\log^m n} \quad m > 0.$$

Here

$$na_n = \frac{n}{\log^m n} \doteq +\infty, \quad \text{by I, 463.}$$

Hence A is divergent.

Example 2.
$$A = \sum \frac{1}{n \log n}.$$

Here

$$na_n = \frac{1}{\log n} \doteq 0.$$

Thus the theorem does not apply. The series is divergent by 82.

Example 3.

$$L = \sum l_n = \sum \log \left(1 + \frac{\mu}{n} + \frac{\theta_n}{n^r} \right), \quad r > 1,$$

where μ is a constant and $|\theta_n| < G$.

From I, 413, we have, setting $r = 1 + s$,

$$l_n = \frac{1}{n} \left(\mu + \frac{\theta_n}{n^s} \right) - \frac{\alpha_n}{n^2} \left(\mu + \frac{\theta_n}{n^s} \right)^2 \quad 0 < \alpha_n < 1.$$

Hence

$$nl_n \doteq \mu, \quad \text{if } \mu \neq 0,$$

and L is divergent. If $\mu > 0$, L is an essentially positive term series. Hence $L = +\infty$. If $\mu < 0$, $L = -\infty$.

Let $\mu = 0$. Then

$$L_n = \frac{\theta_n}{n^r} \left(1 - \frac{\alpha_n \theta_n}{n^r} \right) \quad 0 < \alpha_n < 1,$$

which is comparable with the convergent series

$$\sum \frac{1}{n^r}, \quad r > 1.$$

Thus L is convergent in this case.

Example 4. The harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \dots$$

is divergent. For

$$\lim na_n = 1.$$

Example 5.

$$A = \sum \frac{1}{n^\alpha \log^\beta n} \quad \beta \text{ arbitrary.}$$

Here

$$na_n = \frac{n^{1-\alpha}}{\log^\beta n} \doteq \infty, \quad \alpha < 1$$

by I, 463, 1. Hence A is divergent for $\alpha < 1$.

Example 6.

$$A = \sum \frac{1}{\sqrt[n]{n^{n+1}}}.$$

Here

$$na_n = \frac{1}{\sqrt[n]{n}} \doteq 1 \quad \text{by I, 185, Ex. 8.}$$

Example 7.

$$A = \sum \left(\frac{\log(n+1)}{\log n} - 1 \right).$$

Here, if $\mu > 0$,

$$n^{1+\mu} a_n = n^{1+\mu} \frac{\log\left(1 + \frac{1}{n}\right)}{\log n} = \frac{n^\mu}{\log n} \log\left(1 + \frac{1}{n}\right) \doteq \infty,$$

since $n^\mu > \log n$ and $\left(1 + \frac{1}{n}\right)^n \doteq e$.

Hence A is divergent.

89 D'Alembert's Test. The positive term series $A = a_1 + a_2 + \dots$ converges if there exists a constant $r < 1$ for which

$$\frac{a_{n+1}}{a_n} \leq r, \quad n=1, 2, \dots$$

It diverges if

$$\frac{a_{n+1}}{a_n} \geq 1.$$

This follows from 87, 4, taking for B the geometric series $1 + r + r^2 + r^3 + \dots$

Corollary. Let $\frac{a_{n+1}}{a_n} \doteq l$. If $l < 1$, A converges. If $l > 1$, it diverges.

Example 1. The Exponential Series.

Let us find for what values of x the series

$$E = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad (1)$$

is convergent. Applying D'Alembert's test to its adjoint, we find

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^n}{n!} \cdot \frac{n-1!}{x^{n-1}} \right| = \frac{|x|}{n}.$$

Thus E converges absolutely for every x .

Let us employ 80, 9 to estimate the remainder \bar{E}_n . Let $x > 0$. The terms of E are all > 0 . Since

$$\frac{x^{n+p}}{(n+p)!} = \frac{x^n}{n!} \cdot \frac{x^p}{n+1 \cdot n+2 \cdot \dots \cdot n+p} \leq \frac{x^n}{n!} \left(\frac{x}{n+1} \right)^p,$$

we have

$$\frac{x^n}{n!} < \bar{E}_n < \frac{x^n}{n!} \sum_{p=0}^{\infty} \left(\frac{x}{n+1} \right)^p. \quad (2)$$

However large x may be, we may take n so large that $x < n+1$. Then the series on the right of 2) is a convergent geometric series.

Let $x < 0$. Then however large $|x|$ is, \bar{E}_n is alternating for some m . Hence by 81, 3 for $n \geq m$,

$$|\bar{E}_n| < \frac{|x|^n}{n!}. \quad (3)$$

Example 2. The Logarithmic Series.

Let us find for what values of x the series

$$L = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

is convergent. The adjoint gives

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{n}{n+1} \cdot |x| \doteq |x|.$$

Thus L converges absolutely for any $|x| < 1$, and diverges for $|x| > 1$.

When $x = 1$, L becomes

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

which is simply convergent by 81, 4.

When $x = -1$, L becomes

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

which is the divergent harmonic series.

Example 3. $A = \frac{1}{1^\mu} + \frac{1}{2^\mu} + \frac{1}{3^\mu} + \cdots$

$$\frac{a_{n+1}}{a_n} = \left(\frac{n}{n+1} \right)^\mu = 1.$$

As A is convergent when $\mu > 1$ and divergent if $\mu \leq 1$, we see that D'Alembert's test gives us no information when $l = 1$. It is, however, convergent for this case by 81, 2.

Example 4.

$$\sum_1^{\infty} \frac{n!}{(1+x) \cdots (n+x)} \quad x > 0.$$

Here

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n+1+x} \doteq 1,$$

and D'Alembert's test does not apply.

Example 5.

$$A = \sum n^\mu x^n.$$

Here

$$\left| \frac{a_{n+1}}{a_n} \right| = \left(\frac{n+1}{n} \right)^\mu |x| \doteq |x|.$$

Thus A converges for $|x| < 1$ and diverges for $|x| > 1$. For $|x| = 1$ the test does not apply. For $x = 1$ we know by 81, 2 that A is convergent for $\mu < -1$, and is divergent for $\mu > -1$.

For $x = -1$, A is divergent for $\mu \geq 0$, since a_n does not $\rightarrow 0$. A is an alternating series for $\mu < 0$, and is then convergent.

90. Cauchy's Radical Test. Let $A = a_1 + a_2 + \dots$ be a positive term series. If there exists a constant $r < 1$ such that

$$\sqrt[n]{a_n} < r \quad n = 1, 2, \dots$$

A is convergent. If, on the other hand,

$$\sqrt[n]{a_n} > 1$$

A is divergent.

For on the first hypothesis,

$$a_n < r^n$$

so that each term of A is $<$ the corresponding term in $r + r^2 + r^3 + \dots$ a convergent geometric series. On the second hypothesis, this geometric series is divergent and $a_n > r^n$.

Corollary. If $\lim \sqrt[n]{a_n} = l$, and $l < 1$, A is convergent. If $l > 1$, A is divergent.

Example 1. The series

$$\sum_{n=2}^{\infty} \frac{1}{\log^n n} = a_2 + a_3 + \dots$$

is convergent. For

$$\sqrt[n]{a_n} = \frac{1}{\log n} \rightarrow 0.$$

Example 2.

$$A = \sum \frac{n^n}{(n+1)^n}$$

is convergent. For

$$\sqrt[n]{a_n} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1} \right)^n = \frac{1}{\left(1 + \frac{1}{n} \right)^n} \rightarrow \frac{1}{e} < 1.$$

Example 3. In the elliptic functions we have to consider series of the type

$$\theta(v) = 1 + 2 \sum_1^{\infty} q^{n^2} \cos 2\pi n v \quad 0 < q < 1.$$

This series converges absolutely if

$$q + q^1 + q^0 + \cdots$$

oes. But here

$$\sqrt[n]{a_n} = \sqrt[n]{q^n} = q^n \neq 0.$$

Thus $\theta(v)$ converges absolutely for every v .

Example 4. Let $0 < a < b < 1$. The series

$$A = a + b^2 + a^3 + b^4 + \cdots$$

is convergent. For if

$$n = 2m$$

$$\sqrt[n]{a_n} = \sqrt[2m]{b^{2m}} = b.$$

If $n = 2m + 1$,

$$\sqrt[n]{a_n} = \sqrt[2m+1]{a^{2m+1}} = a.$$

Thus for all n

$$\sqrt[n]{a_n} \leq b < 1.$$

Let us apply D'Alembert's test. Here

$$\frac{a_{n+1}}{a_n} = b \left(\frac{b}{a} \right)^{2m+1} \neq \infty \quad n = 2m + 1,$$

$$= a \left(\frac{a}{b} \right)^{2m} \neq 0 \quad n = 2m.$$

Thus the test gives us no information.

91. Cauchy's Integral Test.

Let $\phi(x)$ be a positive monotone decreasing function in the interval (a, ∞) . The series

$$\Phi = \phi(1) + \phi(2) + \phi(3) + \cdots$$

is convergent or divergent according as

$$\int_a^\infty \phi(x) dx$$

is convergent or divergent.

For in the interval $(n, n+1)$, $n > m \geq a$,

$$\phi(n+1) \leq \phi(x) \leq \phi(n).$$

Hence

$$\phi(n+1) \leq \int_n^{n+1} \phi dx \leq \phi(n).$$

Letting $n = m, m+1, \dots, m+p$, and adding, we have

$$\Phi_{m,p+1} \leq \int_m^{m+p} \phi dx \leq \Phi_{m-1,p+1}.$$

Passing to the limit $p = \infty$, we get

$$\bar{\Phi}_m \leq \int_m^{\infty} \phi dx \leq \bar{\Phi}_{m-1}, \quad (1)$$

which proves the theorem.

Corollary. When Φ is convergent

$$\bar{\Phi}_m \leq \int_m^{\infty} \phi dx.$$

Example 1. We can establish at once the results of 81, 2. For, taking $\phi(x) = \frac{1}{x^\mu}$,

$$\int_1^{\infty} \phi dx = \int_1^{\infty} \frac{dx}{x^\mu}$$

is convergent or divergent according as $\mu > 1$, or $\mu \leq 1$, by I, 635, 636.

We also note that if

$$A = \frac{1}{1^{1+\mu}} + \frac{1}{2^{1+\mu}} + \frac{1}{3^{1+\mu}} + \dots$$

then

$$\bar{A} < \int_1^{\infty} \frac{1}{x^{1+\mu}} = \frac{1}{\mu} \cdot \frac{1}{n^\mu}.$$

Example 2. The logarithmic series

$$\sum_n \frac{1}{n l_1 n l_2 n \dots l_{s-1} n l_s^\mu n} \quad s = 1, 2, \dots$$

are convergent if $\mu > 1$; divergent if $\mu \leq 1$.

We take here

$$\phi(x) = \frac{1}{x l_1 x \dots l_{s-1} x l_s^\mu x}$$

and apply I, 637, 638.

92. 1. One way, as already remarked, to determine whether a given positive term series $A = a_1 + a_2 + \dots$ is convergent or divergent is to compare it with some series whose convergence or divergence is known. We have found up to the present the following *standard series* S :

The geometric series

$$1 + g + g^2 + \dots \quad (1)$$

The general harmonic series

$$\frac{1}{1^\mu} + \frac{1}{2^\mu} + \frac{1}{3^\mu} + \dots \quad (2)$$

The logarithmic series

$$\sum \frac{1}{nl_1^\mu n}, \quad (3)$$

$$\sum \frac{1}{nl_1 nl_2^\mu n}, \quad (4)$$

$$\sum \frac{1}{nl_1 nl_2 nl_3^\mu n} \quad (5)$$

.

We notice that none of these series could be used to determine by comparison the convergence or divergence of the series following it.

In fact, let a_n, b_n denote respectively the n th terms in 1), 2). Then for $g < 1, \mu > 0$,

$$\frac{b_n}{a_{n+1}} = \frac{1}{n^\mu g^n} = \frac{e^{-n \log g}}{n^\mu} \doteq \infty \quad \text{by I, 464,}$$

or using the *infinitary notation* of I, 461,

$$b_n > a_n.$$

Thus the terms of 2) converge to 0 infinitely slower than the terms of 1), so that it is useless to compare 2) with 1) for convergence. Let $g \geq 1$. Then

$$\frac{a_{n+1}}{b_n} = n^\mu g^n \doteq \infty,$$

or
$$a_n > b_n.$$

This shows we cannot compare 2) with 1) for divergence.

Again, if a_n, b_n denote the n th terms of 2), have, if $\mu > 1$,

$$\frac{b_n}{a_n} = \frac{n^{\mu-1}}{\log^\mu n} \rightarrow \infty \quad \text{by 1, 4}$$

or

$$b_n \sim a_n.$$

If $\mu = 1$,

$$\frac{a_n}{b_n} = \log n \rightarrow \infty,$$

or

$$a_n \sim b_n.$$

Thus the convergence or divergence of from 2) by comparison. In the same way with the others.

2. These considerations lead us to introduce notions. Let $A = a_1 + a_2 + \dots$, $B = b_1 + b_2 + \dots$ series. Instead of considering the behavior of a_n and b_n we generalize and consider the ratios $A_n : B_n$ for divergent series. These ratios obviously determine the rate at which A_n and B_n approach their limits. If A and B are divergent and

$$A_n \sim B_n,$$

we say A, B diverge equally fast; if

$$A_n = o(B_n),$$

A diverges slower than B , and B diverges faster than A . In I, 180, we have:

Let A, B be divergent and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = l.$$

B converges faster than A , and A converges slower than B .
I, 184, we have:

Let A, B be convergent and

$$\lim \frac{a_n}{b_n} = l.$$

According as l is 0, $\neq 0$, ∞ , A converges faster, equally fast, or slower than B .

Returning now to the set of standard series S , we see that S converges (diverges) slower than any preceding series of the set S . Such a set may therefore appropriately be called a *scale* of convergent (divergent) series. Thus if we have a decreasing positive term series, whose convergence or divergence is to be ascertained, we may compare it successively with the scale S , until we reach one which converges or diverges equally fast. In practice such a series may always be found. It is easy, however, to show that there exist series which converge or diverge slower than any series in the scale S or indeed any other scale.

For let

$$A, \quad B, \quad C, \dots$$

be any scale of positive term convergent or divergent series.

Then, if convergent,

$$A_n^{-1} > B_n^{-1} > C_n^{-1} > \dots;$$

if divergent,

$$A_n < B_n < C_n < \dots$$

Thus in both cases we are led to a sequence of functions of the type

$$f_1(n) > f_2(n) > f_3(n) > \dots$$

Thus to show the existence of a series O which converges

Then there exist positive increasing functions f_n such that $f_n < f_{n+1}$ for any f_n .

For as $f_1 > f_2$ there exists an $a_1 > a$ such that $x \geq a_1$. Since $f_2 > f_3$, there exists an $a_2 > a_1$ for $x \geq a_2$. And in general there exists an a_n such that $f_n > f_{n+1} + n$ for $x > a_n$. Let now

$$g(x) = f_n(x) + n - 1 \quad \text{in } (a_n, \infty)$$

Then g is an increasing unlimited function. Finally remains below any $f_m(x) + m - 1$, m a fixed integer.

$$\text{Thus } 0 \leq \lim_{x \rightarrow \infty} \frac{g(x)}{f_m(x)} = \lim_{x \rightarrow \infty} \frac{g(x)}{f_m + m - 1} < 1$$

Hence $g < f_m$.

93. From the logarithmic series we can derive various tests, for example, the following:

1. (*Bertram's Tests.*) Let $A = a_1 + a_2 + \dots$ be a series.

Let

$$Q_s(n) = \frac{\log \frac{1}{a_n n l_1 n \cdots l_{s-1} n}}{l_{s+1} n} \quad (s \geq 1)$$

If for some s and m ,

$$Q_s(n) > \mu > 1 \quad n > m,$$

A is convergent. If, however,

$$Q_s(n) < 1,$$

A is divergent.

For multiplying 1) by $l_{s+1} n$, we get

TESTS OF CONVERGENCE FOR POSITIVE TERM SERIES

Thus A is convergent.

The rest of the theorem follows similarly.

2. For the positive term series $A = a_1 + a_2 + \dots$ to converge it is necessary that, for $n = \infty$,

$$\lim a_n = 0, \quad \lim na_n = 0, \quad \lim na_n l_1 n = 0, \quad \lim na_n l_1 n l_2 n$$

We have already noted the first two. Suppose now that

$$\lim na_n l_1 n \dots l_s n > 0.$$

Then by I, 338 there exists an m and a $c > 0$, such that

$$na_n l_1 n \dots l_s n > c, \quad n > m,$$

or

$$a_n > \frac{c}{nl_1 n \dots l_s n}.$$

Hence A diverges.

Example 1.

$$A = \sum \frac{1}{n^\alpha \log^\beta n}.$$

We saw, 88, Ex. 5, that A is divergent for $\alpha < 1$. For A is convergent for $\beta > 1$ and divergent if $\beta \leq 1$, according to 91, Ex. 2.

If $\alpha > 1$, let

$$\alpha = \alpha' + \alpha'' \quad , \quad \alpha'' > 1.$$

Then if $\beta \geq 0$,

$$a_n = \frac{1}{n^\alpha \log^\beta n} \leq \frac{1}{n^{\alpha'}} \quad , \quad n > 2,$$

and A is convergent since $\sum \frac{1}{n^{\alpha'}}$ is. If $\beta < 0$, let

Example 2.

$$A = \sum \frac{1}{n^\mu e^{(1+\frac{1}{2}+\dots+\frac{1}{n})}} = \sum \frac{1}{n^\mu e^{H_n}}.$$

Here

$$\begin{aligned} Q_1 &= \frac{\log \frac{1}{a_n n}}{l_2 n} = \frac{-\log n + \mu \log n + H_n}{l_2 n} \\ &= \frac{\log n}{l_2 n} \left\{ \mu - 1 + \frac{H_n}{\log n} \right\} \doteq \mu \infty \quad \text{by 81, 6).} \end{aligned}$$

Hence A is convergent for $\mu > 0$ and divergent for $\mu < 0$.
test for $\mu = 0$.

But for $\mu = 0$,

$$\begin{aligned} Q_2 &= \frac{\log \frac{1}{a_n n l_1 n}}{l_3 n} = \frac{H_n - l_1 n - l_2 n}{l_3 n} \\ &= \frac{l_1 n}{l_3 n} \left\{ -1 - \frac{l_2 n}{l_1 n} + \frac{H_n}{l_1 n} \right\} \\ &\doteq -\infty, \end{aligned}$$

since $l_2 n > l_3 n$. Thus A is divergent for $\mu = 0$.

94. A very general criterion is due to *Kummer*, viz.:

Let $A = a_1 + a_2 + \dots$ be a positive term series. Let k_1, k_2, \dots be a set of positive numbers chosen at pleasure. A is convergent if there is some constant $k > 0$.

$$K_n = k_n \frac{a_n}{a_{n+1}} - k_{n+1} \geq k \quad n = 1, 2, \dots$$

A is divergent if

$$R = \frac{1}{k_1} + \frac{1}{k_2} + \dots$$

is divergent and

$$K_n \leq 0 \quad n = 1, 2, \dots$$

For on the first hypothesis

and A is convergent by 80, 4.

On the second hypothesis,

$$\frac{a_n}{a_{n+1}} \leq k_{n+1},$$

or

$$\frac{a_{n+1}}{a_n} \geq k_{n+1}^{-1}.$$

Hence A diverges since R is divergent.

95. 1. From Kummer's test we may deduce at once. For take

$$k_1 = k_2 = \dots = 1.$$

Then $A = a_1 + a_2 + \dots$ converges if

$$K_n = \frac{a_n}{a_{n+1}} - 1 > k > 0,$$

i.e. if

$$\frac{a_{n+1}}{a_n} \leq \rho < 1.$$

Similarly A diverges if $\frac{a_{n+1}}{a_n} \geq 1$.

2. To derive Raabe's test we take

$$k_n = n.$$

Then A converges if

$$K_n = n \left(\frac{a_n}{a_{n+1}} - 1 \right) > (n+1) > \infty$$

i.e. if

$$n \left(\frac{a_n}{a_{n+1}} - 1 \right) \geq l > 1.$$

Similarly A diverges if

96. 1. Let $A = a_1 + a_2 + \dots$ be a positive term

$$\lambda_0(n) = n \left(\frac{a_n}{a_{n+1}} - 1 \right)$$

$$\lambda_1(n) = l_1 n \left\{ n \left(\frac{a_n}{a_{n+1}} - 1 \right) - 1 \right\}$$

$$\lambda_2(n) = l_2 n \left\{ l_1 n \left[n \left(\frac{a_n}{a_{n+1}} - 1 \right) - 1 \right] - 1 \right\}$$

$$\dots \dots \dots$$

Then A converges if there exists an s such that

$$\lambda_s(n) \geq \delta > 1 \quad \text{for some } n > m;$$

it diverges if

$$\lambda_s(n) \leq 1 \quad \text{for } n > m.$$

We have already proved the theorem for $\lambda_0(n)$ how to prove it for $\lambda_1(n)$. The other cases follow.

For the Kummer numbers k_n we take

$$k_n = n \log n.$$

Then A converges if

$$k_n = n \log n \cdot \frac{a_n}{a_{n+1}} - (n+1) \log (n+1)$$

As

$$n+1 = n \left(1 + \frac{1}{n} \right),$$

$$\begin{aligned} K_n &= \lambda_1(n) - \log \left(1 + \frac{1}{n} \right)^n - \log \left(1 + \frac{1}{n} \right)^{n+1} \\ &= \lambda_1(n) - \log \left(1 + \frac{1}{n} \right)^{n+1} \end{aligned}$$

TESTS OF CONVERGENCE FOR POSITIVE TERM SERIES

For if this upper limit is not $+\infty$,

$$n \left\{ n \left(\frac{a_n}{a_{n+1}} - 1 \right) - 1 \right\} < G$$

for all n . Hence

$$\lambda_1(n) = \frac{\log n}{n} < G.$$

But the right side > 0 . Hence $\lambda_1(n) < 1$ for $n > \text{some } m$. A is divergent by 1.

Example. We note that Raabe's test does apply to the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

Here

$$n \left(\frac{a_n}{a_{n+1}} - 1 \right) = 1.$$

Hence

$$P_n = 0, \quad \text{and}$$

$$\lim P_n = 0.$$

Hence the series (1) is divergent.

97. Gauss' Test. Let $A = a_1 + a_2 + \cdots$ be a positive term series such that

$$\begin{aligned} a_n &= n^s + a_1 n^{s-1} + \cdots + a_s, \\ a_{n+1} &= n^s + b_1 n^{s-1} + \cdots + b_s, \end{aligned}$$

where $s, a_1, \dots, b_1, \dots$ do not depend on n . Then A is convergent if $a_1 - b_1 > 1$, and divergent if $a_1 - b_1 < 1$.

Using the identity (1, 91, 2), we have

$$\lambda_0(n) = n \left(\frac{a_n}{a_{n+1}} - 1 \right) = \frac{a_1 - b_1 + \frac{1}{n} \{a_2 - b_2 + \cdots\}}{1 + \frac{1}{n} \{b_1 + \cdots\}}.$$

98. Let $A = a_1 + a_2 + \dots$ be a positive

$$\frac{a_n}{a_{n+1}} = 1 + \frac{\alpha}{n} + \frac{\beta_n}{n^\mu} \quad \mu > 1$$

Then A is convergent if $\alpha > 1$ and diverges

For

$$\lambda_0(n) = n \left(\frac{a_n}{a_{n+1}} - 1 \right) = \alpha + \frac{\beta_n}{n^{\mu-1}}$$

and A converges if $\alpha > 1$ and diverges if

$$\lambda_1(n) = l_1 n \{ \lambda_0(n) - 1 \} =$$

and A is divergent.

EXAMPLES

99. *The Binomial Series.* Let us find μ the series

$$B = 1 + \mu x + \frac{\mu \cdot \mu - 1}{1 \cdot 2} x^2 + \frac{\mu \cdot \mu - 1}{1 \cdot 2} x^3 + \dots$$

converges. If μ is a positive integer, B is For $\mu = 0$, $B = 1$. We now exclude the Applying D'Alembert's test to its adjoint

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\mu - n + 1}{n} \right| |x|$$

Thus B converges absolutely for $|x| < 1$.

Let $x = 1$. Then

$$B = 1 + \mu + \frac{\mu \cdot \mu - 1}{1 \cdot 2} + \frac{\mu \cdot \mu - 1}{1 \cdot 2} + \dots$$

Here D'Alembert's test applied to its

Thus B converges absolutely if $\mu > 0$, and its adjoint diverges if $\mu < 0$. Thus B does not converge absolutely for $\mu < 0$.

But in this case we note that the terms of B are alternately positive and negative. Also

$$\frac{a_{n+1}}{a_n} = \left| 1 - \frac{1+\mu}{n} \right|,$$

so that the $|a_n|$ form a decreasing sequence from a certain term on. We investigate now when $a_n \neq 0$. Now

$$a_n = (-1)^n \frac{(-\mu)(-\mu+1)\cdots(-\mu+n-1)}{1 \cdot 2 \cdots n} = (-1)^n Q_n.$$

In I, 143, let $\alpha = -\mu$, $\beta = 1$. We thus find that $\lim a_n = 0$ when $\mu > -1$. Thus B converges when $\mu > -1$ and diverges when $\mu < -1$.

Let $x = -1$. Then

$$B = 1 - \mu + \frac{\mu \cdot \mu - 1}{1 \cdot 2} - \frac{\mu \cdot \mu - 1 \cdot \mu - 2}{1 \cdot 2 \cdot 3} + \dots$$

If $\mu > 0$, the terms of B finally have one sign, and

$$n \left(\left| \frac{a_n}{a_{n+1}} \right| - 1 \right) \doteq 1 + \mu.$$

Hence B converges absolutely.

If $\mu < 0$, let $\mu = -\lambda$. Then B becomes

$$1 + \lambda + \frac{\lambda \cdot \lambda + 1}{1 \cdot 2} + \frac{\lambda \cdot \lambda + 1 \cdot \lambda + 2}{1 \cdot 2 \cdot 3} + \dots$$

Here

100. The Hypergeometric Series

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha \cdot \alpha + 1 \cdot \beta \cdot \beta + 1}{1 \cdot 2 \cdot \gamma \cdot \gamma + 1} x^2 \\ + \frac{\alpha \cdot \alpha + 1 \cdot \alpha + 2 \cdot \beta \cdot \beta + 1 \cdot \beta + 2}{1 \cdot 2 \cdot 3 \cdot \gamma \cdot \gamma + 1 \cdot \gamma + 2} x^3 + \dots$$

Let us find for what values of x this series converges. to the adjoint series, we find

$$\left| \frac{a_{n+2}}{a_{n+1}} \right| = \left| \frac{(\alpha + n)(\beta + n)}{(n + 1)(\gamma + n)} \right| \cdot |x| \doteq |x|.$$

Thus F converges absolutely for $|x| < 1$ and diverges for

Let $x = 1$. The terms finally have one sign, and

$$\frac{a_{n+1}}{a_{n+2}} = \frac{n^2 + n(1 + \gamma) + \gamma}{n^2 + n(\alpha + \beta) + \alpha\beta}.$$

Applying Gauss', test we find F converges when and only when

$$\alpha + \beta - \gamma < 0.$$

Let $x = -1$. The terms finally alternate in sign. Let when $a_n \doteq 0$. We have

$$|a_{n+2}| = \frac{\alpha\beta}{\gamma} \cdot \frac{(\alpha + 1) \dots (\alpha + n)(\beta + 1) \dots (\beta + n)}{(1 + 1) \dots (1 + n)(\gamma + 1) \dots (\gamma + n)}$$

Now

$$\alpha + m = m \left(1 + \frac{\alpha}{m} \right) \quad , \quad \beta + m = m \left(1 + \frac{\beta}{m} \right),$$

$$1 + m = m \left(1 + \frac{1}{m} \right) \quad , \quad \gamma + m = m \left(1 + \frac{\gamma}{m} \right).$$

Thus

$$\left(1 + \frac{\alpha}{m} \right) \left(1 + \frac{\beta}{m} \right)$$

PRINGSHEIM'S THEORY

Hence

$$\begin{aligned} |a_{n+2}| &= \prod_1^n \left(1 + \frac{\alpha}{m}\right) \left(1 + \frac{\beta}{m}\right) \left(1 - \frac{1}{m} + \frac{\sigma_m}{m^2}\right) \left(1 - \frac{\gamma}{m} + \frac{\delta_m}{m^2}\right) \\ &= \prod_1^n \left(1 + \frac{\alpha + \beta - \gamma - 1}{m} + \frac{\eta_m}{m^2}\right). \end{aligned}$$

Hence

$$\log |a_{n+2}| = \sum_1^n \log \left(1 + \frac{\alpha + \beta - \gamma - 1}{m} + \frac{\eta_m}{m^2}\right) = \sum_1^n l_m =$$

and thus

$$L = \lim \log |a_{n+2}| = \sum_1^\infty l_m.$$

Now for a_n to $\doteq 0$ it is necessary that $L_n \doteq -\infty$. In § 100 we saw this takes place when and only when $\alpha + \beta - \gamma - 1 < 0$.

Let us find now when $|a_{n+1}| < |a_n|$. Now 1) gives

$$\left| \frac{a_{n+2}}{a_{n+1}} \right| = 1 + \frac{\alpha + \beta - \gamma - 1}{n} + \frac{\delta_n}{n^2}.$$

Thus when $\alpha + \beta - \gamma - 1 < 0$, $|a_{n+2}| < |a_{n+1}|$. Hence case *F* is an alternating series. We have thus the following theorem:

*The hypergeometric series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$. When $x = 1$, *F* converges only when $\alpha + \beta - \gamma < 0$ and then absolutely. When $x = -1$, *F* converges when $\alpha + \beta - \gamma - 1 < 0$, and absolutely if $\alpha + \beta - \gamma < 0$.*

Pringsheim's Theory

101. 1. In the 35th volume of the *Mathematische Annalen* (1880) Pringsheim has shown that if

A_n^μ , where A is any positive term divergent series

$\bar{B}_n^{-\mu}$ where B is any positive term convergent series

$$e^{\mu n} \quad , \quad (e^{e^n})^\mu \quad , \quad (e^{e^{e^n}})^\mu \quad , \quad \dots e^{\mu}$$

It will be convenient to denote in general a convergent term series by the symbol

$$C = c_1 + c_2 + \dots$$

and a divergent positive term series by

$$D = d_1 + d_2 + \dots$$

2. The series

$$C = \sum_1^\infty \frac{M_{n+1} - M_n}{M_n M_{n+1}} = \sum_1^\infty \left(\frac{1}{M_n} - \frac{1}{M_{n+1}} \right) =$$

is convergent, and conversely every positive term series may be brought into this form.

For

$$\begin{aligned} C_m &= \sum_1^m \left(\frac{1}{M_n} - \frac{1}{M_{n+1}} \right) \\ &= \frac{1}{M_1} - \frac{1}{M_{m+1}} \doteq \frac{1}{M_1} \end{aligned}$$

and C is convergent.

Let now conversely $C = c_1 + c_2 + \dots$ be a convergent positive term series. Let

$$\bar{C}_{n-1} = \frac{1}{M_n}.$$

Then

$$c_n = \frac{1}{M_n} - \frac{1}{M_{n+1}}.$$

3. The series

Let now conversely $D = d_1 + d_2 + \dots$ be a given positive convergent series. Let

$$M_n = D_{n-1}.$$

Then

$$d_n = M_{n+1} - M_n.$$

102. Having now obtained a general form of all convergent and divergent series, we now obtain another general form of convergent or divergent series, but which converges slower than 101, 1) or diverges slower than 101, 2). Let us consider first convergence. Let $M'_n < M_n$, then

$$\sum \left(\frac{1}{M'_n} - \frac{1}{M'_{n+1}} \right)$$

is convergent, and if M'_n is properly chosen, not only is each term of 1) greater than the corresponding term of 101, 1), but all converge slower than 101, 1). For example, for M'_n let us take M_n^μ , $0 < \mu < 1$. Then denoting the resulting series $C' = \sum c'_n$, we have

$$\begin{aligned} \frac{c'_n}{C_n} &= \frac{M_{n+1}^\mu - M_n^\mu}{M_n^\mu M_{n+1}^\mu} \frac{M_n M_{n+1}}{M_{n+1} - M_n} \\ &= \frac{1 - r^\mu}{1 - r} M_n^{1-\mu}, \quad r = \frac{M_n}{M_{n+1}} < 1. \end{aligned}$$

Thus C' converges slower than C . But the preceding analysis shows that C' and

$$\sum \frac{M_{n+1} - M_n}{M_{n+1} M_n^\mu}$$

converge equally fast. In fact 2) states that

$$c' \sim c_n M_n^{1-\mu}.$$

Since M is any positive increasing function of n whose limit

Now by I, 413, for sufficiently large n ,

$$\log M_{n+1} - \log M_n = -\log\left(1 - \frac{M_{n+1} - M_n}{M_{n+1}}\right) >$$

Replacing here M_n by $\log M_n$, we get

$$l_2 M_{n+1} - l_2 M_n > \frac{\log M_{n+1} - \log M_n}{\log M_{n+1}} > \frac{M_{n+1} - M_n}{M_{n+1}}$$

and in general

$$l_2 M_{n+1} - l_2 M_n > \frac{M_{n+1} - M_n}{M_{n+1} l_1 M_{n+1} \cdots l_{r-1} M_{n+1}}$$

Thus the series

$$\sum \frac{M_{n+1} - M_n}{M_{n+1} l_1 M_{n+1} \cdots l_{r-1} M_{n+1} l_r^{1+\mu} M_{n+1}}$$

converges as is seen by comparing with 4). We
the theorem:

The series

$$\sum \frac{M_{n+1} - M_n}{M_n M_{n+1}}, \quad \sum \frac{M_{n+1} - M_n}{M_{n+1} M_n}$$

$$\sum_1^{\infty} \frac{M_{n+1} - M_n}{M_{n+1} l_1 M_{n+1} \cdots l_{r-1} M_{n+1} l_r^{1+\mu} M_{n+1}} \quad r = 1, 2$$

form an infinite set of convergent series; each slower than any preceding it.

The last statement follows from I, 463, 1, 2.

Corollary 1 (Abel). Let $D = d_1 + d_2 + \cdots$ denote a divergent series. Then

$$\sum \frac{d_n}{n^{1+\mu}} \quad \mu > 0.$$

PRINGSHEIM'S THEORY

For by 101, ² we may bring C to the form

$$\sum \frac{M_{n+1} - M_n}{M_n M_{n+1}}.$$

Then any of the series 7) converges slower than C .

103. 1. Let us consider now divergent series. Here problem is simpler and we have at once the theorem:

The series

$$D = \sum_1^{\infty} \frac{M_{n+1} - M_n}{M_n} = \sum d_n$$

converges slower than

$$\sum (M_{n+1} - M_n) = \sum d'_n.$$

That 1) is divergent is seen thus: Consider the product

$$\begin{aligned} P_n &= \prod_1^n \left(1 + \frac{M_{m+1} - M_m}{M_m} \right) = \prod_1^n \frac{M_{m+1}}{M_m} \\ &= \frac{M_{n+1}}{M_1} \end{aligned}$$

which obviously $\doteq \infty$.

Now

$$\begin{aligned} P_n &= (1 + d_1)(1 + d_2) \cdots (1 + d_n) \\ &= 1 + (d_1 + \cdots + d_n) + (d_1 d_2 + d_1 d_3 + \cdots) \\ &\quad + (d_1 d_2 d_3 + \cdots) + \cdots + d_1 d_2 \cdots d_n \\ &< 1 + D_n + \frac{1}{2} D_n^2 + \cdots + \frac{1}{n!} D_n^n < e^{D_n} < e^D. \end{aligned}$$

Hence $D_n \doteq \infty$ and D is divergent.

As

$$\frac{d_n}{n!} = \frac{1}{n!} \doteq 0$$

Then $M_{n+1} > M_n$ and

$$d_n = \frac{M_{n+1} - M_n}{M_n}.$$

Moreover $M_n \doteq \infty$. For

$$\begin{aligned} \frac{M_{n+1}}{M_1} &= (1 + d_1) \cdots (1 + d_n) \\ &> 1 + D_n \quad \text{by I, 90, 1.} \end{aligned}$$

But $D_n \doteq \infty$.

3. *The series*

$$\begin{aligned} \sum_1^{\infty} (M_{n+1} - M_n) &= \sum d_n \\ \sum_1^r M_n l_1 M_n \cdots l_r M_n & \quad r = \end{aligned}$$

form an infinite set of divergent series, each series greater than any preceding it. $l_0 M_n = M_n$.

For

$$\begin{aligned} \log M_{n+1} - \log M_n &= \log \left(1 + \frac{M_{n+1} - M_n}{M_n} \right) \\ &< \frac{M_{n+1} - M_n}{M_n}. \end{aligned}$$

This proves the theorem for $r = 0$. Hence as replacing repeatedly M_n by $\log M_n$,

$$l_{r+1} M_{n+1} - l_{r+1} M_n < \frac{M_{n+1} - M_n}{M_n l_1 M_n \cdots l_r M_n}.$$

Corollary 1. If we take $M_n = n$, we get the series

Corollary 2 (Abel). Let $D = d_1 + d_2 + \cdots$ be a *term series*. Then

$$\sum \frac{d_n}{D_{n-1}}$$

4. In Ex. 3 of I, 454, we have seen that M_{n+1} is not always $\sim M_n$. In case it is we have

The series

$$\sum \frac{M_{n+1} - M_n}{M_n^{1+\mu}} \quad \mu > 0 \quad , \quad M_{n+1} \sim M_n$$

convergent.

It follows from 102, 3).

The series

$$\sum \frac{M_{n+1} - M_n}{e^{\mu M_n}} \quad M_{n+1} \sim M_n$$

convergent if $\mu > 0$; it is divergent if $\mu < 0$.

or

$$e^{\mu M_n} \sim \frac{1}{2} \mu^2 M_n^2 \sim M_n^2 \quad \mu > 0.$$

as

$$\frac{M_{n+1} - M_n}{e^{\mu M_n}} \sim \frac{M_{n+1} - M_n}{M_n^2}.$$

If $\mu < 0$

$$\frac{M_{n+1} - M_n}{e^{\mu M_n}} \gtrsim M_{n+1} - M_n.$$

. If $M_{n+1} \sim M_n$, we have

$$l_{r+1} M_{n+1} - l_{r+1} M_n \sim \frac{M_{n+1} - M_n}{M_n l_1 M_n \cdots l_r M_n} \sim \frac{M_{n+1} - M_n}{M_{n+1} l_1 M_{n+1} \cdots l_r M_{n+1}}.$$

For by 102, 5), 103, 3),

$$l_{r+1} M_{n+1} - l_{r+1} M_n < \frac{M_{n+1} - M_n}{M_n l_1 M_n \cdots l_r M_n},$$

$$\frac{M_{n+1} - M_n}{M_{n+1} l_1 M_{n+1} \cdots l_r M_n} < l_{r+1} M_{n+1} - l_{r+1} M_n.$$

Now since $M_{n+1} \sim M_n$, we have also obviously

$$l_m M_n \sim l_m M_{n+1} \quad m = 1, 2, \dots, r.$$

It diverges if

$$\frac{a_n}{d_n} > G.$$

In the second place, A converges if

$$\frac{a_{n+1}}{a_n} - \frac{c_{n+1}}{c_n} \leq 0,$$

and diverges if

$$\frac{a_{n+1}}{a_n} - \frac{d_{n+1}}{d_n} \geq 0.$$

The tests 1), 2) involve only a single term of A and the comparison series, while the tests 3), 4) involve p terms. With *Du Bois Reymond* such tests we may call *positively* tests of the *first* and *second kinds*. And the relation between p terms

$$a_n, a_{n+1}, \dots a_{n+p-1}$$

of the given series and p terms of a comparison series

$$c_n, c_{n+1}, \dots c_{n+p-1}, \quad \text{or} \quad d_n, d_{n+1}, \dots d_{n+p-1}$$

which serves as a criterion of convergence or divergence, is called a *test of the p^{th} kind*.

Let us return now to the tests 1), 2), 3), 4), where we are testing A for convergence. If for a certain series C

$$\frac{a_n}{c_n} \quad \text{not always} \leq G \quad , \quad n > m$$

it might be due to the fact that $c_n \rightarrow 0$ too fast. We may take another comparison series $C' = \sum c'_n$ which converges more slowly than C . As there always exist series which converge more slowly than any given positive term series, the test 1) must be sufficient for the convergence of A if a proper comparison series is found. In such series we employ series which converge slowly

It diverges if $\lim_{n \rightarrow \infty} \frac{M_n}{M_{n+1} - M_n} a_n > 0$.

This follows at once from 105, 1), 2); and 101, 2; 103, 1.

2. To get tests of greater power we have only to replace series

$$\sum \frac{M_{n+1} - M_n}{M_{n+1} M_n} \quad , \quad \sum \frac{M_{n+1} - M_n}{M_n}$$

just employed in 1), 2) by the series of 102 and 103, 3 which converge (diverge) slower. We thus get from 1:

The positive term series A converges if

$$\lim_{n \rightarrow \infty} \frac{M_{n+1} M_n^\mu}{M_{n+1} - M_n} a_n < \infty \quad \text{or} \quad \lim_{n \rightarrow \infty} \frac{M_n l_1 M_n \cdots l_{r-1} M_{n+1} l_r^{\mu+1} M_n}{M_{n+1} - M_n} a_n < \infty$$

It diverges if $\lim_{n \rightarrow \infty} \frac{M_n l_1 M_n \cdots l_r M_n}{M_{n+1} - M_n} a_n > 0$.

Bonnet's Test. The positive term series A converges if

$$\lim_{n \rightarrow \infty} n l_1 n \cdots l_{r-1} n l_r^{\mu+1} n \cdot a_n < \infty \quad , \quad \mu > 0.$$

It diverges if $\lim_{n \rightarrow \infty} n l_1 n \cdots l_r n \cdot a_n > 0$.

Follows from the preceding setting $M_n = n$.

3. The positive term series A converges or diverges according to

$$\begin{aligned} \frac{e^{\mu M_n l_n}}{M_{n+1} - M_n} &= 1 \quad , \quad \mu > 0, \\ &= 1 \quad , \quad \mu < 0, \end{aligned} \quad M_{n+1} \sim M_n.$$

For in the first case

$$a_n \sim \frac{M_{n+1} - M_n}{e^{\mu M_n}}$$

and in the second case

It diverges if

$$\lim \frac{M_{n+1} - M_n}{a_n} < 0 \quad \text{or} \quad \lim \frac{\log \frac{M_n}{l_{r+1} M_n}}{l_{r+1} M_n} < 0$$

Here $r = 0, 1, 2, \dots$ and as before $l_0 M_n = M_n$

For taking the logarithm of both sides of divergence

$$q = \frac{\log \frac{M_{n+1} - M_n}{a_n}}{M_n} > \mu.$$

As μ is an arbitrarily small but fixed positive number, the series diverges if $\lim q > 0$. Making use of 104, 3 we have the theorem. The rest follows similarly.

Remark. If we take $M_n = n$ we get Cauchy's and Bertram's tests 93.

For if

$$\frac{\log \frac{1}{a_n}}{n} = \log \sqrt[n]{\frac{1}{a_n}} = -\log \sqrt[n]{a_n}$$

it is necessary that

$$\sqrt[n]{a_n} \leq \mu < 1.$$

Also if

$$\begin{aligned} \frac{\log \frac{1}{a_n n l_1 n \cdots l_r n}}{l_{r+1} n} &= \frac{\log \frac{1}{a_n n l_1 n \cdots l_r n}}{l_{r+1} n} \\ &= -1 + \frac{\log a_n n l_1 n}{l_{r+1} n} \end{aligned}$$

it is necessary that

$$\frac{\log \frac{1}{a_n n l_1 n \cdots l_r n}}{l_{r+1} n} \geq \mu + 1$$

PRINGSHEIM'S THEORY

Pringsheim's Criterion. Let $p_1, p_2 \dots$ be a set of positive numbers chosen at pleasure, and let $P_n = p_1 + \dots + p_n$. The series A converges if

$$\lim_{n \rightarrow \infty} \frac{\log \frac{p_n}{a_n}}{P_n} > 0.$$

For A converges if

$$\lim_{n \rightarrow \infty} \frac{\log \frac{M_{n+1} - M_n}{a_n}}{M_n} > 0 \quad , \quad \text{by 106}$$

But $M_{n+1} - M_n = d_n$ is the general term of the divergent series $D = d_1 + d_2 + \dots$

Thus 2) may be written

$$\lim_{n \rightarrow \infty} \frac{\log \frac{d_n}{a_n}}{D_n} > 0.$$

Moreover A converges if

$$\frac{c_n}{a_n} \geq r > 1,$$

that is, if

$$\lim_{n \rightarrow \infty} \frac{c_n}{a_n} > 0,$$

where as usual $C = c_1 + c_2 + \dots$ is a convergent series.

Hence A converges if

$$\lim_{n \rightarrow \infty} \frac{\frac{c_n}{a_n}}{C_n} > 0.$$

But now the set of numbers $p_1, p_2 \dots$ gives rise to a series $P = p_1 + p_2 + \dots$ which must be either convergent or divergent. Thus 3), 4) show that in either case 1) holds.

Or in less general form:

The positive term series A converges if

$$\lim_{n \rightarrow \infty} \left(c_{n+1} \frac{a_n}{a_{n+1}} - c_n \right) > 0.$$

It diverges if

$$\lim_{n \rightarrow \infty} \left(d_{n+1} \frac{a_n}{a_{n+1}} - d_n \right) < 0.$$

Here as usual $C = c_1 + c_2 + \dots$ is a convergent, a divergent series.

2. Although we have already given one proof of Kummer's theorem we wish to show here its more general theory, and also to exhibit it under another form. Let us replace c_n, c_{n+1} in 1) by their values M_n, M_{n+1} . We get

$$\frac{M_{n+2} - M_{n+1}}{M_{n+2}} \cdot \frac{a_n}{a_{n+1}} - \frac{M_{n+1} - M_n}{M_n}$$

or since

$$M_{n+2} > M_{n+1},$$

$$\frac{M_{n+2} - M_{n+1}}{M_{n+1}} \cdot \frac{a_n}{a_{n+1}} - \frac{M_{n+1} - M_n}{M_n}$$

or by 103, 2

$$d_{n+1} \frac{a_n}{a_{n+1}} - d_n > 0,$$

where $D = d_1 + d_2 + \dots$ is *any* divergent positive series. Since any set of positive numbers k_1, k_2, \dots gives rise to a series $k_1 + k_2 + \dots$ which must be either convergent or divergent, from 1) that 5) holds when we replace the c_n by d_n . We have therefore:

Since the k 's are entirely arbitrary positive numbers, the result (6) also gives

converges if

$$k_n \frac{a_n}{a_{n+1}} - k_{n+1} > 0;$$

is seen by writing

$$k_n = \frac{1}{k'_n}$$

and then dropping the accent.

From Kummer's theorem we may at once deduce a series of increasing power, viz.:

The positive term series A is convergent or divergent according

$$\frac{M_{n+2} \cdots M_{n+1}}{M_{n+1} l_1 M_{n+1} \cdots l_r M_{n+1}} \cdot \frac{a_n}{a_{n+1}} - \frac{M_{n+1} - M_n}{M_n l_1 M_n \cdots l_r M_n}$$

is > 0 or < 0 .

For $k_1, k_2 \dots$ we have used here the terms of the divergent series of 103, 3.

Arithmetic Operations on Series

109. 1. Since an infinite series

$$A = a_1 + a_2 + a_3 \dots$$

is not a true sum but the limit of a sum

$$A = \lim_{n \rightarrow \infty} A_n,$$

we now inquire in how far the properties of polynomials hold for infinite polynomials (1). The associative property is expressed by the theorem:

This theorem relates to grouping the terms of a series. The following relate to removing them.

2. Let $B = b_1 + b_2 + \dots$ be convergent and let $b_2 = a_{m_1+1} + \dots + a_{m_2}, \dots$. If 1° $A = a_1 + a_2 + \dots$ converges, $A = B$. 2° If the terms $a_n > 0$, A is convergent, $m_n - m_{n-1} \leq p$ a constant, and $a_n \doteq 0$, A is convergent.

On the first hypothesis we have only to add $A = B$. On the second hypothesis

$$\epsilon > 0, \quad m, \quad B_n < \epsilon, \quad n > m.$$

Then

$$B - A_s < \epsilon, \quad s > m_n.$$

On the third hypothesis we may set

$$A_s = B_r + b'_{r+1}$$

where b'_{r+1} denotes a part of the a -terms in b_r and contains at most p terms of A , $b'_{r+1} \doteq 0$.

Hence

$$\lim A_s = \lim B_r, \quad \text{or} \quad A = B.$$

Example 1. The series

$$B = (1 - 1) + (1 - 1) + (1 - 1) + \dots$$

is convergent. The series obtained by removing the terms

$$A = 1 - 1 + 1 - 1 + \dots$$

is divergent.

Example 2.

$$A = 1 - \frac{1}{1+x} + \frac{1}{2} - \frac{1}{2+x} + \frac{1}{3} - \frac{1}{3+x} + \dots;$$

$$B = \sum (1 - \frac{1}{1+x}) = \sum \frac{x}{1+x}$$

ARITHMETIC OPERATIONS ON SERIES

The terms of a simply convergent series $A = a_1 + a_2 + \dots$ are arranged to form a series S , for which $\lim S_n$ is any prescribed number, or $\pm \infty$.

For let

$$B = b_1 + b_2 + \dots$$

$$C = c_1 + c_2 + \dots$$

be the series formed respectively of the positive and negative terms of A , the relative order of the terms in A being preserved.

To fix the ideas let l be a positive number; the demonstration for the other cases is similar. Since $B_n \doteq +\infty$, there exists a m_1 such that

$$B_{m_1} > l.$$

Let m_1 be the least index for which 1) is true. Since $C_n \doteq +\infty$, there exists an m_2 such that

$$B_{m_1} + C_{m_2} < l.$$

Let m_2 be the least index for which 2) is true. Continuing in this way we take just enough terms, say m_3 terms of B , so that

$$B_{m_1} + C_{m_2} + B_{m_1, m_2} > l.$$

Then just enough terms, say m_4 terms of C , so that

$$B_{m_1} + C_{m_2} + B_{m_1, m_2} + C_{m_2, m_4} < l,$$

etc. In this way we form the series

$$S = B_{m_1} + C_{m_2} + B_{m_1, m_2} + \dots$$

whose sum is l . For

$$|a_n| < \epsilon \quad n > \sigma;$$

also

$$a_1 + a_2 + a_3 + \dots + a_n = a_n$$

We may now take n so large that $A_n - A$ whose index is $< m$. Thus the terms of A of positive sign are a part of A_m and hence

$$|A_n - B_n| < A_m < \epsilon \quad n > m.$$

Thus B is convergent and $B = A$.

The same reasoning shows that B is convergent absolutely convergent.

3. If $A = a_1 + a_2 + \dots$ enjoys the commutative law, it is absolutely convergent.

For if only simply convergent we could arrange the terms to have any desired sum. But this contradicts the theorem above.

Addition and Subtraction

111. Let $A = a_1 + a_2 + \dots$, $B = b_1 + b_2 + \dots$.
The series

$$C = (a_1 \pm b_1) + (a_2 \pm b_2) + \dots$$

are convergent and $C = A \pm B$.

For obviously $C_n = A_n \pm B_n$. We have now the limit.

Example. We saw, 81, 3, Ex. 1, that

$$A = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

We note now that

$$\frac{3}{2}A = A + \frac{1}{2}A$$

$$= \sum \left(\frac{1}{4n-3} - \frac{1}{4n-2} + \frac{1}{4n-1} - \frac{1}{4n} \right) + \frac{1}{2} \sum \left(\frac{1}{2n-1} - \frac{1}{2n} \right)$$

$$= \sum \left(\frac{1}{4n-3} - \frac{1}{4n-2} + \frac{1}{4n-1} - \frac{1}{4n} \right) + \sum \left(\frac{1}{4n-2} - \frac{1}{4n} \right)$$

$$= \sum \left(\frac{1}{4n-1} + \frac{1}{4n-3} - \frac{1}{2n} \right)$$

$$= B \quad \text{by 109, 2.}$$

Thus

$$B = \frac{3}{2}A.$$

This example, due to *Dirichlet*, illustrates the non-commutativity property of simply convergent series. We have shown the convergence of B by actually determining its sum. As an exercise proceed directly as follows :

The series 1) may be written :

$$\sum \frac{8n-3}{2n(4n-1)(4n-3)} = \sum \frac{1}{n^2} \cdot \frac{8 - \frac{3}{n}}{2 \left(4 - \frac{1}{n} \right) \left(4 - \frac{3}{n} \right)}$$

Comparing this with

$$\sum \frac{1}{n^2}$$

we see that it is convergent by 87, 3. Since 1) is convergent, also by 109, 2.

112. 1. Multiplication. We have already seen, 80, 7, that

responds a lattice point ι, κ and conversely.
 it a great help here and later to keep this cor

*Let $A = a_1 + a_2 + \dots$, $B = b_1 + b_2 + \dots$ be a
 Then $C = \Sigma a_\iota b_\kappa$ is absolutely convergent and A*

Let m be taken large at pleasure; we may
 $\Gamma_n - A_m \cdot B_m$ contains no term both of whos

Then

$$\begin{aligned} \Gamma_n - A_m B_m &< \alpha_1 \bar{B}_m + \alpha_2 \bar{B}_m + \dots + \\ &\quad + \beta_1 \bar{A}_m + \beta_2 \bar{A}_m + \dots + \\ &< A_m \bar{B}_m + B_m \bar{A}_m \\ &< \epsilon \text{ for } m \text{ sufficiently} \end{aligned}$$

Hence

$$\lim \Gamma_n = A \cdot B$$

and C is absolutely convergent.

To show that $C = A \cdot B$, we note that

$$|C_n - A_m B_m| \leq \Gamma_n - A_m B_m < \epsilon$$

2. We owe the following theorem to *Merten*

If A converges absolutely and B converges (
lutely), then

$$C = a_1 b_1 + (a_1 b_2 + a_2 b_1) + (a_1 b_3 + a_2 b_2 +$$

is convergent and $C = A \cdot B$.

We set

$$C = c_1 + c_2 + c_3 + \dots$$

where

$$c_1 = a_1 b_1$$

But

$$B_m \leq B - B_m \quad m = 1, 2, \dots$$

Hence

$$\begin{aligned} C_n &= a_1(B - B_n) + a_2(B - B_{n-1}) + \dots + a_n(B - B_1) \\ &= B(a_1 + \dots + a_n) - (a_1B_n + \dots + a_nB_1) \\ &= A_nB - d_n, \end{aligned}$$

where

$$d_n = a_1B_n + a_2B_{n-1} + \dots + a_nB_1.$$

The theorem is proved when we show $d_n \doteq 0$. To this end we consider the two sets of remainders

$$\begin{aligned} B_1, \quad B_2, \quad \dots, B_{n_1} \\ B_{n_1+1}, \quad B_{n_1+2}, \quad \dots, B_{n_1+n_2} \end{aligned} \quad n_1 + n_2 = n.$$

Let * each one in the first set be $\leq M_1$, and each in the second set $\leq M_2$. Then since

$$\begin{aligned} d_n &= (a_1B_n + \dots + a_{n_1}B_{n_1+1}) + (a_{n_1+1}B_{n_1} + \dots + a_n\bar{B}_1) \\ |d_n| &\leq M_2(a_1 + \dots + a_{n_1}) + M_1(a_{n_1+1} + \dots + a_n) \\ &\leq M_2A_{n_1} + M_1A_{n_2} \leq M_2A + M_1A_{n_2}. \end{aligned}$$

Now for each $\epsilon > 0$ there exists an n_1 such that

$$M_2 < \frac{\epsilon}{2A},$$

also a ν , such that

$$\bar{A}_{n_2} < \frac{\epsilon}{2M_1} \quad n_2 > \nu.$$

Thus 1) shows that $|d_n| < \epsilon$.

2. We now show that $a_n \rightarrow B$ uniformly absolutely, the

The series A being alternating is convergent, its adjoint is divergent by 81, 2, since here $\mu = \frac{1}{2}$.

$$C = \frac{1}{\sqrt{1}} \frac{1}{\sqrt{1}} - \left(\frac{1}{\sqrt{1}\sqrt{2}} + \frac{1}{\sqrt{2}\sqrt{1}} \right) + \left(\frac{1}{\sqrt{1}} \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \frac{1}{\sqrt{1}} \right) - \dots$$

$$= c_2 + c_3 + c_4 + \dots$$

and

$$|c_n| = \frac{1}{\sqrt{1}} \frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{2}} \frac{1}{\sqrt{n-2}} + \dots + \frac{1}{\sqrt{n-1}} \frac{1}{\sqrt{1}}$$

By I, 95,

$$\sqrt{m(n-m)} \leq \frac{n}{2}.$$

Hence

$$\frac{1}{\sqrt{m(n-m)}} \geq \frac{2}{n}, \quad |c_n| > \frac{2(n-m)}{n}$$

Hence C is divergent since c_n does not $\rightarrow 0$, were convergent, by 80, 3.

4. In order to have the theorems on multiplication we state here one which we shall prove later.

If all three series A , B , C are convergent, then

113. We have seen, 109, 1, that we may group a convergent series $A = a_1 + a_2 + \dots$ into a series each term b_n containing but a finite number of terms of A . It is easy to arrange the terms of A into a finite or infinite number of infinite series, B' , B'' , $B''' \dots$ For ex

$$B' = a_1 + a_{p+1} + a_{2p+1} + \dots$$

^(m) all terms whose index is the product of m primes. We now show what is the relation between the original series A and the series $B', B'' \dots$

If $A = a_1 + a_2 + \dots$ is absolutely convergent, we may break it up into a finite or infinite number of series B', B'', B''', \dots . Each of these series converges absolutely and

$$A = B' + B'' + B''' + \dots$$

That each $B^{(m)}$ converges absolutely was shown in 80, 6. I now suppose first that there is only a finite number of these series, say p of them. Then

$$A_n = B'_{n_1} + B''_{n_2} + \dots + B^{(p)}_{n_p} \quad n = n_1 + \dots + n_p.$$

As $n \doteq \infty$, each $n_1, n_2 \dots \doteq \infty$. Hence passing to the limit $n = \infty$, the above relation gives

$$A = B' + B'' + \dots + B^{(p)}.$$

Suppose now there are an infinite number of series $B^{(m)}$.

Set

$$B = B' + B'' + B''' + \dots$$

We take ν so large that $A - B_n$, $n > \nu$, contains no term a_n whose index $\leq m$, and m so large that

$$\bar{A}_m < \epsilon.$$

Then

$$|A - B_n| \leq \bar{A}_m < \epsilon. \quad n > \nu.$$

Two-way Series

114. 1. Up to the present the terms of our infinite series have been extended to infinity only one way. It is, however, convenient

Such series we called *two-way series*.
if

$$\lim_{r, s \rightarrow \infty} \sum_{n=r}^{n=s} a_n$$

is finite. If the limit 2) does not exist, the extension of the other terms employed in the present case are too obvious to need any explanation ($n = 0$ is excluded in 1); the fact may be thus $\sum_{n=-\infty}^{\infty} a_n$.

2. Let m be an integer; then while n ranges over the set

$$\dots, -3, -2, -1, 0, 1, 2, \dots$$

$n + m$ will range over the same set with the same order; $n + m$ will be m units ahead or behind n according to the sign of m . This shows that

$$\sum_{n=-\infty}^{\infty} a_n = \sum_{n=-\infty}^{\infty} a_{n+m}.$$

Similarly,

$$\sum_{n=-\infty}^{\infty} a_n = \sum_{n=-\infty}^{\infty} a_{-n}.$$

3. *Example 1.* $(*) = \sum_{n=-\infty}^{\infty} e^{nx+an^2}$

$$= 1 + e^{x+a} + e^{2x+4a} + \dots + e^{-x+a} + e^{-2x+4a} + \dots$$

This series is fundamental in the elliptic function theory.

Example 2. $1 + \sum_{n=1}^{\infty} \frac{1}{n} (1 - (-1)^n)$

TWO-WAY SERIES

115. For a two-way series A to converge, it is necessary that the series B formed with the terms with negative and the series C formed with the terms with non-negative indices be convergent. If A is convergent, $A = B + C$.

It is necessary. For A being convergent,

$$|A - B_r - C_s| < \epsilon/2 \quad , \quad |A - B_r - C_{s'}| < \epsilon/2$$

if $s, s' > \text{some } \sigma$ and $r > \text{some } \rho$. Hence adding,

$$|C_s - C_{s'}| < \epsilon,$$

which shows C is convergent. Similarly we may show B is convergent.

It is sufficient. For B, C being convergent,

$$|B - B_r| < \epsilon/2 \quad , \quad |C - C_s| < \epsilon/2$$

for $r, s > \text{some } p$. Hence

$$|B + C - (B_r + C_s)| < \epsilon,$$

or

$$|B + C - \sum_{n=-r}^{n=s} a_n| < \epsilon.$$

Thus

$$\lim_{n=-r}^{n=s} \sum a_n = B + C.$$

Example 1. The series

$$\frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{x+n} - \frac{1}{n} \right)$$

is absolutely convergent if $x \neq 0, \pm 1, \pm 2, \dots$

For

$$|a_n| = \left| \frac{1}{x+n} - \frac{1}{n} \right| = \frac{|x|}{|n^2 + nx|}.$$

Here

$$n > 0, \quad \sum_n a_n = e^x e^{an} = \infty$$

$$n = -n', \quad n' > 0 \quad \sum_{n'} a_n = e^{-x} e^{an'} = \infty$$

The case $a = 0$ is obvious.

Thus the series defines a one-valued function. As an exercise in manipulation let us prove

1° $\Theta(x)$ is an even function.

For

$$\Theta(-x) = \sum_n e^{-nx + a n^2}$$

If we compare this series with 2) we see that corresponding to $n = m$ and $n = -m$ have similar terms. The reader will see if he actually writes out the series. Cf. 114, 2.

$$2^\circ \quad \Theta(x + 2ma) = e^{-m(x+ma)} \Theta(x), \quad m = 1, 2, \dots$$

For we can write 2) in the form

$$\Theta(x) = e^{\frac{x^2}{4a}} \sum_n e^{-\frac{(x+2na)^2}{4a}}$$

Thus

$$\begin{aligned} \Theta(x + 2ma) &= e^{\frac{(x+2ma)^2}{4a}} \sum_n e^{-\frac{(x+2na)^2}{4a}} \\ &= e^{-m(x+ma)} \cdot e^{\frac{x^2}{4a}} \sum_n e^{-\frac{(x+2na)^2}{4a}} \end{aligned}$$

which with 4) gives 3).

CHAPTER IV

MULTIPLE SERIES

116. Let $x = x_1, \dots, x_m$ be a point in m -way space \mathfrak{R}_m . If the coördinates of x are all integers or zero, x is called a *lattice* point, and any set of lattice points a *lattice system*. If no coördinate of any point in a lattice system is negative, we call it a *non-negative* lattice system, etc. Let $f(x_1 \dots x_m)$ be defined over a lattice system $\iota = \iota_1, \dots, \iota_m$. The set $\{f(\iota_1 \dots \iota_m)\}$ is called an *m -way sequence*. It is customary to set

$$f(\iota_1 \dots \iota_m) = a_{\iota_1 \dots \iota_m}.$$

Then the sequence is represented by

$$A = \{a_{\iota_1 \dots \iota_m}\}.$$

The terms $\lim A = 0$, $\lim A = \infty$, $\lim A = A$

as $\iota_1 \dots \iota_m$ converges to an ideal point have therefore been defined, and some of their elementary properties given in the discussion of I, 314-328; 336-338.

Let $x = x_1 \dots x_m$, $y = y_1 \dots y_m$ be two points in \mathfrak{R}_m . If $y_1 \geq x_1, \dots, y_m \geq x_m$ we shall write more shortly $y \geq x$. If x ranges over a set of points $x' > x'' > x''' \dots$ we shall say that x ranges over a set of points $x' > x'' > x''' \dots$.

117. A very important class of multiple sequences with multiple series as we now show. Let a_1, \dots be a non-negative lattice system. The symbol

$$\sum a_{\iota_1 \dots \iota_m} \quad \iota_1 = 0, 1, \dots, \nu_1 \quad , \quad \dots \quad \iota_m =$$

$$\text{or } \sum_0^{\nu_1 \dots \nu_m} a_{\nu_1 \dots \nu_m}, \quad \text{or } A_{\nu_1 \dots \nu_m}$$

denotes the sum of all the a 's whose lattice point
 belongs to the m -th rectangular cell $0 \leq x_1 \leq \nu_1 \quad \dots \quad 0 \leq x_m \leq \nu_m$

Let us denote this cell by $R_{\nu_1 \dots \nu_m}$ or by R_ν . This can be effected in a variety of ways. To fix the ideas

$$A_{\nu_1\nu_2\nu_3} = \sum_0^{\nu_1} \sum_0^{\nu_2\nu_3} a_{i_1i_2i_3} = \sum_0^{\nu_2\nu_3} \sum_0^{\nu_1} a_{i_1i_2i_3} = \sum_0^{\nu_1} \sum_0^{\nu_2} \sum_0^{\nu_3} a_{i_1i_2i_3}$$

etc. In the first sum, we sum up the terms then add these results. In the second sum, we use parallel lines and then add the results. In the third sum, we sum the terms on the parallel lines lying in a given plane, then add the results; we then sum over the different planes.

Returning now to the general case, the symb

$$A = \sum a_{\iota_1, \dots, \iota_m} \quad \iota_1, \dots, \iota_m = 0, 1, \dots$$

or $A = \sum_0^{\infty} a_{i_1 \dots i_m}$.

is called an m -tuple infinite series. For $m =$
out more fully thus

$$\begin{aligned} & a_{00} + a_{01} + a_{02} + \cdots \\ & + a_{10} + a_{11} + a_{12} + \cdots \\ & + a_{20} + a_{21} + a_{22} + \cdots \end{aligned}$$

is finite, A is *convergent* and the limit 2) is called the sum of the series A . When no confusion will arise, we may denote the series and its sum by the same letter. If the limit 2) is infinite or does not exist, we say A is *divergent*.

Thus every m -tuple series gives rise to an m -tuple sequence $\{A_{v_1 \dots v_m}\}$. Obviously if all the terms of A are ≥ 0 and A is divergent, the limit 2) is $+\infty$. In this case we say A is *infinite*.

Let us replace certain terms of A by zeros, the resulting series may be called the *deleted series*. If we delete A by replacing the terms of the cell $R_{r_1 \dots r_m}$ by zero, the resulting series is called the *remainder* and is denoted by $A_{r_1 \dots r_m}$ or by \bar{A}_r . Similarly, if the cell R_ν contains the cell R_μ , the terms lying in R_ν and not in R_μ may be denoted by $A_{\mu, \nu}$.

The series obtained from A by replacing each term of A by a numerical value is called the *adjoint series*. In a similar manner, most of the terms employed for simple series may be carried over to m -tuple series. In the series $\sum a_{i_1 \dots i_m}$ the indices i all begin with 0. There is no necessity for this; they may each begin with any integer at pleasure.

118. The Geometric Series. We have seen that

$$\frac{1}{1-a} = 1 + a + a^2 + \dots \quad |a| < 1,$$

$$\frac{1}{1-b} = 1 + b + b^2 + \dots \quad |b| < 1.$$

Hence

$$\frac{1}{(1-a)(1-b)} = \sum_0^{\infty} a^m b^n$$

119. 1. It is important to show how any t can be expressed by means of the $A_{v_1 \dots v_m}$.

$$\text{Let } D_{v_1 v_2 \dots v_{m-1}} = A_{v_1 v_2 \dots v_m} - A_{v_1 v_2 \dots v_{m-1} v_m}$$

$$\text{Then } D_{v_1 v_2 \dots v_{m-1-1}} = A_{v_1 v_2 \dots v_{m-1-1} v_m} - A_{v_1 v_2 \dots v_{m-1-1} v_{m-1} v_m}$$

$$\text{Let } D_{v_1 v_2 \dots v_{m-2}} = D_{v_1 v_2 \dots v_{m-1}} - D_{v_1 v_2 \dots v_{m-1} v_m}$$

$$\text{Similarly } D_{v_1 v_2 \dots v_{m-3}} = D_{v_1 v_2 \dots v_{m-2}} - D_{v_1 v_2 \dots v_{m-2} v_m}$$

$$D_{v_1 v_2 \dots v_{m-4}} = D_{v_1 v_2 \dots v_{m-3}} - D_{v_1 v_2 \dots v_{m-3} v_m}$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$\text{Finally } D_{v_1} = D_{v_1 v_2} - D_{v_1 v_2 v_1}$$

$$\text{and } a_{v_1 v_2 \dots v_m} = D_{v_1} - D_{v_1-1}$$

If now we replace the D 's by their values the relation 7) shows that $a_{v_1 \dots v_m}$ may be expressed in terms of a number of $A_{\mu_1 \dots \mu_m}$ where each μ_r is a multiple of v_r .

For $m = 2$ we find

$$a_{v_1 v_2} = A_{v_1 v_2} + A_{v_1-1, v_2-1} - A_{v_1, v_2-1}$$

2. From 1 it follows that we may take $a_{v_1 \dots v_m}$ to form a multiple series

$$A = \sum a_{v_1 \dots v_m}$$

This fact has theoretic importance in studying multiple series present.

120. We have now the following theorem

GENERAL THEORY

3. For A to converge it is necessary and sufficient that

$$\lim_{v \rightarrow \infty} A_v = 0.$$

4. A series whose adjoint converges is convergent.

5. Let A be absolutely convergent. Any deleted series B of absolutely convergent and $|B| < \Lambda$.

6. If $A = \sum a_{i_1 \dots i_m}$ is convergent, so is $B = \sum k a_{i_1 \dots i_m}$ and

$$B = kA, \quad k \text{ a constant.}$$

121. 1. For A to converge it is necessary that

$$D_{v_1 v_2 \dots v_{m-1}} \rightarrow 0, \quad D_{v_1 v_2 \dots v_{m-2}} \rightarrow 0, \quad \dots \rightarrow D_{v_1} \rightarrow 0, \quad a_{v_1 v_2 \dots v_m} \doteq 0, \text{ as } v \doteq \infty$$

For by 120, 1)

$$|A_{\lambda_1 \dots \lambda_m} - A_{\mu_1 \dots \mu_m}| < \epsilon$$

if $\lambda_1 \dots \lambda_m, \mu_1 \dots \mu_m > p$.

Thus by 119, 1)

$$|D_{v_1 v_2 \dots v_{m-1}}| < \epsilon \quad v > p.$$

Hence passing to the limit $p = \infty$,

$$\lim_{v \rightarrow \infty} D_{v_1 \dots v_{m-1}} = 0.$$

As ϵ is small at pleasure, this shows that $D_{v_1 \dots v_{m-1}} \doteq 0$. In the same way we may continue.

2. Although

$$\lim_{v_1 \dots v_m \rightarrow \infty} a_{v_1 \dots v_m} = 0$$

when A converges, we must guard against the error of supposing that $a_{v_1 \dots v_m} \doteq 0$ as $v_1 \dots v_m \rightarrow \infty$.

GENERAL THEORY

But for each α_n there exists a $\gamma_{\alpha_n} \geq \alpha_n$;

hence

$$f(\gamma_{\alpha_n}) \geq f(\alpha_n)$$

and therefore

$$l \geq l.$$

Similarly, for each δ_n there exists an $\alpha_{\delta_n} > \delta_n$;

hence

$$f(\delta_n) < f(\alpha_{\delta_n})$$

and therefore

$$l < l.$$

Thus 2), 3) give

$$\lim_n f(x) = l.$$

Hence by I, 316, 2 the relation 1) holds.

The rest of the theorem follows along the same lines.

2. As a corollary we have

The positive term series $A = \sum a_{i_1, \dots, i_m}$ is convergent if A_{v_1} is limited.

123. 1. Let $A = \sum a_{i_1, \dots, i_n} = \sum a_i$, $B = \sum b_{i_1, \dots, i_n} = \sum b_i$ be two negative term series. If they differ only by a finite number of terms, they converge or diverge simultaneously.

This follows at once from 120, 2.

2. Let A, B be two non-negative term series. Let $r > 0$ be a constant. If $a_i \leq r b_i$, A converges if B is convergent and $A \leq r B$. If $a_i \geq r b_i$, A diverges if B is divergent.

For on the first hypothesis

$$A_\lambda \leq r B_\lambda,$$

and on the second

$$A_\lambda > r B_\lambda.$$

3. Let A, B be two positive term series. Let r, s be positive constants.

4. *The infinite non-negative term series*

$$\sum a_{i_1 \dots i_s} \quad \text{and} \quad \sum \log(1 + a_{i_1 \dots i_s})$$

converge or diverge simultaneously.

This follows from 2.

5. *Let the power series*

$$P = \sum c_{m_1 m_2 \dots m_s} x_1^{m_1} x_2^{m_2} \dots x_s^{m_s}$$

converge at the point $a = (a_1, \dots, a_s)$, then it converges at all points x within the rectangular cell R whose center is a and one of whose vertices is a ; that is for $|x_i| < |a_i|$.

For since P converges at a ,

$$\lim_{m=\infty} c_{m_1 m_2 \dots m_s} a_1^{m_1} \dots a_s^{m_s} = 0.$$

Thus there exists an M such that each term

$$|c_{m_1 \dots m_s} a_1^{m_1} \dots a_s^{m_s}| \leq M.$$

Hence

$$\begin{aligned} |c_{m_1 \dots m_s} x_1^{m_1} \dots x_s^{m_s}| &= |c_{m_1 \dots m_s} a_1^{m_1} \dots a_s^{m_s}| \cdot \left| \frac{x_1}{a_1} \right|^{m_1} \dots \left| \frac{x_s}{a_s} \right|^{m_s} \\ &\leq M \left| \frac{x_1}{a_1} \right|^{m_1} \dots \left| \frac{x_s}{a_s} \right|^{m_s}. \end{aligned}$$

Thus each term of P is numerically \leq than the corresponding term in the convergent geometric series

$$\sum \left| \frac{x_1}{a_1} \right|^{m_1} \dots \left| \frac{x_s}{a_s} \right|^{m_s}.$$

We apply now 2.

We shall call R a *rectangular cell of convergence*.

124. 1. Associated with any m -tuple series $\sum a_{i_1 \dots i_m}$ is an infinite number of simple series called associated

the terms of A arranged in order lying in $R_{\lambda_2} - R_{\lambda_1}$, and so on indefinitely.

Then
$$\mathfrak{A} = a_1 + a_2 + \cdots + a_{s_1} + a_{s_1+1} + \cdots$$

an associate simple series of A .

2. Conversely associated with any simple series $\mathfrak{A} = \sum a_n$ are a family of *associate m -tuple series*. In fact we have only to arrange the terms of \mathfrak{A} over the non-negative lattice points, and call the term a_n which lies at the lattice point $\iota_1 \cdots \iota_m$ the term $a_{\iota_1 \dots \iota_m}$.

3. Let \mathfrak{A} be an associate series of $A = \sum a_{\iota_1 \dots \iota_m}$. If \mathfrak{A} is convergent to A and

$$A = \mathfrak{A}.$$

For
$$A_{\nu_1 \dots \nu_m} = \mathfrak{A}_n.$$

Let now $\nu \doteq \infty$, then $n \doteq \infty$. But $\mathfrak{A}_n \doteq \mathfrak{A}$, hence $A_{\nu_1 \dots \nu_m} \doteq \mathfrak{A}$.

4. If the associate series \mathfrak{A} is absolutely convergent, so is A .

Follows from 3.

5. If $A = \sum a_{\nu_1 \dots \nu_m}$ is a non-negative term convergent series, all its associate series \mathfrak{A} converge.

For, any $\mathfrak{A}_{m,p}$ lies among the terms of some $A_{\mu,\nu}$. But for sufficiently large

$$A_{\mu,\nu} < \epsilon \quad \lambda < \mu < \nu.$$

Hence

$$\mathfrak{A}_{m,p} < \epsilon \quad m > m_0.$$

6. Absolutely convergent series are commutative.

For let B be the series resulting from rearranging the terms of the series A .

Then any associate \mathfrak{B} of B is simply a rearrangement of the associate series \mathfrak{A} of A . But $\mathfrak{A} = \mathfrak{B}$, hence $A = B$.

7. A simply convergent m -tuple series A can be rearranged

Then either $\mathfrak{B}'_n \doteq +\infty$ or $\mathfrak{B}''_n \doteq -\infty$, or both suppose the former. Then we can arrange form a series \mathfrak{C} such that $\mathfrak{C}_n \doteq +\infty$. Let no series of C . Then

$$C_\nu = C_{\nu_1 \nu_2 \dots \nu_m} = \mathfrak{C}_n$$

and thus

$$\lim C_\nu = \lim \mathfrak{C}_n = +\infty.$$

Hence C is divergent.

8. *If the multiple series A is commutative divergent.*

For if simply convergent, we can rearrange resulting series divergent, which contradicts

9. In 121, 2 we exhibited a convergent $a_{i_1 \dots i_m}$ does not need to converge to 0 if $i_1 \dots i_m$ point some of whose coördinates are finite. have the following :

Let A be absolutely convergent. Then for ϵ a λ , such that any finite set of terms B lying a relation

$$|B| < \epsilon;$$

and conversely.

For let \mathfrak{A} be an associate simple series of convergent there exists an n , such that

$$\overline{\mathfrak{A}}_n < \epsilon.$$

But if λ is taken sufficiently large, each which proves 1).

Suppose now A were simply convergent. there exists an associate series \mathfrak{D} which is inf

extended over a lattice system \mathfrak{M} in \mathfrak{R}_m is a simple series in \mathfrak{R}_m . We can generalize as follows. Let $\mathfrak{M} = \{i\}$ be associated with a lattice system $\mathfrak{M} = \{j\}$ in \mathfrak{R}_n such that to each i corresponds a j , conversely.

If $i \sim j$ we set

$$a_{i_1 \dots i_m} = a_{j_1 \dots j_n}.$$

Then A gives rise to an infinity of n -tuple series as

$$B = \sum a_{j_1 \dots j_n}.$$

We say B is a *conjugate n -tuple series*.

We have now the following:

Let A be absolutely convergent. Then the series B is absolutely convergent and $A = B$.

For let A', B' be associate simple series of A, B . Then A' and B' are absolutely convergent and hence $A' = B'$. But $A = A', B = B'$. Hence $A = B$, and B is absolutely convergent.

11. Let $A = \sum a_{i_1 \dots i_m}$ be absolutely convergent. Let $B = \sum a_{j_1 \dots j_n}$ be any p -tuple series formed of a part or all the terms of A . B is absolutely convergent and

$$|B| \leq \text{Adj } A.$$

For let A', B' be associate simple series of A and B . Then A' converges absolutely and $|B'| \leq \text{Adj } A$.

125. 1. Let

$$A = \sum a_{i_1 \dots i_m}.$$

Set

$$f(x_1 \dots x_m) = a_{i_1 \dots i_m}$$

in the cell

$$i_1 - 1 < x_1 \leq i_1, \quad \dots \quad i_m - 1 < x_m \leq i_m.$$

by employing a sequence of rectangular cells $a_v \geq 0$ we may, and we have

For the non-negative term series 1) to converge it is sufficient that the integral 3) converges.

2. Let $f(x_1 \cdots x_m) \geq 0$ be a monotone decreasing function of x in R , the aggregate of points all of whose coordinates are non-negative. Let

$$a_{i_1 \dots i_m} = f(i_1 \cdots i_m).$$

The series

$$A = \sum a_{i_1 \dots i_m}$$

is convergent or divergent with

$$J = \int_R f dx_1 \cdots dx_m.$$

For let R_1, R_2, \dots be a sequence of rectangular cells contained in R_{n+1} .

Let

$$R_{n,s} = R_s - R_n \quad s > n.$$

Then λ, μ being taken at pleasure but $>$ some l, m such that

$$A_{\lambda\mu} < \int_{R_{lm}} f_{lm}.$$

But the integral on the right can be made small by taking $l > m >$ some n . Hence A is convergent if J is. Similarly the other half of the theorem is proved.

Iterated Summation of Multiple Series

126. Consider the finite sum

$$\sum a_{i_1 \dots i_m} \quad i_1 = 0, 1, \dots, n_1 \quad \dots \quad i_m = 0, 1, \dots, n_m$$

and so on arriving finally at

$$\sum_{t_m=0}^{m_m} \cdots \sum_{t_1=0}^{m_1} a_{t_1, \dots, t_m}$$

whose value is that of 1). We call this process *iterated summation*. We could have taken the indices $t_1 \cdots t_m$ in any order instead of the one just employed; in each case we would have arrived at the same result, due to the commutative property of finite sums.

Let us see how this applies to the infinite series,

$$A = \sum a_{t_1, \dots, t_m}, \quad t_1 \cdots t_m = 0, 1, \dots \infty.$$

The corresponding process of iterated summation would be applied to a series

$$\mathfrak{A} = \sum_{t_m=0}^{\nu_m} \sum_{t_{m-1}=0}^{\nu_{m-1}} \cdots \sum_{t_1=0}^{\nu_1} a_{t_1, \dots, t_m},$$

which is an *m-tuple iterated series*. Now by definition

$$\begin{aligned} \mathfrak{A} &= \lim_{\nu_m \rightarrow \infty} \sum_{t_m=0}^{\nu_m} \lim_{\nu_{m-1} \rightarrow \infty} \sum_{t_{m-1}=0}^{\nu_{m-1}} \cdots \lim_{\nu_1 \rightarrow \infty} \sum_{t_1=0}^{\nu_1} a_{t_1, \dots, t_m} \\ &= \lim_{\nu_m \rightarrow \infty} \lim_{\nu_{m-1} \rightarrow \infty} \cdots \lim_{\nu_1 \rightarrow \infty} A_{\nu_1, \dots, \nu_m}, \end{aligned}$$

while

$$A = \lim_{t_1, \dots, t_m} A_{t_1, \dots, t_m}.$$

Thus A is defined by a general limit while \mathfrak{A} is defined by an iterated limit. These two limits may be quite different. In 6) we have passed to the limit in a certain order. (If we changed this order in 6) we would give us another iterated series of the same type 4) with a sum which may be quite different. However it is not difficult to see that the iterated limit is independent of the order of summation.)

Let

$$A = a_{00} + a_{01} + a_{02} + \dots + a_{10} + a_{11} + a_{12} + \dots$$

be a double series. The m^{th} row forms a series

$$R^{(m)} = a_{m,0} + a_{m,1} + \dots = \sum_{n=0}^{\infty} a_{mn}$$

and the n^{th} column, the series

$$C^{(n)} = a_{0n} + a_{1n} + \dots = \sum_{m=0}^{\infty} a_{mn}.$$

Then

$$R = \sum_{m=0}^{\infty} R^{(m)} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn},$$

$$C = \sum_{n=0}^{\infty} C^{(n)} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn}$$

are the series formed by *summing by rows* and *by columns* respectively.

2. *A double series may converge although every row and every column is divergent.*

This is illustrated by the series considered in § 1. The series $\sum_{r=0}^{\infty} a_{rs}$ is convergent while $\sum_{r=0}^{\infty} a_{rs}$, $\sum_{s=0}^{\infty} a_{rs}$ are divergent, since a_{rs} does not evanescent.

3. *A double series A may be divergent although A is convergent when summed by rows or the series C obtained by summing A by columns is convergent.*

For let

$$A_{rs} = 0 \quad \text{if } r \text{ or } s = 0$$

$$= \frac{r}{r+s} \quad \text{if } r, s > 0.$$

4. In the last example R and C converged but their sums were different. We now show :

A double series may diverge although both R and C converge and have the same sum.

$$\text{For let} \quad A_{r,s} = 0 \quad \text{if } r \text{ or } s = 0 \\ = \frac{rs}{r^2 + s^2} \quad \text{if } r, s > 0.$$

Then by I, 319, $\lim A_{rs}$ does not exist and A is divergent. On the other hand,

$$R = \lim_{r \rightarrow \infty} \lim_{s \rightarrow \infty} A_{rs} = 0,$$

$$C = \lim_{s \rightarrow \infty} \lim_{r \rightarrow \infty} A_{rs} = 0.$$

Then R and S both converge and have the same sum.

28. We consider now some of the cases in which iterated summation is permissible.

Let $A = \sum_0^{\infty} a_{i_1, \dots, i_m}$ be convergent. Let i'_1, i'_2, \dots, i'_m be any permutation of the indices i_1, i_2, \dots, i_m . If all the $m-1$ -tuple series

$$\sum_{i'_2=0}^{\infty} \sum_{i'_3=0}^{\infty} \dots \sum_{i'_m=0}^{\infty} a_{i_1, \dots, i_m}$$

$$\text{are convergent,} \quad A = \sum_{i'_1=0}^{\infty} \dots \sum_{i'_m=0}^{\infty} a_{i_1, \dots, i_m}.$$

This follows at once from I, 324. For simplicity the theorem is there stated only for two variables; but obviously the demonstration applies to any number of variables.

For if f is limited, $\lim_{x_{i_s}=a_{i_s}} f$, $x_{i_s} < a_{i_s}$

exists by 122, 1. Moreover 3) is a monotone function of the remaining $m-1$ variables.

Hence similarly $\lim_{x_{i_{s-1}}=a_{i_{s-1}}} \lim_{x_{i_s}=a_{i_s}} f$

exists and is a monotone function of the remaining variables, etc. The rest of the theorem follows as in 122, 1.

2. As a corollary we have

Let A be a non-negative term m -tuple series. If its m -tuple iterated series is convergent, A and all its m -tuple series are convergent and have the same sum. If its m -tuple iterated series is divergent, they all are.

3. *Let a be a non-negative term m -tuple series. If the s -tuple iterated series of A are convergent if A is convergent and if these iterated series is divergent, so is A .*

130. 1. *Let $A = \sum a_{i_1 \dots i_m}$ be absolutely convergent. If the s -tuple iterated series $s=1, 2 \dots m$, converge to A , then all m -tuple iterated series all $= A$.*

For as usual let $\alpha_{i_1 \dots i_m} = |a_{i_1 \dots i_m}|$. Since A is convergent, all the s -tuple iterated series of A are convergent. Thus $s_1 = \sum_{i_1=0}^{\infty} \alpha_{i_1 \dots i_m}$ is convergent since $\sum_{i_1=0}^{\infty} \alpha_{i_1 \dots i_m}$

$|s_1| < \sigma_1$. Similarly $\sum_{i_2=0}^{\infty} \sum_{i_1=0}^{\infty} \alpha_{i_1 \dots i_m} = \sum_{i_2} s_1$ is convergent. $\sum_{i_2=0}^{\infty} \sum_{i_1=0}^{\infty} \alpha_{i_1 \dots i_m} = \sum_{i_2} \sigma_1$ is convergent; etc. Thus e

Consider the series

$$\begin{aligned} A &= 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \dots \\ &+ 1 + 2a + \frac{(2a)^2}{2!} + \frac{(2a)^3}{3!} + \dots \\ &+ 1 + 3a + \frac{(3a)^2}{2!} + \frac{(3a)^3}{3!} + \dots \\ &+ \dots \end{aligned}$$

Here

$$R^{(n)} = 1 + \frac{na}{1!} + \frac{(na)^2}{2!} + \dots = e^{na}$$

1

$$R = e^a + e^{2a} + e^{3a} + \dots$$

This is a geometric series and converges *absolutely* for $a < \infty$. Thus one of the double iterated series of A is absolutely convergent. We cannot, however, infer from this that A is convergent. The theorem of 2 requires that one of the iterated series form the *adjoint* of A should converge. Now both those series are divergent. The series A is divergent, for $|a_{rs}| \doteq \infty$, $s \doteq \infty$.

31. 1. Up to the present the series

$$\sum a_{i_1 \dots i_m}$$

have been extended only over non-negative lattice points. This restriction was imposed only for convenience; we show now how it may be removed. Consider the signs of the coördinates of a point $x = (x_1, \dots, x_m)$. Since each coördinate can have two signs, there are 2^m combinations of signs. The set of points x whose coördinates belong to a given one of these combinations forms a *lattice* for $m = 2$, an *octant* for $m = 3$, and a *2^m -tant* or *poly-*

If
$$\lim_{\lambda \rightarrow \infty} A_\lambda$$

exists, we say A is *convergent*, otherwise A is *divergent*. In a similar manner the other terms employed in the preceding section may be extended to the present case. The rectangular figures in the above definition may without loss of generality be replaced by the cube

$$|x_1| \leq \lambda_0 \quad \cdots \quad |x_m| \leq \lambda_0$$

Moreover the condition necessary and sufficient for the existence of the limit 3) is that

$$|A_\lambda - A_\mu| < \epsilon \quad \lambda, \mu \geq \lambda_0$$

132. The properties of series lying in the preceding section may be readily extended to series lying in several variables. For the convenience of the reader we bring the most important of them, omitting the proof when it follows along the same lines.

1. *For A to converge it is necessary and sufficient that*

$$\lim_{\lambda \rightarrow \infty} \bar{A}_\lambda = 0.$$

2. *A series whose adjoint converges is convergent.*

3. *Any deleted series B of an absolutely convergent series A is absolutely convergent and*

$$|B| < \text{Adj } A.$$

4. *If $A = \sum a_{i_1 \dots i_n}$ is convergent, so is $B = \sum a_{i_1 \dots i_n}$ for any n .*

5. *The non-negative term series A is convergent if and only if $\lambda \doteq \infty$.*

Then $A = \sum a_{i_1, \dots, i_m}$
converges or diverges with
$$\int_{\mathfrak{R}} f dx_1 \dots dx_m,$$

the integration extended over all space containing terms of A .

133. 1. Let $B, C, D \dots$ denote the series formed of the terms lying in the different polyants. For A to converge it is sufficient although not necessary that B, C, \dots converge. When they do

$$A = B + C + D + \dots$$

For if $B_\lambda, C_\lambda \dots$ denote the terms of $B, C \dots$ which lie in the rectangular cell R_λ ,

$$A_\lambda = B_\lambda + C_\lambda + \dots$$

Passing to the limit we get 1).

That A may converge when B, C, \dots do not is shown by the following example. Let all the terms of $A = \sum a_{i_1, \dots, i_m}$ vanish except those lying next to the coördinate axes. Let these have value $+1$ if $i_1, i_2 \dots i_m > 0$ and let two a 's lying on opposite sides of the coördinate planes have the same numerical value but opposite signs. Obviously, $A_\lambda = 0$, hence A is convergent. On the other hand, every $B, C \dots$ is divergent.

2. Thus when $B, C \dots$ converge, the study of the given series A may be referred to series whose terms lie in a single polyant. But obviously the theory of such series is identical with that of the series lying in the first polyant.

3. The preceding property enables us at once to extend the theorems of 129, 130 to series lying in more than one polyant. The iterated series will now be made up, in general of two or more simple series.

CHAPTER V

SERIES OF FUNCTIONS

134. 1. Let $\iota = (\iota_1, \iota_2 \cdots \iota_p)$ run over an infinite set \mathfrak{A} . Let the one-valued functions

$$f_{\iota_1 \dots \iota_p}(x_1 \cdots x_m) = f_{\iota}(x) = f_{\iota}$$

be defined over a domain \mathfrak{A} , finite or infinite.

$$F = F(x) = F(x_1 \cdots x_m) = \sum f_{\iota_1 \dots \iota_p}(x)$$

extended over the lattice system \mathfrak{A} is convergent. The one-valued function $F(x_1 \cdots x_m)$ over \mathfrak{A} . We prove the properties of this function with reference to differentiation and integration.

2. Here, as in so many parts of the theory of series, the result is independent of the order of summation, depending on changing the order of an iterated limit, is fundamental.

We shall therefore take this opportunity to prove the properties in an entirely general manner so that they apply not only to infinite series, but to infinite products, integrals, etc.

3. In accordance with the definition of I, § 3, 1) is *uniformly convergent* in \mathfrak{A} when F_{μ} converges to the limit F . Or in other words when for each ϵ

GENERAL THEORY

135. 1. Let

$$\lim_{t=\tau} f(x_1 \cdots x_m, t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$$

in \mathfrak{A} . Here \mathfrak{A} , τ may be finite or infinite. If there exists $\eta > 0$ such that $f \doteq \phi$ uniformly in $V_\eta(a)$, a finite or infinite, we shall say f converges uniformly at a ; if there exists no η we say f does not converge uniformly at a .

2. Let now a range over \mathfrak{A} . * Let \mathfrak{B} denote the points of \mathfrak{A} at which no η exists or those points, they may lie in \mathfrak{A} or outside of \mathfrak{A} , whose vicinity the minimum of η is 0. Let D denote a division of space of norm d . Let \mathfrak{B}_D denote as usual the division of D containing points of \mathfrak{B} . Let \mathfrak{C}_D denote the points of \mathfrak{A} in \mathfrak{B}_D . Then $f \doteq \phi$ uniformly in \mathfrak{C}_D however small d is taken, then fixed. The converse is obviously true.

3. If f converges uniformly in \mathfrak{A} , and if moreover it converges uniformly in a finite number of other points \mathfrak{B} , it converges uniformly in $\mathfrak{A} \cup \mathfrak{B}$.

For if $f \doteq \phi$ uniformly in \mathfrak{A} ,

$$|f - \phi| < \epsilon \quad x \text{ in } \mathfrak{A}, \quad t \text{ in } V_{\delta_0}^*(\tau).$$

Then also at each point b_s of \mathfrak{B} ,

$$|f - \phi| < \epsilon \quad x = b_s \quad t \text{ in } V_{\delta_s}^*(\tau).$$

If now $\delta < \delta_0, \delta_1, \delta_2 \cdots$ these relations hold for any x in $\mathfrak{A} \cup \mathfrak{B}$ and any t in $V_\delta^*(\tau)$.

4. Let $f(x_1 \cdots x_m, t_1 \cdots t_n) \doteq \phi(x_1 \cdots x_m)$ uniformly in \mathfrak{A} and f be limited in \mathfrak{A} for each t in $V_\delta^*(\tau)$. Then ϕ is limited in \mathfrak{A} .

For $\phi = f(x, t) + \epsilon' \quad |\epsilon'| < \epsilon$

for any x in \mathfrak{A} and t in $V_\delta^*(\tau)$. Let us therefore fix

2. If the power series $P = a_0 + a_1x + a_2x^2 + \dots$ converges at the end point of its interval of convergence, it converges at this point.

Suppose P converges at the end point $x = R$.

$$|a_{m+1}R^{m+1} + \dots + a_nR^n| < \epsilon$$

however large n is taken. But for $0 < x \leq R$

$$\begin{aligned} |a_{m+1}x^{m+1} + \dots + a_nx^n| \\ = \left| a_{m+1}R^{m+1} \left(\frac{x}{R}\right)^{m+1} + \dots + a_nR^n \left(\frac{x}{R}\right)^n \right| \\ < \epsilon \quad \text{by Abel's identity.} \end{aligned}$$

Thus the convergence is uniform at $x = R$. In a similar manner we may treat $x = -R$.

3. Let $f_n(x_1 \dots x_m)$, $n = 1, 2, \dots$ be defined over \mathfrak{A} . If $|f_n| < \text{some constant } c_n$ in \mathfrak{A} , f_n is limited in \mathfrak{A} . If the c_n are all $< \text{some constant } C$, we say the f_n are uniformly limited in \mathfrak{A} . In general if each function in the set $\{f\}$ defined over at point set \mathfrak{A} satisfy the relation

$$|f| < \text{a fixed constant } C, \text{ } x \text{ in } \mathfrak{A}$$

we say the f 's are uniformly limited in \mathfrak{A} .

The series $F = \sum g_n h_n$ is uniformly convergent in \mathfrak{A} if $\sum g_n$ is uniformly convergent in \mathfrak{A} , while $\sum |h_{n+1} - h_n|$ is uniformly limited in \mathfrak{A} .

This follows at once from Abel's identity as

4. The series $F = \sum g_n h_n$ is uniformly convergent in \mathfrak{A} if $\sum g_n$ is uniformly convergent, h_n is uniformly limited in \mathfrak{A} , and the G_n uniformly limited.

GENERAL THEORY

6. *The series $F = \sum g_n h_n$ is uniformly convergent in \mathfrak{A} if $G_2 = g_1 + g_2 + \dots$ are uniformly limited in \mathfrak{A} and if h_1, h_2, \dots form a monotone decreasing sequence for x in \mathfrak{A} but also formly evanescent.*

For by 83, 1, $|F_{n,p}| \leq |h_{n+1}| G_2$.

Example. Let $A = a_1 + a_2 + \dots$ be convergent. Let b_1, b_2, \dots be a limited monotone sequence. Then

$$F(x) = \sum_{n=1}^{\infty} \frac{a_n}{1 + b_n x}$$

converges uniformly in any interval \mathfrak{A} which does not contain a point of $\left\{ \frac{1}{b_n} \right\}$.

For obviously the numbers

$$h_n = \frac{1}{1 + b_n x}$$

form a monotone sequence at each point of \mathfrak{A} . We now have

7. As an application of these theorems we have, using the results of 84,

The series $a_0 + a_1 \cos x + a_2 \cos 2x + \dots$

converges uniformly in any complete interval not containing the points $1 \pm 2m\pi$ provided $\sum |a_{n+1} - a_n|$ is convergent and hence in particular if $a_1 + a_2 + \dots \neq 0$.

8. *The series* $a_0 + a_1 \cos x + a_2 \cos 2x + \dots$

10. The series $a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \dots$ converges uniformly in any complete interval not containing the points $\pm(2m-1)\pi$ provided $\sum |a_n| < \infty$, $a_n \rightarrow 0$, and hence in particular if $a_1 \geq a_2 \geq a_3 \geq \dots \geq 0$.

138. 1. Let
$$F = \sum f_{i_1, \dots, i_n}(x_1 \dots x_m)$$

be uniformly convergent in \mathfrak{A} . Let A, B be two constants such that

$$Af_i(x) \leq g_i(x) \leq Bf_i(x) \quad \text{in } \mathfrak{A}$$

Then

$$G = \sum g_{i_1, \dots, i_n}(x_1 \dots x_m)$$

is uniformly convergent in \mathfrak{A} .

For then

$$AF_{\lambda, \mu} \leq G_{\lambda, \mu} \leq BF_{\lambda, \mu}.$$

But F being uniformly convergent,

$$|F_{\lambda, \mu}| < \epsilon.$$

2. Let

$$F = \sum f_{i_1, \dots, i_n}(x_1 \dots x_m) \quad f_i \geq 0$$

converge uniformly in \mathfrak{A} . Then

$$L = \sum \log(1 + f_i)$$

is uniformly convergent in \mathfrak{A} . Moreover if F is L .

For $f_i \geq 0$ in \mathfrak{A} , hence

$$|f_i| < e$$

for any i outside some rectangular cell R_λ .

Thus for such i

$$Af_i \leq \log(1 + f_i) \leq Bf_i \quad \text{in } \mathfrak{A}$$

For if $f \doteq \phi$ uniformly in \mathfrak{A} ,

$$\epsilon > 0, \quad \delta > 0 \quad |f - \phi| < \epsilon$$

for any x in \mathfrak{A} and any t in $V_\delta^*(\tau)$, δ independent of x .

But then $|\Delta| < \epsilon \quad t \text{ in } V_\delta^*(\tau).$

2. As a corollary we have :

Let $a_1, a_2, \dots \doteq a$. Let $F = \sum f_n$ be uniformly convergent at a .

$$\overline{F}_n(a_n) \doteq 0.$$

140. *Example 1.*

$$\lim_{u=0} f = \lim_{u=0} \frac{\sin u \sin 2u}{\sin^2 u + x \cos^2 u} = \phi(x) = \begin{cases} 2 & \text{for } x=0, \\ 0 & \text{for } x \neq 0. \end{cases}$$

The convergence is not uniform at $x=0$. For

$$f = \frac{2 \cos u}{1 + x \cot^2 u}.$$

Hence if we set $x = u^2$

$$\lim_{u=0} f = 1, \quad \text{since } u^2 \cot^2 u \doteq 1.$$

Thus on this assumption

$$\lim |f - g| = |1 - 2| = 1.$$

Example 2. $F = 1 - x + x(1 - x) + x^2(1 - x) + x^3(1 - x) + \dots$

Here

$$F = \sum_0^\infty (1 - x) \cdot x^n.$$

Hence F is uniformly convergent in any $(-r, r)$, $0 < r < 1$, by 3, 2.

We can see this directly. For

We show now that F does not converge uniformly. For let

$$a_n = 1 - \frac{1}{n}.$$

Then

$$|\bar{F}_n(a_n)| = \left(1 - \frac{1}{n}\right)^n \doteq \frac{1}{e}$$

and F does not converge uniformly at $x = 1$, by 13.

Example 3.

$$F(x) = \sum_1^{\infty} \frac{x^2}{(1 + nx^2)(1 + (n+1)x^2)}$$

Here

$$f_n = \frac{1}{1 + nx^2} - \frac{1}{1 + (n+1)x^2}$$

and F is telescopic. Hence

$$\begin{aligned} F_n &= \frac{1}{1 + x^2} - \frac{1}{1 + (n+1)x^2} \\ &\doteq \frac{1}{1 + x^2}, \quad x \neq 0 \\ &\doteq 0, \quad x = 0. \end{aligned}$$

Thus

$$|\bar{F}_n| = \frac{1}{1 + (n+1)x^2}, \quad x \neq 0.$$

Let us take

$$a_n = \frac{1}{\sqrt{n+1}}.$$

Then

$$\bar{F}_n(a_n) = \frac{1}{2}$$

and F is not uniformly convergent at $x = 0$. It $(-\infty, \infty)$ except at this point. For let us take

Here

$$f_n = x \left\{ \frac{n}{1 + n^2 x^2} - \frac{n+1}{1 + (n+1)^2 x^2} \right\}$$

F is telescopic. Hence

$$\begin{aligned} F_n &= \frac{x}{1 + x^2} - \frac{(n+1)x}{1 + (n+1)^2 x^2} \\ &\doteq \frac{x}{1 + x^2} \quad \text{in } \mathfrak{A} = (-R, R). \end{aligned}$$

The convergence is not uniform at $x = 0$.

For set $a_n = \frac{1}{n+1}$. Then

$$|F_n(a_n)| = \frac{1}{2}, \text{ does not } \rightarrow 0.$$

It is, however, uniformly convergent in \mathfrak{A} except at 0. For $|x| > \delta$,

$$\begin{aligned} |F_n(x)| &= \left| \frac{(n+1)x}{1 + (n+1)^2 x^2} \right| < \frac{(n+1)R}{1 + (n+1)^2 \delta^2} \\ &< \epsilon \quad \text{for } n > \text{some } m. \end{aligned}$$

41. Let us suppose that the series F converges absolutely and uniformly in \mathfrak{A} . Let us rearrange F , obtaining the series G . If F is absolutely convergent, so is G and $F = G$. We can, however, state that G is uniformly convergent in \mathfrak{A} , as *Bôcher* has shown.

Example.

$$F = \frac{1-x}{x} \{ 1 - 1 + x - x + x^2 - x^2 + x^3 - x^3 + \dots \}$$

Here

$$F_{2n} = 0.$$

$$F_{2n+1} = x^{n+1}(1-x).$$

Let

$$a_n = 1 - \frac{1}{n}.$$

Then

$$\begin{aligned} G_{2n+2}(a_n) &= \left(1 - \frac{1}{n}\right)^n \left\{ 1 - \left(1 - \frac{1}{n}\right)^n \right\} \\ &\doteq \frac{1}{e} \left(1 - \frac{1}{e}\right) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Hence G does not converge uniformly at $x = 1$.

142. 1. Let $f \doteq \phi$ uniformly in a finite set $\mathfrak{A}_1, \dots, \mathfrak{A}_p$. Then f converges uniformly in their union.

For by definition

$$\epsilon > 0, \delta_s > 0, |f - \phi| < \epsilon \quad x \text{ in } \mathfrak{A}_s,$$

Since there are only p aggregates, the minimum δ_s is > 0 . Then 1) holds if we replace δ_s by δ .

2. The preceding theorem may not be true if the set of aggregates $\mathfrak{A}_1, \mathfrak{A}_2, \dots$ is infinite. For consider

$$F = \sum (1 - x)x^n,$$

which converges uniformly in $\mathfrak{A} = (0, 1)$ except at $x = 1$.

$$\mathfrak{A}_s = \left(\frac{s-1}{s}, \frac{s}{s+1} \right) \quad s = 1, 2, \dots$$

Then F is uniformly convergent in each \mathfrak{A}_s , but not in their union, which is \mathfrak{A} .

3. Let $f \doteq \phi, g \doteq \psi$ uniformly in \mathfrak{A} .

Then $f \pm g \doteq \phi \pm \psi$ uniformly in \mathfrak{A} .

GENERAL THEORY

4. To show that 1), 2) may be false if ϕ, ψ are not limited.

Let

$$f = g = \frac{1}{x} + t, \quad \mathfrak{A} = (0^*, 1), \quad \tau = 0.$$

Then $\phi = \psi = \frac{1}{x}$ and the convergence is uniform.

But

$$\Delta = fg - \phi\psi = \frac{2t}{x} + t^2.$$

Let $x = t$. Then $\Delta \doteq 2$ as $t \doteq 0$, and fg does not converge uniformly.

Again, let

$$f = \frac{1}{x} + t, \quad g = x + t,$$

the rest being as before.

Then

$$\phi = \frac{1}{x}, \quad \psi = x.$$

But setting $x = t$,

$$|\Delta| = \left| \frac{f}{g} - \frac{\phi}{\psi} \right| = \frac{t^2 - 1}{2t^2} \doteq -\infty \text{ as } t \doteq 0$$

and $\frac{f}{g}$ does not converge uniformly to $\frac{\phi}{\psi}$.

143. 1. As an extension of I, 317, 2 we have :

Let

$$\lim_{y \rightarrow \eta} f(x_1 \cdots x_m, y_1 \cdots y_p) = \phi(x_1 \cdots x_m)$$

uniformly in \mathfrak{A} . Let

$$\lim_{t \rightarrow \tau} y_1(t_1 \cdots t_n) = \eta_1 \cdots \lim_{t \rightarrow \tau} y_p(t_1 \cdots t_n) = \eta_p.$$

form a limited set \mathfrak{B} . Let $F(u_1 \cdots u_p)$ be continuous set containing \mathfrak{B} . Then

$$\lim_{t=\tau} F(u_1 \cdots u_p) = F(v_1 \cdots v_p)$$

uniformly in \mathfrak{A} .

For F , being continuous in the complete set uniformly continuous. Hence for a given $\epsilon > 0$ fixed $\sigma > 0$, such that

$$|F(u) - F(v)| < \epsilon \quad u \text{ in } V_\sigma(v),$$

But as $u_i \doteq v_i$ uniformly there exists a fixed $\delta > 0$

$$|u_i - v_i| < \epsilon' \quad , \quad x \text{ in } \mathfrak{A} \quad , \quad t \text{ in } V_\delta^*$$

Thus if ϵ' is sufficiently small, $u = (u_1, \cdots u_p)$ when x is in \mathfrak{A} and t in $V_\delta^*(\tau)$.

144. 1. Let $\lim_{t=\tau} f(x_1 \cdots x_m, t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$

uniformly in \mathfrak{A} . Then

$$\lim_{t=\tau} e^f = e^\phi$$

uniformly in \mathfrak{A} , if ϕ is limited.

This is a corollary of 143, 2.

2. Let

$$\lim_{t=\tau} f(x_1 \cdots x_m, t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$$

uniformly in \mathfrak{A} . Let ϕ be greater than some positive

Then

$$\lim_{t=\tau} \log f = \log \phi,$$

it by 2), $\log f \equiv \log \phi$ uniformly in \mathfrak{A} ; and by 142, $f \equiv \psi \log \phi$, uniformly in \mathfrak{A} . Hence 2) gives 1) by 1.

5. 1. The definition of uniform convergence may be given in a slightly different form which is sometimes useful. The function

$$f(x_1 \cdots x_m, t_1 \cdots t_n)$$

is a function of two sets of variables x and t , one ranging in an \mathfrak{A} , the other in an \mathfrak{B}_n .

Let us set now $w = (x_1 \cdots x_m, t_1 \cdots t_n)$ and consider w as a point in p -way space.

As x ranges over \mathfrak{A} and t over $\Gamma_\delta^*(\tau)$, let w range over \mathfrak{B}_δ .

$$\lim_{t \rightarrow \tau} f = \phi$$

uniformly in \mathfrak{A} when and only when

$$\epsilon > 0, \quad \delta > 0 \quad |f - \phi| < \epsilon \quad w \text{ in } \mathfrak{B}_\delta, \quad \delta \text{ fixed.}$$

By means of this second definition we obtain at once the following theorem:

Instead of the variables $x_1 \cdots x_m, t_1 \cdots t_n$ let us introduce the variables $y_1 \cdots y_m, u_1 \cdots u_n$ so that as w ranges over \mathfrak{B}_δ ,

$$z = (y_1 \cdots y_m, u_1 \cdots u_n)$$

as z ranges over \mathfrak{C}_δ , the correspondence between $\mathfrak{B}_\delta, \mathfrak{C}_\delta$ being uniform, $f \equiv \phi$ uniformly in \mathfrak{A} when and only when

$$\epsilon > 0, \quad \delta > 0 \quad |f - \phi| < \epsilon, \quad z \text{ in } \mathfrak{C}_\delta, \quad \delta \text{ fixed.}$$

Example. Let $f(x, u) = \frac{n^A x^A}{e^{n^A u}}$

As the term on the right $\doteq 0$ as $n \doteq \infty$, we have in (a, b) .

When, however, $a = 0$, or $b = \infty$, this reason fails. In this case we set

$$t = e^{n^\mu x^\beta},$$

which gives

$$x = \frac{\log^{1/\beta} t}{n^{\mu/\beta}}.$$

As the point (x, n) ranges over \mathfrak{X} defined by

$$x \geq 0, \quad n > 1,$$

the point (t, n) ranges over a field \mathfrak{T} defined by

$$t \geq 1, \quad n \geq 1,$$

and the correspondence between \mathfrak{X} and \mathfrak{T} is univalent.

$$|f - \phi| = \frac{1}{n^{\frac{\mu}{\beta} - \lambda}} \frac{\log^{\mu/\beta} t}{t}.$$

The relation 2) shows that when $x > 0$, $t > 1$, when $x = 0$, $t = 1$ for any n . Thus the convergence is uniform when

$$\frac{\alpha}{\beta} > \frac{\lambda}{\mu}.$$

The convergence is not uniform at $x = 0$ when $\frac{\alpha}{\beta} \leq \frac{\lambda}{\mu}$. For take

$$x = \frac{1}{n^{\lambda/\alpha}}, \quad n = 1, 2, \dots$$

For these values of x

$$|f - \phi| = e^{n^{\frac{\beta\lambda}{\alpha} - \mu}},$$

which does not $\doteq 0$ as $n \doteq \infty$.

146. 1. (Moore, Osgood.) Let

We first show Φ exists. To this end we show that

$$0 < \delta < 0 \quad , \quad |\phi(x') - \phi(x'')| < \epsilon \quad x', x'' \text{ in } V_\delta^*(a). \quad (1)$$

Now since $f(x, t)$ converges uniformly, there exists an $\eta > 0$ such that for any x', x'' in \mathfrak{A}

$$|\phi(x') - f(x', t)| < \epsilon' \quad t \text{ in } I_\eta^*(\tau) \quad (2)$$

$$|\phi(x'') - f(x'', t)| < \epsilon'' \quad |\epsilon'|, |\epsilon''| < \frac{\epsilon}{4}. \quad (3)$$

On the other hand, since $f \rightarrow \psi$ there exists a $\delta > 0$ such that

$$|f(x', t) - \psi(t)| < \epsilon''' \quad (4)$$

$$|f(x'', t) - \psi(t)| < \epsilon^{iv} \quad |\epsilon'''|, |\epsilon^{iv}| < \frac{\epsilon}{4} \quad (5)$$

any x', x'' in $V_\delta^*(a)$; t fixed.

From 2), 3), 4), 5) we have at once 1). Having established the existence of Φ , we show now that $\Phi = \Psi$. For since f converges uniformly to ϕ , we have

$$|f(x, t) - \phi(x)| < \frac{\epsilon}{3} \quad x \text{ in } \mathfrak{A} \quad , \quad t \text{ in } V_\eta^*(\tau). \quad (6)$$

Since $f \rightarrow \psi$, we have

$$|f(x, t) - \psi(t)| < \frac{\epsilon}{3} \quad x \text{ in } V_{\delta'}^*(a) \quad , \quad t \text{ fixed in } V_\eta^*(\tau). \quad (7)$$

Since $\phi \equiv \Phi$,

$$|\phi(x) - \Phi| < \frac{\epsilon}{3} \quad x \text{ in } V_{\delta''}^*(a). \quad (8)$$

As 7), 8) hold simultaneously for $\delta < \delta', \delta''$.

Hence

$$|\psi(t) - \Phi| < \epsilon \quad t \text{ in } I_\eta^*(\tau),$$

3. The theorem in 1 obviously holds when we replace the restricted limits, by limits which are subjected to some condition. The variables are to approach their limits along some curve.

4. As a corollary we have :

Let $F = \Sigma f_s(x_1 \dots x_m)$ be uniformly convergent in \mathfrak{A} , of which x is a limiting point. Let $\lim_{x=a} f_s = l_s$, and set $L = \Sigma l_s$. Then

$$\lim_{x=a} F = L; \quad \text{a finite or infinite,}$$

in other words

$$\lim \Sigma f_s = \Sigma \lim f_s.$$

Example 1.

$$F(x) = \sum \frac{e^{nx} - 1}{2^n e^{nx}}$$

converges uniformly in $\mathfrak{A} = (0, \infty)$ as we saw 136, 2, Ex. 1. Here

$$\lim_{x=\infty} f_n = \frac{1}{2^n} = l_n,$$

and

$$L = \Sigma l_n = \sum \frac{1}{2^n} = 1.$$

Hence

$$\lim_{x=\infty} F(x) = 1.$$

Also

$$R \lim_{x=0} f_n = 0;$$

hence

$$R \lim_{x=0} F(x) = 0.$$

Example 2.

$$F(x) = 1 + \sum_1 \frac{1}{n!} \left(\frac{\sin x}{x} \right)^n$$

Example 3.

$$F(x) = \sum_1^{\infty} \frac{x^2}{(1 + nx^2)(1 + (n+1)x^2)} = \sum f_n(x) \\ = \frac{1}{1 + x^2} \quad \text{for } x \neq 0, \\ = 0 \quad \text{for } x = 0.$$

Here $\lim_{x \rightarrow 0} F(x) = 1,$

while $\sum \lim_{x \rightarrow 0} f_n(x) = \sum 0 = 0.$

Thus here $\lim_{x \rightarrow 0} \sum f_n(x) \neq \sum \lim_{x \rightarrow 0} f_n(x),$

But F does not converge uniformly at $x = 0$.
 On the other hand, it does converge uniformly at $x = \pm \infty$.

Now $\lim_{x \rightarrow \pm \infty} F(x) = 0, \quad \lim_{x \rightarrow \pm \infty} f_n(x) = 0,$

and $\lim_{x \rightarrow \pm \infty} \sum f_n(x) = \sum \lim_{x \rightarrow \pm \infty} f_n(x),$

as the theorem requires.

Example 4. $F(x) = \sum_1^{\infty} \left\{ \frac{nx^2}{e^{nx^2}} - \frac{(n+1)x^2}{e^{(n+1)x^2}} \right\} = \frac{x^2}{e^{x^2}},$

which converges about $x = 0$ but not uniformly.

However, $\lim_{x \rightarrow 0} \sum f_n(x) = \sum \lim_{x \rightarrow 0} f_n(x) = 0.$

Thus the uniform convergence is not a necessary

A direct proof may be given as follows:

$$f(x, t) = \phi(x) + \epsilon' \quad |\epsilon'| < \epsilon, x \text{ in } V$$

$$\phi(x') - \phi(x'') = f(x', t) - f(x'', t) + \epsilon$$

But $|f(x'', t) - f(x', t)| < \epsilon$, if $|x' - x''| < \delta$

2. Let $F = \sum f_{s_1 \dots s_p}(x_1 \dots x_m)$ be uniformly convergent. Let each $f_{s_1 \dots s_p}$ be continuous at a . Then $F(x_1 \dots x_m)$ is continuous at $x = a$.

Follows at once from 1).

3. In Ex. 3 of 140 we saw that

$$F = \sum \frac{x^2}{(1+nx^2)(1+(n+1)x^2)}$$

is discontinuous at $x = 0$ and does not converge uniformly.

In Ex. 4 of 140 we saw that

$$F = \sum x \frac{n(n+1)x^2 - 1}{(1+n^2x^2)(1+(n+1)^2x^2)}$$

does not converge uniformly at $x = 0$ and yet is continuous.

We have thus the result: *The condition of uniform convergence is sufficient but not necessary.*

Finally, let us note that

$$F(x) = \sum \left\{ \frac{nx^\alpha}{e^{nx^\beta}} - \frac{(n+1)x^\alpha}{e^{(n+1)x^\beta}} \right\} \quad 0 < \alpha < \beta$$

$$= \frac{x^\alpha}{e^{x^\beta}}, \quad x \geq 0$$

is a series which is not uniformly convergent at $x = 0$. $F(x)$ is continuous at this point.

GENERAL THEORY

5. The power series $P = \sum a_{s_1 \dots s_m} x_1^{s_1} \dots x_m^{s_m}$ is continuous at every point of its rectangular cell of convergence.

For we saw P converges uniformly at this point.

6. The power series $P = a_0 + a_1x + a_2x^2 + \dots$ is a continuous function of x in its interval of convergence.

For we saw P converges uniformly in this interval. In particular we note that if P converges at an end point $x = c$ of its interval of convergence, P is continuous at c .

This fact enables us to prove the theorem on multiplication of series which we stated 112, 4; viz.:

148. Let $A = a_0 + a_1 + a_2 + \dots$, $B = b_0 + b_1 + b_2 + \dots$
 $C = a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \dots$
 converge. Then $AB = C$.

For consider the auxiliary series

$$F(x) = a_0 + a_1x + a_2x^2 + \dots$$

$$G(x) = b_0 + b_1x + b_2x^2 + \dots$$

$$H(x) = a_0b_0 + (a_0b_1 + a_1b_0)x + \dots$$

Since A, B, C converge, F, G, H converge for $x = 1$, and hence absolutely for $|x| < 1$. But for all $|x| < 1$,

$$H(x) = F(x) G(x).$$

Thus

$$L \lim_{x \rightarrow 1} H(x) = L \lim_{x \rightarrow 1} F(x) \cdot L \lim_{x \rightarrow 1} G(x),$$

$$C = A \cdot B.$$

149. 1. We have seen that we cannot say that $f \doteq \phi$ uniformly

Suppose now $|\psi(x, t')| \leq |\psi(x, t)|$ for any t' in the same cell one of whose vertices is t and whose center is ϕ . Then that the convergence of f to ϕ is *steady* or *uniform*. If for each x in \mathfrak{A} , there exists a rectangular cell in which the above inequality holds, we say the convergence is *steady in \mathfrak{A}* .

The modification in this definition for the case of a point is obvious. See I, 314, 315.

2. We may now state *Dini's* theorem.

Let $f(x_1 \dots x_m, t_1 \dots t_n) \doteq \phi(x_1 \dots x_m)$ steadily in a complete field \mathfrak{A} as $t \doteq \tau$; τ finite or ideal. Let f and ϕ be functions of x in \mathfrak{A} . Then f converges uniformly to ϕ .

For let x be a given point in \mathfrak{A} , and

$$f(x, t) = \phi(x) + \psi(x, t).$$

We may take t' so near τ that $|\psi(x, t')| < \frac{\epsilon}{3}$.

Let x' be a point in $V_\eta(x)$. Then

$$f(x', t') = \phi(x') + \psi(x', t').$$

As f is continuous in x ,

$$|f(x', t') - f(x, t')| < \frac{\epsilon}{3}.$$

Similarly,

$$|\phi(x') - \phi(x)| < \frac{\epsilon}{3}.$$

Thus

$$|\psi(x', t')| < \epsilon \quad x' \text{ in } V_\eta(x).$$

Hence

$$|\psi(x', t)| < \epsilon \quad \text{for any } x' \text{ in } V_\eta(x)$$

and for any t in the rectangular cell determined by

uous at a . Then G and a fortiori $F = \Sigma f_{i_1 \dots i_s}$ converge uniformly at a .

5. Let $G = \Sigma |f_{i_1 \dots i_s}(x_1 \dots x_m)|$ converge in the limited complete metric space \mathfrak{A} , having a as limiting point. Let $\lim_{x=a} G$ and each $\lim_{x=a} f_{i_1 \dots i_s}$ exist. Moreover, let $\lim G = \Sigma \lim f_{i_1 \dots i_s}$.

Then G is uniformly convergent at a .

For if in 4 the function had values assigned them at $x = a$ different from their limits, we could redefine them so that they are continuous at a .

150. 1. Let $\lim_{t=\tau} f(x_1 \dots x_m, t_1 \dots t_n) = \phi(x_1 \dots x_m)$ uniformly in the limited field \mathfrak{A} . Let ϕ be limited in \mathfrak{A} . Then

$$\lim_{t=\tau} \int_{\mathfrak{A}} f = \int_{\mathfrak{A}} \phi = \int_{\mathfrak{A}} \lim_{t=\tau} f.$$

For let

$$f = \phi + \psi.$$

Since $f \doteq \phi$ uniformly

$$|\psi| < \epsilon$$

for any t in some $V^*(\tau)$ and for any x in \mathfrak{A} .

Thus

$$\left| \int_{\mathfrak{A}} f - \int_{\mathfrak{A}} \phi \right| < \epsilon \bar{\mathfrak{A}}.$$

Remark. Instead of supposing ϕ to be limited we may suppose that $f(x, t)$ is limited in \mathfrak{A} for each t near τ .

2. As corollary we have

Let $\lim_{t=\tau} f(x_1 \dots x_m, t_1 \dots t_n) = \phi(x_1 \dots x_m)$ uniformly in the limited field \mathfrak{A} . Let f be limited and integrable in \mathfrak{A} for each t in $V_\delta^*(\tau)$. Then ϕ is integrable in \mathfrak{A} and

If the f_{i_1}, \dots, f_{i_n} are not integrable, we have

$$\int_{\mathfrak{A}} F = \sum \int_{\mathfrak{A}} f_{i_1, \dots, i_n}.$$

Example.

$$F = \sum_0^{\infty} \frac{x^2}{(1 + nx^2)(1 + (n+1)x^2)}$$

does not converge uniformly at $x = 0$. (Cf. 140, Ex. 3.)

Here

$$F_n = 1 - \frac{1}{1 + nx^2}$$

and

$$F(x) = \begin{cases} 1 & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Hence

$$\int_0^1 F dx = 1,$$

$$\begin{aligned} \int_0^1 F_n dx &= 1 - \int_0^1 \frac{dx}{1 + nx^2} \\ &= 1 - \operatorname{arctg} \frac{\sqrt{n}}{\sqrt{n}} = 1. \end{aligned}$$

Thus we can integrate F termwise although F does not converge uniformly in $(0, 1)$.

151. That uniform convergence of the series

$$F(x) = f_1(x) + f_2(x) + \dots$$

with integrable terms, in the interval $\mathfrak{A} = (a, b)$ is a condition for the validity of the relation

$$\int_a^b F dx = \int_a^b f_1 dx + \int_a^b f_2 dx + \dots$$

GENERAL THEORY

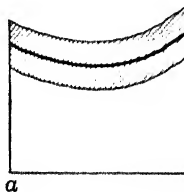
In the figure, the graph of $F(x)$ is drawn heavy. On either side of it are drawn the curves $F - \epsilon$, $F + \epsilon$ giving the band which we call the ϵ -band.

From 2), 3) we see that the graph of each F_n , $n \geq m$ lies in the ϵ -band. The figure thus shows at once that

$$\int_a^b F dx \quad \text{and} \quad \int_a^b F_n dx$$

can differ at most by the area of the ϵ -band, i.e. by at most

$$\int_a^b 2 \epsilon dx = 2 \epsilon (b - a).$$



152. 1. Let us consider a case where the convergence is uniform, as

$$F(x) = \sum_1^{\infty} \left\{ \frac{nx}{e^{nx^2}} - \frac{(n-1)x}{e^{(n-1)x^2}} \right\} = 0.$$

Here

$$F'_n(x) = \frac{nx}{e^{nx^2}}.$$

If we plot the curves $y = F'_n(x)$, we observe that they grow out more and more as $n \rightarrow \infty$, and approach the x -axis



near the origin, they have peaks which increase indefinitely in height. The $F'_n(x)$, $n \geq m$, and m sufficiently large, lie in an ϵ -band about

SERIES OF FUNCTIONS

$$\int_0^a F_n dx = \frac{1}{2} \int_0^a \frac{d}{dx} \left[-\frac{1}{e^{nx^2}} \right] dx = \frac{1}{2} \left[-\frac{1}{e^{nx^2}} \right]_0^a = \frac{1}{2} \left(1 - \frac{1}{e^{na^2}} \right) \\ \doteq \frac{1}{2} \quad \text{as } n \doteq \infty.$$

thus we cannot integrate the F series termwise.

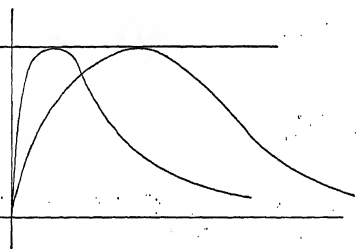
As another example in which the convergence is not uniform we consider

$$F(x) = \sum_0^\infty \left\{ \frac{(n+1)x}{e^{(n+1)x}} - \frac{nx}{e^{nx}} \right\} = 0.$$

here

$$F_n = \frac{nx}{e^{nx}}.$$

the convergence of F is uniform in $\mathfrak{A} = (0, 1)$ except at $x = 0$. The peaks of the curves $F_n(x)$ all have the height e^{-1} .



Obviously the area of the region under the peaks can be made small at pleasure if m is taken sufficiently large. Thus in this case we can obviously integrate termwise, although the convergence is not uniform in \mathfrak{A} .

We may verify this analytically. For

$$\int_0^x F_n dx = \int_0^x \frac{nx}{e^{nx}} dx = \frac{1}{n} - \frac{1+nx}{ne^{nx}} \doteq 0 \quad \text{as } n \doteq \infty.$$

Finally let us consider

$$F(x) = \sum_0^\infty \left(\frac{(n+1)^2 x}{e^{(n+1)^2 x}} - \frac{n^2 x}{e^{n^2 x}} \right)$$

Their height thus increases indefinitely with n . But at the same time they become so slender that the area under them tends to a limit. In fact

$$\begin{aligned}\int_0^a E_n(x) dx &= \int_0^a \frac{1}{2n} d \log (1 + n^3 x^2) \\ &= \frac{1}{2n} \left[\log (1 + n^3 x^2) \right]_0^a = \frac{1}{2} \frac{\log (1 + n^3 a^2)}{n} \rightarrow 0.\end{aligned}$$

We can therefore integrate termwise in $(0 < a)$.

153. 1. Let $\lim_{t \rightarrow \tau} G(x, t_1 \dots t_n) = g(x)$ in $\mathfrak{A} = (a, a + \delta)$, or infinite. Let each $G'_r(x, t)$ be continuous in \mathfrak{A} ; also let G'_r converge to a limit uniformly in \mathfrak{A} as $t \rightarrow \tau$. Then

$$\lim_{t \rightarrow \tau} G'_r(x, t) = g'(x) \quad \text{in } \mathfrak{A},$$

and $g'(x)$ is continuous.

For by 150, 2,

$$\lim_{t \rightarrow \tau} \int_a^x G'_x dx = \int_a^x \lim_{t \rightarrow \tau} G'_x dx.$$

By I, 538,

$$\int_a^x G'_x dx = G(x, t) - G(a, t).$$

Also by hypothesis, $\lim_{t \rightarrow \tau} \{G(x, t) - G(a, t)\} = g(x) - g(a)$.

Hence

$$g(x) - g(a) = \int_a^x \lim_{t \rightarrow \tau} G'_x(x, t) dx.$$

But by 147, 1, the integrand is continuous in \mathfrak{A} .

Hence by I, 537, the derivative of the right side of 2) is

3. The more general case that the terms f_1, \dots, f_s are functions of several variables x_1, \dots, x_m follows readily from 2.

154. Example.

$$F(x) = \sum_0^{\infty} \left\{ \frac{n^{\lambda} x^{\alpha}}{e^{n^{\mu} x^{\beta}}} - \frac{(n+1)^{\lambda} x^{\alpha}}{e^{(n+1)^{\mu} x^{\beta}}} \right\} = \Sigma f_n; \quad \alpha, \beta, \lambda \geq 0, \mu > 0$$

Here

$$F_n = -\frac{n^{\lambda} x^{\alpha}}{e^{n^{\mu} x^{\beta}}},$$

function whose uniform convergence was studied, 145, 3. We see

$$F(x) = 0 \quad \text{for any } x \geq 0.$$

Hence

$$F'(x) = 0 \quad x \geq 0.$$

Let

$$G(x) = \Sigma f'_n(x).$$

Then

$$G_n(x) = F'_n(x) = -\frac{\alpha n^{\lambda} x^{\alpha-1}}{e^{n^{\mu} x^{\beta}}} + \frac{\beta n^{\lambda+\mu} x^{\alpha+\beta-1}}{e^{n^{\mu} x^{\beta}}}.$$

If $x > 0$,

$$G_n(x) \doteq 0,$$

hence

$$F'(x) = \Sigma f'_n(x),$$

and we may differentiate the series termwise.

If $x = 0$, and $\alpha = 1, \lambda > 0$; $G_n(0) = -n^{\lambda} \doteq -\infty$ as $n \doteq \infty$.

In this case 2) does not hold, and we cannot differentiate termwise.

For $x = 0$, and $\alpha > 1$, $G_n(0) = 0$, and now 2) holds; we may therefore differentiate the series termwise. But if we look at the uniform convergence of the series 1), we see this takes place only when

$$\frac{\alpha-1}{\beta} > \frac{\lambda}{\mu}.$$

or simplicity let us take $s = 1$. Let the series on the right be denoted by $\phi(x)$. For each x in \mathfrak{A} we show that

$$\epsilon > 0, \quad \delta > 0, \quad D = \left| \frac{\Delta F}{\Delta x} - \phi(x) \right| < \epsilon, \quad |\Delta x| < \delta.$$

or

$$\frac{\Delta F}{\Delta x} = \frac{\sum f_n(x + \Delta x) - f_n(x)}{\Delta x} = \sum f'_n(\xi_n)$$

where ξ_n lies in $I_\delta(x)$.

thus

$$D = \sum_1^{\infty} \{f'_n(\xi_n) - f'_n(x)\} = D_m + D_\infty.$$

As G being convergent, $G_m < \epsilon/3$ if m is taken sufficiently large.

$$|D_m| \leq \sum_{m+1}^{\infty} |f'_n(\xi_n)| + \sum_{m+1}^{\infty} |f'_n(x)| \leq 2 G_m < \frac{2}{3} \epsilon.$$

On the other hand, since $\frac{\Delta f_n}{\Delta x} = f'_n(x)$ and since there are only m terms in D_m , we may take δ so small that

$$|D_m| < \epsilon/3.$$

thus

$$|D| < \epsilon \quad \text{for } |\Delta x| < \delta.$$

Example 1. Let

$$F(x) = \sum_0^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{1 + a^n x} = \sum f_n(x) \quad a > 1.$$

This series converges uniformly in $\mathfrak{A} = (0 < b)$, since

$$|f_n(x)| \leq \frac{1}{n!}.$$

Also

$$f'_n(x) = \frac{(-1)^{n+1}}{n!} \cdot \frac{a^n}{(1 + a^n x)^2}.$$

3. *Example 2. The \mathfrak{A} functions.*

These are defined by

$$\begin{aligned}\mathfrak{A}_1(x) &= 2 \sum_0^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} \sin(2n+1)\pi x \\ &= 2q^{\frac{1}{4}} \sin \pi x - 2q^{\frac{9}{4}} \sin 3\pi x + \dots\end{aligned}$$

$$\begin{aligned}\mathfrak{A}_2(x) &= 2 \sum_0^{\infty} q^{(n+\frac{1}{2})^2} \cos(2n+1)\pi x \\ &= 2q^{\frac{1}{4}} \cos \pi x + 2q^{\frac{9}{4}} \cos 3\pi x + \dots\end{aligned}$$

$$\begin{aligned}\mathfrak{A}_3(x) &= 1 + 2 \sum_1^{\infty} q^{n^2} \cos 2n\pi x \\ &= 1 + 2q \cos 2\pi x + 2q^4 \cos 4\pi x + \dots\end{aligned}$$

$$\begin{aligned}\mathfrak{A}_0(x) &= 1 + 2 \sum_1^{\infty} (-1)^n q^{n^2} \cos 2n\pi x \\ &= 1 - 2q \cos 2\pi x + 2q^4 \cos 4\pi x - \dots\end{aligned}$$

Let us take

$$|q| < 1.$$

Then these series converge uniformly at every x . Let us consider as an example \mathfrak{A}_1 . The series

$$T = |q| + |q|^4 + |q|^9 + \dots$$

is convergent since the ratio of two successive terms

$$\frac{q^{(n+1)^2}}{q^{n^2}} = q^{2n+1};$$

and this $\rightarrow 0$. Now each term in \mathfrak{A}_1 is numerically

$$\leq |q|^{(n+\frac{1}{2})^2} < |q|^{n^2},$$

and hence $<$ the corresponding term in T .

Thus $\mathfrak{A}_1(x)$ is a continuous function of x for every

GENERAL THEORY

$$\begin{aligned} g_2'(x) &= -2\pi \sum_0^{\infty} (2n+1) q^{(n+1)^2} \sin (2n+1)\pi x \\ &= -2\pi (q^1 \sin \pi x + 3q^4 \sin 3\pi x + \dots) \end{aligned}$$

$$\begin{aligned} g_3'(x) &= -4\pi \sum_1^{\infty} nq^{n^2} \sin 2n\pi x \\ &= -4\pi (q \sin 2\pi x + 2q^4 \sin 4\pi x + \dots) \end{aligned}$$

$$\begin{aligned} g_0'(x) &= 4\pi \sum_1^{\infty} (-1)^n nq^{n^2} \sin 2n\pi x \\ &= 4\pi (q \sin 2\pi x - 2q^4 \sin 4\pi x + \dots) \end{aligned}$$

To show the uniform convergence of these series, let us consider the first and compare it with

$$S = 1 + 3|q| + 5|q|^4 + 7|q|^9 + \dots$$

The ratio of two successive terms of this series is

$$\frac{2n+3}{2n+1} \frac{|q|^{(n+1)^2}}{|q|^{n^2}} = \frac{2n+3}{2n+1} |q|^{2n+1},$$

which $\rightarrow 0$. Thus S is convergent. The rest follows before.

156. 1. *Let*

$$\lim_{t \rightarrow \tau} \frac{G(a+h, t_1 \dots t_n) - G(a, t_1 \dots t_n)}{h} = g(a+h) - g(a)$$

uniformly for $0 < |h| < \eta$, τ *finite or infinite.*

Let

$$G'_t(a, t) \text{ exist}$$

for each t *near* τ . *Then* $g'(a)$ *exists and*

$$\lim_{t \rightarrow \tau} G'_t(a, t) = g'(a).$$

converge for each x in \mathfrak{A} which has $x = a$ as a p. Let $f'_i(a)$ exist for each $i = (i_1, \dots, i_n)$. Let

$$\sum \frac{f_i(a+h) - f_i(a)}{h}$$

converge uniformly with respect to h . Then

$$F''(a) = \sum f'_{i_1, \dots, i_n}(a).$$

CHAPTER VI

POWER SERIES

57. On account of their importance in analysis we devote a separate chapter to power series.

We have seen that without loss of generality we may employ series

$$a_0 + a_1x + a_2x^2 + \dots$$

instead of the formally more general one

$$a_0 + a_1(x-a) + a_2(x-a)^2 + \dots$$

We have seen that if 1) converges for $x=c$ it converges absolutely and uniformly in $(-\gamma, \gamma)$ where $0 < \gamma < |c|$. Finally we saw that if c is an end point of its interval of convergence it is unilaterally continuous at this point. The series 1) is, of course, a continuous function of x at any point *within* its interval of convergence.

58. 1. Let $P(x) = a_0 + a_1x + a_2x^2 + \dots$ converge in the interval $\mathfrak{B} = (-\alpha, \alpha)$ which may not be complete. The series

$$P_n = 1 \cdot 2 \cdot \dots \cdot na_n + 2 \cdot 3 \cdot \dots \cdot (n+1)a_{n+1}x + \dots$$

obtained by differentiating each term of P n times is absolutely and uniformly convergent in $\mathfrak{B} = (-\beta, \beta)$, $\beta < \alpha$, and

$$\frac{d^n P}{dx^n} = P_n(x), \quad \text{in } \mathfrak{B}.$$

But the series whose general term is the last term of the preceding inequality is convergent.

$$2. \text{ Let } P = a_0 + a_1x + a_2x^2 + \dots$$

converge in the interval \mathfrak{A} . Then

$$Q = \int_a^x P dx = \int_a^x a_0 dx + \int_a^x a_1 x dx + \dots$$

where a, x lie in \mathfrak{A} . Moreover Q considered as a function of x converges uniformly in \mathfrak{A} .

For by 137, P is uniformly convergent in (a, x) . We may therefore integrate termwise by 150, 3. To show that Q is uniformly convergent in \mathfrak{A} we observe that P being uniformly convergent in \mathfrak{A} we may set

$$P = \overset{*}{P}_m + \bar{P}_m$$

where

$$|\bar{P}_m| < \sigma, \quad m > m_0, \quad \sigma \text{ small at pleasure.}$$

Then

$$Q = Q_m + \bar{Q}_m$$

where

$$|\bar{Q}_m| = \left| \int_a^x \bar{P}_m dx \right| \leq \sigma \widehat{\mathfrak{A}} < \epsilon$$

on taking σ sufficiently small.

159. 1. Let us show how the theorems in 2 may be used to obtain the developments of some of the elementary functions in power series.

The Logarithmic Series. We have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

for any x in $\mathfrak{A} = (-1^*, 1^*)$. Thus

$$\int_0^x \frac{dx}{1-x} = -\log(1-x) = \int_0^x dx + \int_0^x x dx + \dots$$

Hence

$$\log(1-x) = - \left\{ x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \right\} \quad ; \quad x \text{ in } \mathfrak{A}.$$

This gives also

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \quad ; \quad x \text{ in } \mathfrak{A}. \quad (1)$$

The series 1) is also valid for $x = 1$. For the series is convergent for $x \leq 1$, and $\log(1+x)$ is continuous at $x=1$. We now apply 117, 6.

For $x = 1$, we get

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

2. *The Development of arcsin x .* We have by the Binomial Series

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \dots$$

for x in $\Re = (-1^+, 1^+)$. Thus

$$\int_0^x \frac{dx}{\sqrt{1-x^2}} = \arcsin x = x + \frac{1}{2 \cdot 3}x^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5}x^5 + \dots \quad (2)$$

It is also valid for $x = 1$. For the series on the right is convergent for $x \leq 1$. We can thus reason as in 1.

For $x = 1$ we get

$$\frac{\pi}{2} = 1 + \frac{1}{2 \cdot 3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} + \dots$$

3. *The Arctan Series.* We have

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$

for x in $\Re = (-1^+, 1^+)$. Thus

$$\begin{aligned} \int_0^x \frac{dx}{1+x^2} &= \arctan x = \int_0^x dx - \int_0^x x^2 dx + \dots \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \end{aligned} \quad (3)$$

valid in \Re . The series 3) is valid for $x=1$ for the same reason as in 2.

For $x = 1$ we get $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

4. *The Development of e^x .* We have seen that

$$E(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

converges for any x . Differentiating, we get

$$E'(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$$

Hence

$$E'(x) = E(x)$$

for any x . Let us consider now the function

$$f(x) = \frac{E(x)}{e^x}.$$

We have

$$f'(x) = \frac{e^x E' - E e^x}{e^{2x}} = \frac{E' - E}{e^x} = 0$$

by (a). Thus by I, 400, $f(x)$ is a constant. For

Hence

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

valid for any x .

5. *Development of* $\cos x$, $\sin x$.

The series

$$C = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

converges for every x . Hence, differentiating,

$$C' = -\frac{x}{1!} + \frac{x^3}{3!} - \frac{x^5}{5!} + \dots$$

$$C'' = -1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \dots$$

Hence adding,

$$C + C'' = 0.$$

Let us consider now the function

$$f(x) = C \sin x + C' \cos x.$$

Then

$$\begin{aligned} f'(x) &= C \cos x + C' \sin x - C' \sin x + C'' \cos x \\ &= (C + C'') \cos x \end{aligned}$$

If we multiply (c) by $\sin x$ and (d) by $\cos x$ and add, we get $\cos x = \cos x$. Similarly we get $C' = -\sin x$. Thus finally

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

valid for any x .

160. 1. Let $P = a_m x^m + a_{m+1} x^{m+1} + \dots$, $a_m \neq 0$, converge in some interval \mathfrak{A} about the origin. Then there exists an interval $\mathfrak{B} \subset \mathfrak{A}$ in which P does not vanish except at $x = 0$.

For

$$\begin{aligned} P &= x^m (a_m + a_{m+1} x + \dots) \\ &= x^m Q. \end{aligned}$$

Obviously Q converges in \mathfrak{A} . It is thus continuous at $x = 0$. Since $Q \neq 0$ at $x = 0$ it does not vanish in some interval \mathfrak{B} about $x = 0$ by I, 351.^o

In analogy to polynomials, we say P has a *zero or root of order* m at the origin.

2. Let $P = a_0 + a_1 x + a_2 x^2 + \dots$ vanish at the points $b_1, b_2, \dots \neq 0$. Then all the coefficients $a_n = 0$. The points b_n are supposed to be different from each other and from 0.

For by hypothesis $P(b_n) = 0$. But P being continuous at $x = 0$

$$P(0) = \lim P(b_n).$$

Hence

$$P(0) = 0,$$

and thus

$$a_0 = 0.$$

be equal for the points of an infinite sequence B whose points are $a_n = b_n$, $n = 0, 1, 2 \dots$

For $P - Q$ vanishes at the points B .

Hence
$$a_n - b_n = 0 \quad , \quad n = 0, 1, 2 \dots$$

4. Obviously if the two series P , Q are equal in a little interval about the origin, the coefficients of P and Q are equal; that is

$$a_n = b_n \quad , \quad n = 0, 1, 2 \dots$$

161. 1. Let

$$y = a_0 + a_1x + a_2x^2 + \dots$$

converge in an interval \mathfrak{A} . As x ranges over \mathfrak{A} , y ranges over an interval \mathfrak{B} . Let

$$z = b_0 + b_1y + b_2y^2 + \dots$$

converge in \mathfrak{B} . Then z may be considered as a function of x defined in \mathfrak{A} . We seek to develop z in a power series in x .

To this end let us raise 1) to the 2°, 3°, 4° ... powers. We get series

$$y^2 = a_{20} + a_{21}x + a_{22}x^2 + \dots$$

$$y^3 = a_{30} + a_{31}x + a_{32}x^2 + \dots$$

$$\dots \dots \dots$$

which converge absolutely within \mathfrak{A} .

We note that a_{mn} is a polynomial.

$$a_{m,n} = F_{m,n}(a_0, a_1 \dots a_n)$$

in $a_0, \dots a_n$ with coefficients which are positive integers.

If we put 3) in 2), we get a double series

$$D = (b_0 + b_1a_0) + b_1a_1x + b_1a_2x^2 + \dots$$

GENERAL THEORY

where

$$c_0 = b_0 + b_1 a_0 + b_2 a_{20} + b_3 a_{30} + \dots$$

$$c_1 = b_1 a_1 + b_2 a_{21} + b_3 a_{31} + \dots$$

$$\dots \dots \dots$$

We may now state the following theorem, which is a solution of our problem.

Let the adjoint y -series,

$$\eta = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2 + \dots$$

converge for $\xi = \xi_0$ to the value $\eta = \eta_0$. Let the adjoint z series

$$\zeta = \beta_0 + \beta_1 \eta + \beta_2 \eta^2 + \dots$$

converge for $\eta = \eta_0$. Then the z series 2) can be developed as a power series in x , viz. the series 5), which is valid for $|x| \leq \xi_0$

For in the first place, the series 8) converges for $\eta \leq \eta_0$. We show now that the positive term series

$$\begin{aligned} \mathfrak{D} = & (\beta_0 + \beta_1 \alpha_0) + \beta_1 \alpha_1 \xi + \beta_1 \alpha_2 \xi^2 + \dots \\ & + \beta_2 \alpha_{20} + \beta_2 \alpha_{21} \xi + \beta_2 \alpha_{22} \xi^2 + \dots \\ & + \dots \dots \dots \end{aligned}$$

converges for $\xi \leq \xi_0$. We observe that \mathfrak{D} differs from A at most by its first term. To show the convergence of \mathfrak{D} we have, raising 7) to successive powers,

$$\eta^2 = A_{20} + A_{21} \xi + A_{22} \xi^2 + \dots$$

$$\eta^3 = A_{30} + A_{31} \xi + A_{32} \xi^2 + \dots$$

Putting these values of $\eta, \eta^2, \eta^3 \dots$ in 8), we get

$$\begin{aligned}\Delta = & (\beta_0 + \beta_1 \alpha_0) + \beta_1 \alpha_1 \xi + \beta_1 \alpha_2 \xi^2 + \dots \\ & + \beta_2 A_{20} + \beta_2 A_{21} \xi + \beta_2 A_{22} \xi^2 + \dots \\ & + \dots \dots \dots\end{aligned}$$

Summing by rows we get a convergent series whose terms are \leq than the corresponding terms in 8). But this series converges for $\xi \leq \xi_0$ since the series $\sum \alpha_n \xi^n$ and 8) converges by hypothesis for $\eta = \eta_0$. Now the n -th term of \mathfrak{D} is \leq than the corresponding term in Δ . Hence \mathfrak{D} converges for $\xi \leq \xi_0$.

2. As a corollary of 1 we have :

Let

$$y = a_0 + a_1 x + a_2 x^2 + \dots$$

converge in \mathfrak{A} , and

$$z = b_0 + b_1 y + b_2 y^2 + \dots$$

converge for all $-\infty < y < +\infty$. Then z can be developed in a power series in x ,

$$z = c_0 + c_1 x + c_2 x^2 + \dots = C$$

for all x within \mathfrak{A} .

3. Let the series

$$y = a_m x^m + a_{m+1} x^{m+1} + \dots, \quad m \geq 1$$

converge for some $x > 0$. If the series

$$z = b_0 + b_1 y + b_2 y^2 + \dots$$

converges for some $y > 0$, it can be developed in a power series in x ,

$$z = c_0 + c_1 x + c_2 x^2 + \dots$$

convergent for some $x > 0$.

GENERAL THEORY

converge in $\mathfrak{A} = (-A, A)$. Then y can be developed in series about any point c of \mathfrak{A} ,

$$y = c_0 + c_1(x - c) + c_2(x - c)^2 + \dots$$

which is valid in an interval \mathfrak{B} whose center is c and lying

162. 1. As an application of the theorem 161, let us

$$z = 1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \dots$$

$$y = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

As the reader already knows,

$$z = e^y, \quad y = \sin x,$$

hence z considered as a function of x is

$$z = e^{\sin x}.$$

We have

$$\begin{aligned} z = 1 + x + 0 \cdot x^2 + \frac{1}{6} x^3 + 0 \cdot x^4 + \frac{1}{120} x^5 + 0 \cdot x^6 + \dots \\ + \frac{1}{2} x^2 + 0 = \frac{1}{6} x^4 + 0 + \frac{1}{48} x^6 + \dots \\ + \frac{1}{6} x^3 + 0 = \frac{1}{12} x^5 + 0 + \dots \\ + \frac{1}{24} x^4 + 0 = \frac{1}{36} x^6 + \dots \\ + \frac{1}{120} x^5 + 0 + \dots \\ + \frac{1}{720} x^6 + \dots \end{aligned}$$

Summing by columns, we get

$$z = e^{\sin x} = 1 + x + \frac{1}{2} x^2 + \frac{1}{8} x^4 + \frac{1}{16} x^5 + \frac{1}{240} x^6 + \dots$$

2. As a second application let us consider the power

$$z = P(y) = a_0 + a_1 y + a_2 y^2 + \dots$$

This may be regarded as a double series. It is summed by columns. Hence

$$\begin{aligned} P(x+h) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \dots \\ &\quad + h(a_1 + 2a_2x + 3a_3x^2 + \dots) \\ &\quad + \frac{h^2}{2!}(2a_2 + 2 \cdot 3a_3x + 3 \cdot 4a_4x^2 + \dots) \\ &\quad + \dots \\ &= P(x) + hP'(x) + \frac{h^2}{2!}P''(x) + \dots \end{aligned}$$

on using 158, 1.

This, as the reader will recognize, is Taylor's series (1) about the point x . We thus have

A power series (1) may be developed in Taylor's series about any point x within its interval of convergence. If h is such that $x+h$ lies within the interval of convergence,

163. 1. The addition, subtraction, and multiplication of power series may be effected at once by the principles just stated. We have if

$$P = a_0 + a_1x + a_2x^2 + \dots$$

$$Q = b_0 + b_1x + b_2x^2 + \dots$$

they converge in a common interval \mathfrak{A} :

$$P + Q = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \dots$$

$$P - Q = (a_0 - b_0) + (a_1 - b_1)x + (a_2 - b_2)x^2 + \dots$$

$$P \cdot Q = a_0b_0 + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1)x^2 + \dots$$

These are valid within \mathfrak{A} , and the first two in

2. Let us now consider the division of P by

$$P \quad 1$$

GENERAL THEORY

Then $1/P$ can be developed in a power series

$$\frac{1}{P} = c_0 + c_1x + c_2x^2 + \dots$$

valid in \mathfrak{B} . The first coefficient $c_0 = \frac{1}{a_0}$.

For

$$\begin{aligned} \frac{1}{P} &= \frac{1}{a_0 + Q} = \frac{1}{a_0} \cdot \frac{1}{1 + \frac{Q}{a_0}} \\ &= \frac{1}{a_0} \left\{ 1 - \frac{Q}{a_0} + \frac{Q^2}{a_0^2} - \frac{Q^3}{a_0^3} + \dots \right\} \end{aligned}$$

for all x in \mathfrak{B} . We have now only to apply 161, 1.

3. Suppose $P = a_mx^m + a_{m+1}x^{m+1} + \dots \quad a_m \neq 0$.

To reduce this case to the former, we remark that

$$P = x^m Q$$

where

$$Q = a_m + a_{m+1}x + \dots$$

Then

$$\frac{1}{P} = \frac{1}{x^m} \cdot \frac{1}{Q}.$$

But $1/Q$ has been treated in 2.

164. 1. Although the reasoning in 161 affords us a means of determining the coefficients in the development of the quotient of two power series, there is a more expeditious method applicable also to many other problems, called the method of *undetermined coefficients*. It rests on the hypothesis that $f(x)$ can be developed in a power series in a certain interval about some point, let

2. If $f(x) = a_0 + a_1x + a_2x^2 + \dots$

is an even function, the right-hand side can
of x ; if $f(x)$ is odd, only odd powers occur.

For if f is even, $f(x) = f(-x)$.

But $f(-x) = a_0 - a_1x + a_2x^2 - \dots$

Subtracting 3) from 1), we have by 2)

$$0 = 2(a_1x + a_3x^3 + a_5x^5 + \dots)$$

for all x near the origin. Hence by 160,

$$a_1 = a_3 = a_5 = \dots = 0$$

The second part of the theorem is similar.

165. Example 1. $f(x) = \tan x$.

Since

$$\tan x = \frac{\sin x}{\cos x},$$

and

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

we have

$$\tan x = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots}$$

$\tan x$ is an odd function. From 1), 2) we have, clearing fractions,

$$\begin{aligned} \frac{x^3}{3!} + \frac{x^5}{5!} + \dots &= (a_1x + a_3x^3 + \dots) \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) \\ &= a_1x + \left(a_3 - \frac{a_1}{2!}\right)x^3 + \left(a_5 - \frac{a_3}{2!} + \frac{a_1}{4!}\right)x^5 \\ &\quad + \left(a_7 - \frac{a_5}{2!} + \frac{a_3}{4!} - \frac{a_1}{6!}\right)x^7 + \left(a_9 - \frac{a_7}{2!} + \frac{a_5}{4!} - \frac{a_3}{6!} + \frac{a_1}{8!}\right)x^9 + \dots \end{aligned}$$

Comparing coefficients on each side of this equation gives

$$a_1 = 1.$$

$$a_3 - \frac{a_1}{2!} = 0, \quad \therefore a_3 = \frac{1}{3!}.$$

$$a_5 - \frac{a_3}{2!} + \frac{a_1}{4!} = 0, \quad \therefore a_5 = \frac{2}{5!}.$$

$$a_7 - \frac{a_5}{2!} + \frac{a_3}{4!} - \frac{a_1}{6!} = 0, \quad \therefore a_7 = \frac{17}{315}.$$

$$a_9 - \frac{a_7}{2!} + \frac{a_5}{4!} - \frac{a_3}{6!} + \frac{a_1}{8!} = 0, \quad \therefore a_9 = \frac{62}{2835}.$$

Thus $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \dots$

lies in $\left(-\frac{\pi^*}{2}, \frac{\pi^*}{2}\right)$.

Example 2.

$$f(x) = \operatorname{cosec} x = \frac{1}{\sin x}$$

$$= \frac{1}{x \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right)} = \frac{1}{xP} = x(1 - Q)$$

Since

$$Q = 1 - \frac{1}{\sin x}$$

Hence

$$\begin{aligned}
 1 &= (a_0 + a_2x^2 + \dots) \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots \right) \\
 &= a_0 + \left(a_2 - \frac{a_0}{3!} \right) x^2 + \left(a_4 - \frac{a_2}{3!} + \frac{a_0}{5!} \right) x^4 \\
 &\quad + \left(a_6 - \frac{a_4}{3!} + \frac{a_2}{5!} - \frac{a_0}{7!} \right) x^6 + \left(a_8 - \frac{a_6}{3!} + \frac{a_4}{5!} - \frac{a_2}{7!} + \frac{a_0}{9!} \right) x^8 + \dots
 \end{aligned}$$

Comparing like coefficients gives

$$a_0 = 1.$$

$$a_2 - \frac{a_0}{3!} = 0. \quad \therefore a_2 = \frac{1}{6}.$$

$$a_4 - \frac{a_2}{3!} + \frac{a_0}{5!} = 0. \quad \therefore a_4 = \frac{7}{360}.$$

$$a_6 - \frac{a_4}{3!} + \frac{a_2}{5!} - \frac{a_0}{7!} = 0. \quad \therefore a_6 = \frac{1}{1260}.$$

Thus

$$\frac{1}{\sin x} = \frac{1}{x} + \frac{1}{6}x + \frac{7}{360}x^3 + \frac{31}{3 \cdot 7!}x^5 + \dots$$

valid in $(-\pi^*, \pi^*)$.

166. Let

$$F(x) = f_1(x) + f_2(x) + \dots$$

where

$$f_n(x) = a_{n0} + a_{n1}x + a_{n2}x^2 + \dots \quad n = 1, 2, 3, \dots$$

Let the adjoint series

$$\alpha_{n0} + \alpha_{n1}\xi + \alpha_{n2}\xi^2 + \dots$$

converge for $\xi = R$ and have ϕ_n as sums for the

Let

$$\Phi = \phi_1 + \phi_2 + \dots$$

GENERAL THEORY

F converges uniformly in \mathfrak{A} . For as $|x| \leq \xi$,

$$\begin{aligned} |f_n(x)| &\leq \alpha_{n0} + \alpha_{n1}|x| + \alpha_{n2}|x|^2 + \dots \\ &\leq \alpha_{n0} + \alpha_{n1}\xi + \alpha_{n2}\xi^2 + \dots = \phi_n. \end{aligned}$$

We now apply 136, 2 as $\sum \phi_n$ is convergent for $\xi = R$.

To prove the latter part of the theorem we observe that

$$\begin{aligned} &\alpha_{10} + \alpha_{11}R + \alpha_{12}R^2 + \dots \\ &+ \alpha_{20} + \alpha_{21}R + \alpha_{22}R^2 + \dots \\ &+ \dots \dots \dots \end{aligned}$$

is convergent, since summing it by rows it has Φ as sum. The double series 1) converges absolutely for $|x| \leq \xi$, by 130, 1. Thus the series 1) may be summed by columns by 130, 1. $F(x)$ as sum, since 1) has F as sum on summing by rows.

167. Example.

$$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{1+a^n x} = \sum f_n(x) \quad a > 1.$$

This series we have seen converges in $\mathfrak{A} = (0, b)$, b positive and arbitrarily large.

Since it is impossible to develop the $f_n(x)$ in a power series about the origin which will have a common interval of convergence, we develop F in a power series about $x_0 > 0$.

We have

$$\frac{1}{1+a^n x} = \frac{1}{1+a^n x_0} + \frac{1}{1+a^n(x-x_0)}$$

Thus F give rise to the double series

$$\begin{aligned} D = & A'_{00} + A'_{01}(x - x_0) + A'_{02}(x - x_0)^2 + \dots \\ & + A'_{10} + A'_{11}(x - x_0) + A'_{12}(x - x_0)^2 + \dots \\ & + \dots \end{aligned}$$

where

$$A'_{n\kappa} = \frac{(-1)^n}{n!} A_{n\kappa}.$$

The adjoint series to $f_n(x)$ is, setting $\xi = |x - x_0|$,

$$\phi_n(\xi) = \frac{1}{n!} \left(\frac{1}{1 + a^n x_0} + \frac{a^n \xi}{(1 + a^n x_0)^2} + \frac{a^{2n} \xi^2}{(1 + a^n x_0)^3} + \dots \right)$$

This is convergent if

$$\frac{a^n \xi}{1 + a^n x_0} < 1 \quad \text{or if} \quad \xi < x_0,$$

that is, if

$$0 < x < 2x_0,$$

For any x such that $x_0 \leq x < 2x_0$, $\xi = x - x_0$.

Then for such an x

$$\phi_n = \frac{1}{n!} \frac{1}{1 + a^n(2x_0 - x)}$$

and the corresponding series

$$\Phi = \sum \phi_n$$

is evidently convergent, since $\phi_n < \frac{1}{n!}$.

This method enables us thus to determine 3) such that this series when put in 2) satisfies the relation. We shall call the series 3) which satisfies the relation the values given in 6), the *inverse series* for 2).

Suppose now the inverse series 3) can be found. Can we say it satisfies 2) for values of u other than $u=0$? The answer is, Yes. For by 161, 3, we may construct a double series which results by replacing in 2)

$$x, \quad x^2, \quad x^3, \quad \dots$$

by their values in 3), 4). But when we do this, the series 2) goes over into the right side of 5), and the left side $= 0$ by 6) except the first.

We have therefore only to show that the series 2) converges for some $u \neq 0$. To show this we need only show that 2) converges for $x = 1$. Then $a_n \neq 0$, and

$$|a_n| < \text{some } \alpha \quad n = 2, 3, \dots$$

On the other hand, the relations 6) show

$$c_n = f_n(a_2, a_3, \dots, a_n)$$

is a polynomial with integral positive coefficients. To replace a_2, a_3, \dots by α , getting

$$\gamma_n = f_n(\alpha, \alpha, \dots, \alpha).$$

Obviously

$$|c_n| < \gamma_n.$$

Let us now replace all the a 's in 2) by α . The resulting series

$$\begin{aligned} u &= x - \alpha x^2 + \alpha^2 x^3 - \alpha^3 x^4 + \dots \\ &= x - \frac{\alpha x^2}{1 - \alpha x}. \end{aligned}$$

Let us set $1 - 2(2\alpha + 1)u + u^2 = 1 - v$. For u near $u = 0$, $v < 1$. Then by the Binomial Theorem

$$\sqrt{1 - v} = 1 + d_1 v + d_2 v^2 + \dots$$

Replacing v by its value in \mathcal{A} , this becomes a power series in u which holds for u near the origin, by 161, 3. Thus 14) shows that x can be developed in a power series about the origin. Thus x converges about $u = 0$. But then by 10) the inverse series converges in some interval about $u = 0$.

We may, therefore, state the theorem:

Let $u = b + b_1 x + b_2 x^2 + b_3 x^3 + \dots$, $b_1 \neq 0$,

converge about the point $x = 0$. Then this relation defines x as a function of u which admits the development

$$x = (u - b) \left\{ \frac{1}{b_1} + a_1(u - b) + a_2(u - b)^2 + \dots \right\}$$

about the point $u = b$. The coefficients a may be obtained from the method of undetermined coefficients.

Example. We saw that

$$u = \log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$$

If we set

$$u = x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

we have $a_2 = -\frac{1}{2}$, $a_3 = \frac{1}{3}$, $a_4 = -\frac{1}{4}$, $a_5 = \frac{1}{5}$...

If we invert 2), we get

$$x = u + c_2 u^2 + c_3 u^3 + \dots$$

Thus we get

$$x = u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!} + \dots$$

But from 1) we have

$$1 + x = e^u = 1 + \frac{u}{1!} + \frac{u^2}{2!} + \dots$$

which agrees with 3).

Taylor's Development

169. 1. We have seen, I, 409, that if the first n derivatives are continuous in \mathfrak{A}

$$\begin{aligned} f(a+h) = & f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \dots \\ & + \frac{h^n}{n!} f^{(n)}(a + \theta h) \end{aligned}$$

where

$$a \leq a + h \leq b, \quad 0 < \theta < 1.$$

Consider the infinite power series in h .

$$T = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \dots$$

We call it the *Taylor's series* belonging to f . The terms of 1) and 2) are the same. Let

$$R_n = \frac{h^n}{n!} f^{(n)}(a + \theta h)$$

We observe that R_n is a function of n , the variable θ being between 0 and 1.

Then

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \dots$$

The above theorem is called *Taylor's theorem*; (5) is the development of $f(x)$ in the interval \mathfrak{A} .

Another form of (5) is

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$$

When the point a is the origin, that is, when $a=0$, (5) gives

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots$$

This is called *Maclaurin's development* and the series is called *Maclaurin's series*. It is of course only a special case of Taylor's development.

2. Let us note the content of Taylor's Theorem.

1° $f(x)$ can be developed in this form in the interval \mathfrak{A} ;

$\mathfrak{A} = (a, b)$;

2° if $f(x)$ and all its derivatives are known at the point

$x=a$;

then the value of f and all its derivatives are known at every point x within \mathfrak{A} .

The remarkable feature about this result is that the condition 1° requires a knowledge of the values of $f(x)$ and all its derivatives at the point a (or $a+\delta$) as small as we please. Since the value of a real variable takes on in a part of its interval all the values that $f(x)$ may have in this interval (or $a+\delta$), the condition 1° must impose a con-

For differentiating 1) n times, we get

$$f^{(n)}(x) = n! a_n + \frac{n+1!}{2!} a_{n+1}(x - a) + \dots$$

Setting here $x = a$, we get 2).

The above theorem says that if $f(x)$ can be developed in power series about $x = a$, this series can be no other series.

171. 1. *Let $f^{(n)}(x)$ exist and be numerical constant M for all $a \leq x \leq b$, and for every n . Then $f(x)$ can be developed in Taylor's series for all x in (a, b) .*

For then

$$|R_n| < M \frac{h^n}{n!}.$$

But obviously

$$\lim_{n \rightarrow \infty} \frac{h^n}{n!} = 0.$$

2. The application of the preceding theorem to the functions

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

which are valid for every x .

Since

$$a^x = e^{x \log a}, \quad a > 0,$$

we have

$$a^x = 1 + x \frac{\log a}{1!} + x^2 \frac{\log^2 a}{2!} + \dots$$

We introduce the auxiliary function defined over (a, b) .

$$g(t) = f(t) + f'(t)(x-t) + \dots + f^{(n-1)}(t) \frac{(x-t)^{n-1}}{n-1!}.$$

Then

$$g(x) = f(x)$$

and

$$g(a) = f(a) + f'(a)(x-a) + \dots + f^{(n-1)}(a) \frac{(x-a)^{n-1}}{n-1!}.$$

Hence

$$R_n = g(x) - g(a).$$

If we differentiate 1), we find the terms cancel in pairs, leaving

$$g'(t) = \frac{(x-t)^{n-1}}{n-1!} f^{(n)}(t).$$

We apply now Cauchy's theorem, I, 448, introducing another arbitrary auxiliary function $G(x)$ which satisfies the conditions of that theorem.

Then

$$\frac{g(x) - g(a)}{G(x) - G(a)} = \frac{g'(c)}{G'(c)}, \quad a < c < x.$$

Using 2) and 3), we get, since $x = a + h$,

$$R_n = \frac{G(a+h) - G(a)}{G'(a+\theta h)} \frac{h^{n-1}(1-\theta)^{n-1}}{n-1!} f^{(n)}(a+\theta h)$$

where $0 < \theta < 1$.

2. If we set

$$G(x) = (b-x)^\mu, \quad \mu \neq 0,$$

we have a function which satisfies our conditions. Then 4) becomes

$$R_n = \frac{h^n(1-\theta)^{n-\mu}}{n!} f^{(n)}(a+\theta h)$$

For $\mu = n$, we get from 5)

$$R_n = \frac{h^n}{n!} f^{(n)}(a + \theta h),$$

which is Lagrange's formula already obtained.

173. 1. We consider now the development of

$$(1+x)^\mu \quad x > -1, \quad \mu \text{ arbitrary}$$

The corresponding Taylor's series is

$$T = 1 + \frac{\mu}{1}x + \frac{\mu \cdot \mu - 1}{1 \cdot 2}x^2 + \frac{\mu \cdot \mu - 1 \cdot \mu}{1 \cdot 2 \cdot 3}x^3 + \dots$$

We considered this series in 99, where we saw

T converges for $|x| < 1$ and diverges for $|x| > 1$.

When $x = 1$, T converges only when $\mu > 0$.
 T converges only when $\mu \geq 0$.

We wish to know when

$$(1+x)^\mu = 1 + \frac{\mu}{1}x + \frac{\mu \cdot \mu - 1}{1 \cdot 2}x^2 + \dots$$

The cases when T diverges are to be thrown aside.
 Consider in succession the cases that T converges.
 Investigate when $\lim R_n = 0$.

Case 1°. $0 < |x| < 1$. It is convenient to take the
 form of the remainder. This gives

$$\begin{aligned} R_n &= (1-\theta)^{n+1} \frac{\mu \cdot \mu - 1 \cdot \dots \mu - n + 1}{1 \cdot 2 \cdot \dots n} x^n \\ &= \frac{1}{n} S_n U_n W_n, \end{aligned}$$

TAYLOR'S DEVELOPMENT

In U_n ,

$$|1 + \theta x| < 1 + |x|,$$

which is finite. Hence U_n is $<$ some constant M .

To show that $\lim S_n = 0$, we make use of the fact that T converges for the values of x under consideration. every μ

$$\lim \frac{\mu \cdot \mu - 1 \cdot \dots \mu - n + 2}{1 \cdot 2 \cdot \dots n - 1} x^{n-1} = 0,$$

since the limit of the n^{th} term of a convergent series this formula replace μ by $\mu - 1$, then

$$\lim \frac{\mu - 1 \cdot \mu - 2 \cdot \dots \mu - n + 1}{1 \cdot 2 \cdot \dots n - 1} x^{n-1} = \lim \frac{S_n}{\mu x} = 0.$$

Hence

$$\lim S_n = 0.$$

Thus

$$\lim R_n = 0.$$

Hence 1) is valid for $|x| < 1$.

Case 2. $x = 1$, $\mu > -1$. We employ here Lagrange's the remainder, which gives

$$\begin{aligned} R_n &= \frac{\mu \cdot \mu - 1 \cdot \dots \mu - n + 1}{1 \cdot 2 \cdot \dots n} (1 + \theta)^{\mu-n} \\ &= U_n W_n, \end{aligned}$$

setting

$$U_n = \frac{\mu \cdot \mu - 1 \cdot \dots \mu - n + 1}{1 \cdot 2 \cdot \dots n},$$

$$W_n = (1 + \theta)^{\mu-n}.$$

Consider W_n . Since n increases without limit, $\mu - n$

Case 3. $x = -1$, $\mu \geq 0$. We use here for μ Roche form of the remainder (172, 5). We set a

$$\begin{aligned} R_n &= (-1)^n \frac{(1-\theta)^n}{n-1!} \mu^{\frac{n}{\mu}} \mu \cdot \mu - 1 \cdot \dots \mu - n \\ &= (-1)^n \frac{\mu - 1 \cdot \mu - 2 \cdot \dots \mu - n + 1}{1 \cdot 2 \cdot \dots n - 1}. \end{aligned}$$

Applying I, 143, we see that $\lim R_n = 0$.

Hence 1) is valid here if $\mu > 0$.

When $\mu = 0$ equation 1) is evidently true reduce to 1.

Summing up, we have the theorem :

The development of $(1+x)^\mu$ in Taylor's series is valid for $|x| < 1$ for all μ . When $x = +1$ it is necessary that $\mu > 0$. When $x = -1$ it is necessary that $\mu \geq 0$.

2. We note the following formulas obtained for $x = 1$ and -1 .

$$2^\mu = 1 + \frac{\mu}{1} + \frac{\mu \cdot \mu - 1}{1 \cdot 2} + \frac{\mu \cdot \mu - 1 \cdot \mu - 2}{1 \cdot 2 \cdot 3} + \dots$$

$$0 = 1 - \frac{\mu}{1} + \frac{\mu \cdot \mu - 1}{1 \cdot 2} - \frac{\mu \cdot \mu - 1 \cdot \mu - 2}{1 \cdot 2 \cdot 3} + \dots$$

174. 1. We develop now $\log(1+x)$. Taylor's series is

$$T = 1 + \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

We saw, 89, Ex. 2, that T converges with

TAYLOR'S DEVELOPMENT

Let $-1 < x < 0$. We use here Cauchy's remainder, setting $x = -\xi$, $0 < \xi < 1$,

$$\begin{aligned} |R_n| &= \xi^n \cdot \frac{1}{1 - \theta\xi} \cdot \left(\frac{1 - \theta}{1 - \theta\xi} \right)^{n-1} \\ &= S_n U_n W_n, \end{aligned}$$

if

$$S_n = \xi^n,$$

$$U_n = \frac{1}{1 - \theta\xi},$$

$$W_n = \left(\frac{1 - \theta}{1 - \theta\xi} \right)^{n-1}.$$

Evidently $\lim S_n = 0$.

Also

$$U_n < \frac{1}{1 - \xi}.$$

Finally

$$\lim W_n = 0 \quad \text{since} \quad \frac{1 - \theta}{1 - \theta\xi} < 1.$$

We can thus sum up in the theorem :

Taylor's development of $\log(1+x)$ is valid when $|x| < 1$ or $x = 1$. That is, for such values of x

$$\log(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

2. We note the following special case :

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \log 2.$$

The series on the left we have already met with.

175. We add for completeness the development of other functions for which it can be shown that $\lim R_n = 0$.

$$\text{Example 1. } e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$$

176. We wish now to call attention to some fallacies which are prevalent regarding the development of a function in Taylor's series.

Criticism 1. It is commonly supposed, that if a function $f(x)$ is convergent

$$f(x) = T,$$

That this is not always true we proceed to give a few examples.

Example 1. For $f(x)$ take Cauchy's function

$$C(x) = \lim_{n \rightarrow \infty} e^{-x^2 n^{\frac{1}{n}}}$$

$$\text{For } x \neq 0 \quad C(x) = e^{-x^2} \quad ; \quad \text{for } x = 0 \quad C(x) = 1$$

$$1^{\circ} \text{ derivative.} \quad \text{For } x \neq 0, \quad C'(x) = -\frac{2}{x^2} e^{-x^2}$$

$$\text{For } x = 0, \quad C'(0) = \lim_{h \rightarrow 0} \frac{C(h) - C(0)}{h}$$

$$2^{\circ} \text{ derivative.} \quad x \neq 0, \quad C''(x) = C'(x) \left\{ \begin{array}{l} \end{array} \right.$$

$$x = 0, \quad C''(0) = \lim_{h \rightarrow 0} \frac{C''(h) - C''(0)}{h}$$

$$3^{\circ} \text{ derivative.} \quad x \neq 0, \quad C'''(x) = C''(x)$$

$$x = 0, \quad C'''(0) = \lim_{h \rightarrow 0} \frac{C'''(h) - C'''(0)}{h}$$

TAYLOR'S DEVELOPMENT

That is, T is convergent for every x , but vanishes identically. It is thus obvious that $U(x)$ cannot be developed about the origin in Taylor's series.

Example 2. Because the Taylor's series about the origin belonging to $U(x)$ vanishes identically, the reader may be inclined to regard this example with suspicion, yet without reason. Let us consider therefore the following function,

$$f(x) = U(x) + e^x = U(x) + g(x).$$

Then $f(x)$ and its derivatives of every order are continuous

$$\text{Since } f^{(n)}(x) = U^{(n)}(x) + g^{(n)}(x) \qquad n = 1, 2, \dots$$

$$\text{and } U^{(n)}(0) = 0$$

$$\text{we have } f^{(n)}(0) = 1.$$

Hence Taylor's development for $f(x)$ about the origin is

$$T = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

This series is convergent, but it does not converge to the right value since

$$T = e^x.$$

177. 1. *Example 3.* The two preceding examples leave nothing to be desired from the standpoint of rigor and simplicity. They involve, however, a function, namely, $U(x)$, which is not defined in the usual way; it is therefore interesting to have examples of functions defined in one of the ordinary everyday ways, e.g. as infinite series. Such examples have been given by *Sieringheim*.

The Taylor's series about the origin for f

$$T(x) = \sum_{\lambda=0}^{\infty} \frac{x^{\lambda}}{\lambda!} F^{(\lambda)}(0) \quad ; \quad \lambda! =$$

and by 2)

$$\frac{F^{(\lambda)}(0)}{\lambda!} = (-1)^{\lambda} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

Hence

$$T(x) = \sum_{\lambda=0}^{\infty} \frac{(-1)^{\lambda}}{e^{a\lambda}} x^{\lambda} = \sum$$

As $t_{\lambda} \geq 0$ and $\lim t_{\lambda} = 0$, $t_{\lambda+1} < t_{\lambda}$, this so for any x in \mathfrak{A} . Hence T converges in \mathfrak{A} .

2. Readers familiar with the elementary properties of a complex variable will know well that our Taylor's series T given in §3 for a function F in any interval \mathfrak{A} , however small, is the Taylor's series of an analytic function for which the origin is a singular point, since F has the poles $-\frac{1}{a}$ and the limiting point is 0.

3. To show by elementary means that T is the Taylor's series of F about the origin in a Taylor's series we prove now, however, with *Pringsheim*:

If we take $a \geq \left(\frac{e+1}{e-1}\right)^2 = 4.68 \dots$, T converges throughout any interval $\mathfrak{A} = (0, b)$, however small.

We show 1° that if $F(x) = T(x)$ throughout $\mathfrak{B} = (0, 2b^*)$.

In fact let $0 < x_0 < b$.

By 1.61, 4 we can develop T about x_0 .

TAYLOR'S DEVELOPMENT

equal for $0 \leq x < 2x_0$. As we can take x_0 as near b as we like, $F = T$ in \mathfrak{B} .

By repeating the operation often enough, we can show $F = T$ in any interval $(0, B)$ where $B > 0$ is arbitrarily large.

To prove our theorem we have now only to show that for some one $x > 0$.

Since

$$F(x) = \left(\frac{1}{1+x} - \frac{1}{1+ax} \right) + \left(\frac{1}{2!} \frac{1}{1+a^2x} - \frac{1}{3!} \frac{1}{1+a^3x} \right) + \dots$$

we have

$$F(x) > \frac{1}{1+x} - \frac{1}{1+ax} = G(x).$$

On the other hand

$$T(x) = \frac{1}{e} - \left(\frac{x}{e^a} - \frac{x^2}{e^{a^2}} \right) - \left(\frac{x^3}{e^{a^3}} - \frac{x^4}{e^{a^4}} \right) - \dots$$

Hence

$$T(x) < \frac{1}{e}.$$

To find a value of x for which $G \geq \frac{1}{e}$ take $x = a^{-\frac{1}{2}}$.
value of x

$$G = \frac{a^{\frac{1}{2}} - 1}{a^{\frac{1}{2}} + 1}.$$

Observe that G considered as a function of a is an increasing function. For

$$a = \left(\frac{e+1}{e-1} \right)^2, \quad G = \frac{1}{e}.$$

Hence $F > T$ for $x > a^{-\frac{1}{2}}$.

178. Criticism 2. It is commonly thought if $f(x)$ and its derivatives of every order are continuous in an interval, then the corresponding Taylor's series is convergent in that interval.

That this is not always so is shown by the following

Taylor's series about the origin is

$$T = \sum_{\lambda=0}^{\infty} \frac{1}{\lambda!} (-1)^{\lambda} (e^{a^{\lambda}} + e^{-a^{\lambda}})$$

since

$$F^{(\lambda)}(0) = (-1)^{\lambda} \lambda! \sum_{n=0}^{\infty} \frac{a^{2n\lambda}}{(2n)!}$$

The series T is divergent for $x > 0$, as is

179. Criticism 3. It is commonly thought that if all the derivatives vanish for a certain value of x , then $f(x)$ vanishes identically. One reason

The development of $f(x)$ about $x = a$ is

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$$

As f and all its derivatives vanish at a , the

$$\begin{aligned} f(x) &= 0 + 0 \cdot (x-a) + 0 \cdot (x-a)^2 + \dots \\ &= 0 \text{ whatever } x \text{ is.} \end{aligned}$$

There are two tacit assumptions which in

First, one assumes because f and all its derivatives are finite at $x = a$, that therefore $f(x)$ has a Taylor's series. An example to the contrary is $C(x)$. We have seen that $C(x)$ and all its derivatives are 0 at $x = 0$, yet $C(x)$ is not identically 0; in fact, viz. at $x = 0$.

Secondly, suppose $f(x)$ were developable in a certain interval $\mathfrak{A} = (a-h, a+h)$. Then f is not 0 out \mathfrak{A} , but we cannot infer that it is there. In fact, from Dirichlet's definition of a function,

Since Taylor's series T is a power series, it converges not only in \mathfrak{A} , but also within $\mathfrak{B} = (2a - b, a)$. It is commonly supposed that $f(x) = T$ also in \mathfrak{B} . A moment's reflection shows such an assumption is unjustified without further conditions on f .

2. *Example.* We construct a function by the method contained in I, §33, viz.

$$f(x) = \lim_{n \rightarrow \infty} \frac{(1+x)^n \cos x + 1 + \sin x}{1 + (1+x)^n}.$$

$$\begin{aligned} \text{Then } f(x) &= \cos x, & \text{in } \mathfrak{A} &= (0, 1) \\ &= 1 + \sin x, & \text{within } \mathfrak{B} &= (0, -1). \end{aligned}$$

We have therefore as a development in Taylor's series in \mathfrak{A} ,

$$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = T.$$

It is obviously not valid within \mathfrak{B} , although T converges in \mathfrak{B} .

3. We have given in 1) an arithmetical expression for $f(x)$. Our example would have been just as conclusive if we had

$$\begin{aligned} \text{Let } f(x) &= \cos x & \text{in } \mathfrak{A}, \\ \text{and } f(x) &= 1 + \sin x & \text{within } \mathfrak{B}. \end{aligned}$$

181. 1. *Criticism 5.* The following error is sometimes made. Suppose Taylor's development

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$$

valid in $\mathfrak{A} = (a < b)$.

It may happen that T is convergent in a larger interval $\mathfrak{B} = (a < B)$.

One must not therefore suppose that 1) is also valid in \mathfrak{B} .

2. *Example.*

182. Let $f(x)$ have finite derivative $f'(x)$ in the interval $\mathfrak{A} = (a < b)$. In order that $f(x)$ can be developed in a power series

$$f(x) = f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots$$

valid in the interval \mathfrak{A} we saw that it is necessary that

$$\lim_{n \rightarrow \infty} R_n = 0.$$

But R_n is not only a function of the index n of the unknown variable θ which lies within the interval \mathfrak{A} and is a function of n and h .

Pringsheim has shown how the above condition can be replaced by the following one in which θ is an independent variable.

For the relation 1) to be valid for all h and θ it is necessary and sufficient that Cauchy's form of the remainder be bounded.

$$R_n(h, \theta) = \frac{(1-\theta)^{n-1} h^n}{n-1!} f^{(n)}(a + \theta h)$$

the h and θ being independent variables, considered in the rectangle D whose points (h, θ) satisfy the conditions

$$0 \leq h \leq H$$

$$0 \leq \theta \leq 1.$$

1° It is sufficient. For then there exists a number ϵ such that

$$|R_n(h, \theta)| \leq \epsilon$$

for every point (h, θ) of D .

Let us fix h ; then $|R_n| \leq \epsilon$ for every θ in the interval $[0, 1]$.

TAYLOR'S DEVELOPMENT

The demonstration depends upon the fact that $R_n(h)$, times the n^{th} term $f_n(\alpha, k)$ of the development of $f'(x)$ at point $a + \alpha$. In fact let $h = \alpha + k$. Then by 158

$$f'(a + h) = f'(a + \alpha + k) = f'(a + \alpha) + \dots + \frac{k^{n-1}}{n-1!} f^{(n)}(a + \alpha)$$

whose n^{th} term is

$$f_n(\alpha, k) = \frac{k^{n-1}}{n-1!} f^{(n)}(a + \alpha).$$

Let $\alpha = \theta h$, then

$$R_n(h, \alpha) = hf_n(\alpha, k)$$

as stated.

The image Δ_0 , of D_0 is the half of a square of side h_0 , be diagonal.

To show that R_n converges uniformly to 0 in D_0 we have to show that

$$f_n(\alpha, k) \doteq 0 \quad \text{uniformly in } \Delta_0$$

To this end we have from 1) for all t in \mathfrak{A}

$$f'(a + t) = f'(a) + tf''(a) + \frac{t^2}{2!} f'''(a) + \dots$$

Its adjoint

$$G(t) = |f'(a)| + t|f''(a)| + \dots$$

also converges in \mathfrak{A} .

By 161, 4 we can develop 4) about $t = \alpha$, which gives

$$G(\alpha, k) = G(\alpha) + kG'(\alpha) + \dots + \frac{k^{n-1}}{n-1!} G^{(n-1)}(\alpha) + \dots$$

But obviously $G(\alpha, k)$ is continuous in Δ_0 , and evidently

To prove 6) we have from 1)

$$f^{(n)}(a + \alpha) = f^{(n)}(a) + \alpha f^{(n+1)}(a) + \frac{\alpha^2}{2} f^{(n+2)}(a) + \dots$$

and from 4)

$$G^{(n-1)}(\alpha) = [f^{(n)}(a)] + \alpha [f^{(n+1)}(a)] + \dots$$

The comparison of 7), 8) proves 6).

Circular and Hyperbolic Functions

183. 1. We have defined the circular functions as the functions of certain lines; from this definition their properties may be deduced as is shown in trigonometry.

From this geometric definition we have derived the analytical expression for these functions. In particular,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

valid for every x .

As an interesting and instructive exercise we propose now to develop some of the properties of these functions purely from their definition as infinite series. We shall consider these series respectively S and C .

Let us also define $\tan x = \frac{\sin x}{\cos x}$, $\sec x = \frac{1}{\cos x}$.

2. To begin, we observe that both S and C are

Since S involves only odd powers of x , and C only even powers,

$\sin x$ is an *odd*, $\cos x$ is an *even* function.

Since S and C are power series which converge for every x they have derivatives of every order. In particular

$$\frac{dS}{dx} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = C.$$

$$\frac{dC}{dx} = -\frac{x}{1} + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \dots = -S.$$

Hence

$$\frac{d \sin x}{dx} = \cos x, \quad \frac{d \cos x}{dx} = -\sin x.$$

To get the *addition theorem*, let an index as x, y attached to C indicate the variable which occurs in the series. Then

$$S_x C_y = x - \left(\frac{x^3}{3!} + \frac{xy^2}{2!} \right) + \left(\frac{x^5}{5!} + \frac{x^3 y^2}{3! 2!} + \frac{xy^4}{1! 4!} \right)$$

$$- \left(\frac{x^7}{7!} + \frac{x^5 y^2}{5! 2!} + \frac{x^3 x^4}{3! 4!} + \frac{xy^6}{6!} \right) + \dots$$

$$C_x S_y = y - \left(\frac{y^3}{3!} + \frac{x^2 y}{2!} \right) + \left(\frac{y^5}{5!} + \frac{y^3 x^2}{3! 2!} + \frac{x^4 y}{4! 1!} \right)$$

$$- \left(\frac{y^7}{7!} + \frac{y^5 x^2}{5! 2!} + \frac{y^3 x^4}{3! 4!} + \frac{y x^6}{1! 6!} \right) + \dots$$

Adding,

$$C_y + C_x S_y = x + y - \frac{1}{3!} \left\{ x^3 + \binom{3}{1} x^2 y + \binom{3}{1} xy^2 + y^3 \right\}$$

$$+ \frac{1}{5!} \left\{ x^5 + \binom{5}{1} x^4 y + \binom{5}{2} x^3 y^2 + \binom{5}{3} x^2 y^3 + \binom{5}{4} xy^4 + y^5 \right\} + \dots$$

7. We can get now the important relation

$$\sin^2 x + \cos^2 x = 1$$

directly from the addition theorem. Let $x = 0$. We have

$$\begin{aligned} S^2 &= \frac{x^2}{1} - x^4 \left(\frac{1}{3!} + \frac{1}{2!} \right) + x^6 \left(\frac{1}{5!} + \frac{1}{3!} \right) \\ &\quad - x^8 \left(\frac{1}{7!} + \frac{1}{3!5!} + \frac{1}{5!3!} + \frac{1}{7!} \right) \\ C^2 &= 1 - x^2 \left(\frac{1}{2!} + \frac{1}{2!} \right) + x^4 \left(\frac{1}{4!} + \frac{1}{2!} \right) \\ &\quad - x^6 \left(\frac{1}{6!} + \frac{1}{4!2!2!} + \frac{1}{4!} + \frac{1}{6!} \right) \\ &\quad + x^8 \left(\frac{1}{8!} + \frac{1}{6!2!} + \frac{1}{4!4!} + \frac{1}{6!} \right) \end{aligned}$$

Hence

$$\begin{aligned} S^2 + C^2 &= 1 - \frac{x^2}{2!} \left(1 - \binom{2}{1} + 1 \right) + \frac{x^4}{4!} \left(1 - \binom{4}{1} + \binom{4}{2} - 1 \right) \\ &\quad - \frac{x^6}{6!} \left(1 - \binom{6}{1} + \binom{6}{2} - \binom{6}{3} + \binom{6}{4} - 1 \right) + \dots \end{aligned}$$

Now by I, 96,

$$1 - \binom{m}{1} + \binom{m}{2} - \binom{m}{3} + \dots$$

Thus

$$S^2 + C^2 = \sin^2 x + \cos^2 x = 1$$

CIRCULAR AND HYPERBOLIC FUNCTIONS

Similarly we see $y = 1$ for $x = 0$. Γ crosses the y -axis and is parallel to the x -axis.

Since

$$S = x\left(1 - \frac{x^2}{2 \cdot 3}\right) + \frac{x^5}{5!}\left(1 - \frac{x^2}{6 \cdot 7}\right) + \dots$$

and each parenthesis is positive for $0 < x^2 < 6$,

$$\sin x > 0 \quad \text{for } 0 < x < \sqrt{6} = 2.449 \dots$$

Since

$$C = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}\left(1 - \frac{x^2}{5 \cdot 6}\right) + \frac{x^8}{8!}\left(1 - \frac{x^2}{9 \cdot 10}\right) + \dots$$

we see

$$\cos x > 0 \quad \text{for } 0 < x < \sqrt{2} = 1.414 \dots$$

Since

$$C = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\left(1 - \frac{x^2}{7 \cdot 8}\right) - \frac{x^{10}}{10!}\left(1 - \frac{x^2}{11 \cdot 12}\right) - \dots$$

$$\cos x < 0 \quad \text{for } x = 2.$$

Since $D_x \cos x = -\sin x$ and $\sin x > 0$ for $0 < x < \sqrt{6}$, $\cos x$ is a decreasing function for these values of x . As it is continuous and > 0 for $x = \sqrt{2}$, but < 0 for $x = 2$, $\cos x$ vanishes and only once in $(\sqrt{2}, 2)$.

This root, uniquely determined, of $\cos x$ we denote by $\frac{\pi}{2}$. As a first approximation, we have

$$\sqrt{2} < \frac{\pi}{2} < 2.$$

From 4) we have $\sin^2 \frac{\pi}{2} = 1$. As we saw $\sin x > 0$ for $0 < x < \frac{\pi}{2}$, we have

$$\sin \frac{\pi}{2} = +1.$$

Thus $\sin x$ increases constantly from 0 to 1 while $\cos x$ decreases from 1 to 0. At $x = \frac{\pi}{2}$ we have $\sin x = 1$ and $\cos x = 0$.

Knowing how $\sin x$, $\cos x$ march in I_1
how they march in $I_2 = \left(\frac{\pi}{2}, \pi\right)$.

From the addition theorem,

$$\sin(\pi + x) = -\sin x, \quad \cos(\pi + x) = -\cos x$$

Knowing how $\sin x$, $\cos x$ march in $(0, \pi)$
us about their march in $(0, 2\pi)$.

The addition theorem now gives

$$\sin(x + 2\pi) = \sin x, \quad \cos(x + 2\pi) = \cos x$$

Thus the functions $\sin x$, $\cos x$ are periodic

The graph of $\sin x$, $\cos x$ for negative x
recalling that $\sin x$ is odd and $\cos x$ is even

10. As a first approximation of π we find

$$\sqrt{2} < \frac{\pi}{2} < 2.$$

By the aid of the development given 15)

$$\operatorname{arctg} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

we can compute π as accurately as we please

In fact, from the addition theorem we deduce

$$\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}, \quad \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

Hence

$$\tan \frac{\pi}{4} = 1.$$

This in 5) gives *Leibnitz's formula*,

11. To get a more rapid means of computation, we make use of the addition theorem.

To start with, let

$$\alpha = \operatorname{arctg} \frac{1}{5}.$$

Then 5) gives

$$\alpha = \frac{1}{5} - \frac{1}{3} \frac{1}{5^3} + \frac{1}{5} \frac{1}{5^5} - \frac{1}{7} \frac{1}{5^7} + \dots$$

rapidly converging series.

The error E_n committed in breaking off the summation at the n th term is

$$E_n < \frac{1}{2n-1} \frac{1}{5^{2n-1}}.$$

By virtue of the formula for duplicating the argument

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha},$$

we have

$$\tan 2\alpha = \frac{5}{12}.$$

Similarly

$$\tan 4\alpha = \frac{120}{119}.$$

Let

$$\beta = 4\alpha - \frac{\pi}{4}.$$

The addition theorem gives

$$\tan \beta = \frac{\tan 4\alpha - 1}{1 + \tan 4\alpha} = \frac{1}{239}.$$

Then 5) gives

$$\beta = \frac{1}{239} - \frac{1}{3} \frac{1}{239^3} + \frac{1}{5} \frac{1}{239^5} - \dots$$

so a very rapidly converging series.

184. *The Hyperbolic Functions.* Circular functions are the hyperbolic functions by the equations

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

$$\operatorname{sech} x = \frac{1}{\cosh x}, \quad \operatorname{cosech} x = \frac{1}{\sinh x}.$$

Since

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

we have

$$\sinh x = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

valid for every x . From these equations

$$\sinh(-x) = -\sinh x; \quad \cosh(-x) = \cosh x$$

$$\sinh 0 = 0, \quad \cosh 0 = 1.$$

$$\frac{d}{dx} \sinh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots = \cosh x.$$

THE HYPERGEOMETRIC FUNCTION

Evidently from 3), 4)

$$\lim_{x \rightarrow -\infty} \sinh x = -\infty, \quad \lim_{x \rightarrow +\infty} \cosh x = +\infty.$$

At $x=0$, $\cosh x$ has a minimum, and $\sinh x$ cuts the x -axis at 45° .

For $x > 0$, $\cosh x > \sinh x$ since

$$e^x + e^{-x} > e^x - e^{-x}.$$

The two curves approach each other asymptotically as $x \rightarrow \infty$. For the difference of their ordinates is e^{-x} which $\rightarrow 0$ as $x \rightarrow \infty$.

The addition theorem is easily obtained from that of e^x .

$$\begin{aligned} \sinh x \cosh y &= \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} \\ &= \frac{1}{4}(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}). \end{aligned}$$

$$\text{Similarly } \cosh x \sinh y = \frac{1}{4}(e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y}).$$

Hence

$$\sinh x \cosh y + \cosh x \sinh y = \frac{1}{2}(e^{x+y} - e^{-(x+y)}) = \sinh(x+y).$$

Similarly we find

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y.$$

In the same way we may show that

$$\cosh^2 x - \sinh^2 x = 1.$$

The Hypergeometric Function

185. This function, although known to Wallis, Euler, and other earlier mathematicians, was first studied in detail by Gauss. It may be defined by the following power series in x :

$$F(\alpha, \beta, \gamma; x) = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha \cdot \alpha + 1 \cdot \beta \cdot \beta + 1}{1 \cdot 2 \cdot \gamma \cdot \gamma + 1} x^2 + \dots$$

The convergence of the series F was the main result obtained there is that F converges for $|x| < 1$, whatever values the parameters α, β, γ have, γ a negative integer or zero.

186. For special values of the parameters the elementary functions in the following cases :

1. If α or β is a negative integer $-\alpha$ or $-\beta$ of degree n .

$$2. \quad F(1, 1, 2; -x) = \frac{1}{x} \log(1+x)$$

$$\text{For} \quad F(1, 1, 2, -x) = 1 - \frac{x}{2} + \frac{x^2}{3}$$

$$\text{Also} \quad \log(1+x) = x(1 - \frac{x}{2} + \frac{x^2}{3} - \dots)$$

The relation 1) is now obvious.

Similarly we have

$$F(1, 1, 2; x) = -\frac{1}{x} \log(1-x)$$

$$F(\frac{1}{2}, 1, \frac{3}{2}, x^2) = \frac{1}{2x} \log \frac{1+x}{1-x}$$

$$3. \quad F(-\alpha, \beta, \beta; x) = 1 - \frac{\alpha}{1}x + \frac{\alpha(\alpha-1)}{1 \cdot 2}x^2 - \dots = (1-x)^\alpha.$$

$$4. \quad xF(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, x^2) = \arcsin x.$$

$$5. \quad xF(\frac{1}{2}, 1, \frac{3}{2}, -x^2) = \arctan x.$$

Let $0 < \gamma < \beta$. Then

$$F\left(\beta, 1, 1; \frac{\gamma}{\beta}\right) = 1 + \frac{\gamma}{1!} + \left(1 + \frac{1}{\beta}\right) \frac{\gamma^2}{2!} + \left(1 + \frac{1}{\beta}\right) \left(1 + \frac{\gamma}{\beta}\right) \frac{\gamma^3}{3!} + \dots$$

is convergent since its argument is numerically < 1 . In 3) and 4) we see each term of 3) is numerically \leq the term of 4) for any $|x| < \gamma$ and any $\alpha \geq \beta$. The series is considered as a function of α is uniformly convergent on the interval $(\beta + \infty)$ by 136, 2; and hereby x may be taken in $(-\gamma, \gamma)$. Applying now 146, 4 to 3) and 4) we see 3) goes over into 2).

$$7. \quad \lim_{\alpha \rightarrow +\infty} x F\left(\alpha, \alpha, \frac{3}{2}; -\frac{x^2}{4\alpha^2}\right) = \sin x.$$

For

$$x F\left(\alpha, \alpha, \frac{3}{2}; -\frac{x^2}{4\alpha^2}\right) = x - \frac{x^3}{3!} + \left(1 + \frac{1}{\alpha}\right) \frac{x^5}{5!} - \left(1 + \frac{1}{\alpha}\right) \left(1 + \frac{\gamma}{\alpha}\right) \frac{x^7}{7!} + \dots$$

Let $x = \gamma > 0$ and $\alpha = \gamma$. Then

$$\gamma F\left(\gamma, \gamma, \frac{3}{2}; -\frac{1}{4}\right) = \gamma + \frac{\gamma^3}{3!} + \left(1 + \frac{1}{\gamma}\right) \frac{\gamma^5}{5!} - \left(1 + \frac{1}{\gamma}\right) \left(1 + \frac{\gamma}{\gamma}\right) \frac{\gamma^7}{7!} + \dots$$

is convergent by 185. We may now reason as in 6).

8. Similarly we may show:

$$\lim_{\alpha \rightarrow +\infty} F\left(\alpha, \alpha, \frac{1}{2}; -\frac{x^2}{4\alpha^2}\right) = \cos x.$$

$$\lim_{\alpha \rightarrow +\infty} F\left(\alpha, \alpha, \frac{3}{2}, \frac{x^2}{4\alpha^2}\right) = \sinh x.$$

$$\lim_{\alpha \rightarrow +\infty} F\left(\alpha, \alpha, \frac{1}{2}, \frac{x^2}{4\alpha^2}\right) = \cosh x.$$

Between F and two of its contiguous relation. As the number of such pairs of

$$\frac{6 \cdot 5}{1 \cdot 2} = 15,$$

there are 15 such linear relations. Let
We set

$$Q_n = \frac{\alpha + 1 \cdot \alpha + 2 \cdot \dots \alpha + n - 1 \cdot \gamma}{1 \cdot 2 \cdot \dots n \cdot \gamma \cdot \gamma + 1}$$

Then the coefficient of x^n in $F(\alpha, \beta, \gamma, x)$ is

$$\alpha(\beta + n - 1)Q_n;$$

in $F(\alpha + 1, \beta, \gamma, x)$ it is

$$(\alpha + n)(\beta + n - 1)Q_n$$

in $F(\alpha, \beta, \gamma - 1, x)$ it is

$$\frac{\alpha(\beta + n - 1)(\gamma + n - 1)}{\gamma - 1}$$

Thus the coefficient of x^n in

$$(\gamma - \alpha - 1)F(\alpha, \beta, \gamma, x) + \alpha F(\alpha + 1, \beta, \gamma, x) + (1 - \gamma)F(\alpha, \beta, \gamma - 1, x)$$

is 0. This being true for each n , we have

$$(\gamma - \alpha - 1)F(\alpha, \beta, \gamma, x) + \alpha F(\alpha + 1, \beta, \gamma, x) + (1 - \gamma)F(\alpha, \beta, \gamma - 1, x) = 0$$

Again, the coefficient of x^n in $F(\alpha, \beta, \gamma, x)$ is $\alpha(\beta + n - 1)Q_n$.
in $xF(\alpha + 1, \beta, \gamma, x)$ it is $n(\gamma + n - 1)Q_n$.

Hence using the above coefficients, we get

THE HYPERGEOMETRIC FUNCTION

Eliminating $F(\alpha, \beta, \gamma - 1, x)$ from 1), 3) gives

$$(\beta - \alpha)F(\alpha, \beta, \gamma, x) + \alpha F(\alpha + 1, \beta, \gamma, x) - \beta F(\alpha, \beta + 1, \gamma, x) = 0$$

Permuting α, β in 2) gives

$$(\gamma - \alpha - \beta)F(\alpha, \beta, \gamma, x) + \beta(1 - x)F(\alpha, \beta + 1, \gamma, x) + (\alpha - \gamma)F(\alpha - 1, \beta, \gamma, x) = 0$$

From 3), 5) let us eliminate $F(\alpha, \beta + 1, \gamma, x)$, getting

$$(\alpha - 1 - (\gamma - \beta - 1)x)F(\alpha, \beta, \gamma, x) + (\gamma - \alpha)F(\alpha - 1, \beta, \gamma, x) + (1 - \gamma)(1 - x)F(\alpha, \beta, \gamma - 1, x) = 0$$

In 1) let us replace α by $\alpha - 1$ and γ by $\gamma + 1$; we get

$$(\gamma - \alpha + 1)F(\alpha - 1, \beta, \gamma + 1, x) + (\alpha - 1)F(\alpha, \beta, \gamma + 1, x) - \gamma F(\alpha - 1, \beta, \gamma, x) = 0.$$

In 6) let us replace γ by $\gamma + 1$; we get

$$(\alpha - 1 - (\gamma - \beta)x)F(\alpha, \beta, \gamma + 1, x) + (\gamma + 1 - \alpha)F(\alpha - 1, \beta, \gamma + 1, x) - \gamma(1 - x)F(\alpha, \beta, \gamma, x) = 0$$

Subtracting (b) from (a), eliminates $F(\alpha - 1, \beta, \gamma + 1, x)$, gives

$$\gamma(1 - x)F(\alpha, \beta, \gamma, x) - \gamma F(\alpha - 1, \beta, \gamma, x) + (\gamma - \beta)x F(\alpha, \beta, \gamma + 1, x) = 0$$

From 6), 7) we can eliminate $F(\alpha - 1, \beta, \gamma, x)$, getting

$$\begin{aligned} \gamma(\gamma - 1) + (\alpha + \beta + 1 - 2\gamma)x \{ F(\alpha, \beta, \gamma, x) \\ + (\gamma - \alpha)(\gamma - \beta)x F(\alpha, \beta, \gamma + 1, x) \\ + \gamma(1 - \gamma)(1 - x)F(\alpha, \beta, \gamma - 1, x) = 0 \end{aligned}$$

In this manner we may proceed, getting the remaining

For let p, q, r be any three integers.

$$F(\alpha\beta\gamma x), \quad F(\alpha+1, \beta, \gamma, x) \\ F(\alpha+p, \beta+1, \gamma, x), \quad F(\alpha+p, \beta+2, \gamma, x) \\ F(\alpha+p, \beta+q, \gamma+1, x), \quad F(\alpha+p, \beta+q, \gamma+2, x)$$

We have $p+q+r+1$ functions, and these are contiguous. There are thus $p+q+r$ relations between them. We can thus by elimination eliminate between any three of these functions.

189. *Derivatives.* We have

$$\begin{aligned} F''(\alpha, \beta, \gamma, x) &= \sum_{n=1}^{\infty} n \frac{\alpha \cdot \alpha+1 \cdots \alpha+n-1}{1 \cdot 2 \cdots n \cdot \gamma} x^{n-1} \\ &= \sum_{n=0}^{\infty} \frac{\alpha \cdot \alpha+1 \cdots \alpha+n}{1 \cdot 2 \cdots n+1 \cdot \gamma} x^n \\ &= \frac{\alpha\beta}{\gamma} \sum_{n=0}^{\infty} \frac{\alpha+1 \cdots \alpha+n}{1 \cdot 2 \cdots n+1 \cdot \gamma} x^n \\ &= \frac{\alpha\beta}{\gamma} F(\alpha+1, \beta+1, \gamma, x) \end{aligned}$$

Hence

$$\begin{aligned} F'''(\alpha, \beta, \gamma, x) &= \frac{\alpha \cdot \beta}{\gamma} F''(\alpha+1, \beta+1, \gamma, x) \\ &= \frac{\alpha \cdot \alpha+1 \cdot \beta \cdot \beta+1}{\gamma \cdot \gamma+1} F(\alpha+2, \beta+2, \gamma+1, x) \end{aligned}$$

and so on for the higher derivatives, and so on for the higher order functions.

THE HYPERGEOMETRIC FUNCTION

The coefficient of x^n in $x^2 P'''$ is

$$n(n-1)P_n,$$

in $-xP''$ it is

$$-\frac{n(\alpha+n)(\beta+n)}{\gamma+n}P_n,$$

in $(\alpha+\beta+1)xP'$ it is

$$n(\alpha+\beta+1)P_n,$$

in $-\gamma P'$ it is

$$-\gamma \frac{n(\alpha+n)(\beta+n)}{\gamma+n}P_n,$$

in $\alpha\beta P$ it is

$$\alpha\beta P_n.$$

Adding all these gives the coefficient of x^n in the left side. We find it is 0.

191. Expression of $F(\alpha\beta\gamma x)$ as an Integral.

We show that for $|x| < 1$,

$$B(\beta, \gamma - \beta) \cdot F(\alpha\beta\gamma x) = \int_0^1 u^{\beta-1} (1-u)^{\gamma-\beta-1} (1-xu)^{-\alpha} du$$

where $B(p, q)$ is the Beta function of I, 692,

$$B(p, q) = \int_0^1 u^{p-1} (1-u)^{q-1} du.$$

For by the Binomial Theorem

$$(1-xu)^{-\alpha} = 1 + \frac{\alpha}{1}xu + \frac{\alpha \cdot \alpha + 1}{1 \cdot 2} \frac{1}{2}x^2u^2 + \frac{\alpha \cdot \alpha + 1 \cdot \alpha + 2}{1 \cdot 2 \cdot 3} x^3u^3$$

for $|xu| < 1$. Hence

$$J = \int_0^1 u^{\beta-1} (1-u)^{\gamma-\beta-1} (1-xu)^{-\alpha} du$$

Now from I, 692, 10)

$$B(\beta + 1, \gamma - \beta) = \frac{\beta}{\gamma} B(\beta, \gamma - \beta)$$

Hence

$$B(\beta + 2, \gamma - \beta) = \frac{\beta + 1}{\gamma + 1} B(\beta + 1, \gamma - \beta)$$

etc. Putting these values in 2) we get

192. Value of $F(\alpha, \beta, \gamma, x)$ for $x = 1$.

We saw that the F series converges when $\alpha + \beta - \gamma < 0$. The value of F when $x = 1$ is interesting. As it is now a function of α, β, γ we denote it by $F(\alpha, \beta, \gamma)$. The relation between F and B function may be established, as Gauss showed, viz.:

$$\begin{aligned} \gamma \{ \gamma - 1 + (\alpha + \beta + 1 - 2\gamma)x \\ + (\gamma - \alpha)(\gamma - \beta)x^2 \\ + \gamma(1 - \gamma)(1 - x)^2 \} F(\alpha, \beta, \gamma, x) \end{aligned}$$

Assuming that $\alpha + \beta - \gamma < 0$

we see that the first and second terms are positive but we cannot say this in general for the third term for this that $\alpha + \beta - (\gamma - 1) > 0$. We

$$L \lim_{x \rightarrow 1} (1 - x) F(\alpha, \beta, \gamma, x)$$

supposing 2) to hold. For if $|x| < 1$,

$$F(\alpha, \beta, \gamma - 1, x) = a_0 + a_1 x + a_2 x^2 + \dots$$

Now by 100, this series also converges

THE HYPERGEOMETRIC FUNCTION

and this establishes 3). Thus passing to the limit $x = 1$ gives

$$\gamma(\alpha + \beta - \gamma) F(\alpha, \beta, \gamma) + (\gamma - \alpha)(\gamma - \beta) F(\alpha, \beta, \gamma + 1)$$

or
$$F(\alpha, \beta, \gamma) = \frac{(\gamma - \alpha)(\gamma - \beta)}{\gamma(\gamma - \alpha - \beta)} F(\alpha, \beta, \gamma + 1).$$

Replacing γ by $\gamma + 1$, this gives

$$F(\alpha, \beta, \gamma + 1) = \frac{(\gamma + 1 - \alpha)(\gamma + 1 - \beta)}{(\gamma + 1)(\gamma + 1 - \alpha - \beta)} F(\alpha, \beta, \gamma + 2)$$

etc. Thus in general

$$F(\alpha, \beta, \gamma) = \frac{(\gamma - \alpha)(\gamma + 1 - \alpha) \cdots (\gamma + n - 1 - \alpha) \cdot (\gamma - \beta)(\gamma + 1 - \beta) \cdots (\gamma + n - 1 - \beta)}{\gamma(\gamma + 1) \cdots (\gamma + n - 1) (\gamma - \alpha - \beta)(\gamma - \alpha - \beta + 1) \cdots (\gamma - \alpha - \beta + n - 1)} F(\alpha, \beta, \gamma + n)$$

Gauss sets now

$$\Pi(n, x) = \frac{n! n^x}{(x + 1)(x + 2) \cdots (x + n)}.$$

Hence the above relation becomes

$$F(\alpha, \beta, \gamma) = \frac{\Pi(n, \gamma - 1) \Pi(n, \gamma - \alpha - \beta - 1)}{\Pi(n, \gamma - \alpha - 1) \Pi(n, \gamma - \beta - 1)} F(\alpha, \beta, \gamma + n)$$

Now

$$\lim_{n \rightarrow \infty} F(\alpha, \beta, \gamma + n) = 1.$$

For the series

$$F(\alpha, \beta, \gamma) = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} + \frac{\alpha \cdot \alpha + 1 \cdot \beta \cdot \beta + 1}{1 \cdot 2 \cdot \gamma \cdot \gamma + 1} + \dots$$

converges absolutely when 2) holds. Hence

We shall show in the next chapter that

$$\lim_{n \rightarrow \infty} \Pi(n, x)$$

exists for all x different from a negative integer, and is denoted by $\Gamma(x)$; as we shall see,

$$\Gamma(x) = \Pi(x-1),$$

Letting $n \doteq \infty$, (6) gives

$$F(\alpha, \beta, \gamma) = \frac{\Pi(\gamma-1)\Pi(\gamma-\alpha-\beta)}{\Pi(\gamma-\alpha)\Pi(\gamma-\beta)}$$

We must of course suppose that

$$\gamma, \quad \gamma - \alpha, \quad \gamma - \beta,$$

are not negative integers or zero, as otherwise Π or F function are not defined.

Bessel Function

193. 1. The infinite series

$$J_n(x) = x^n \sum_{s=0}^{\infty} \frac{(-1)^s}{2^{n+2s} s! (n+s)!} x^{2s}$$

converges for every x . For the ratio of the adjoint series is

$$\frac{|x|^2}{2^2(s+1)(n+s+1)}$$

which $\doteq 0$ as $s \doteq \infty$ for any given x .

The series (1) thus define functions of x which are continuous. They are called Bessel functions of order n .

$$n = 0, 1, 2, \dots$$

BESSEL FUNCTIONS

2. The following linear relation exists between three consecutive Bessel functions:

$$xJ_{n+1}(x) = \frac{x^{2n}}{x}J_n(x) - J_{n-1}(x) \quad n > 0.$$

$$\text{For } J_{n-1} = \frac{x^{n-1}}{2^{n-1}(n-1)!} + \sum_{s=1}^{\infty} (-1)^s \frac{x^{2s+n-1}}{2^{n+2s-1}s!(n-1+s)!}$$

$$J_{n+1} = \sum_{s=1}^{\infty} (-1)^s \frac{x^{2s+n-1}}{2^{n+2s-1}(s-1)!(n+s)!}.$$

Hence

$$J_{n-1} + J_{n+1}$$

$$= \frac{x^{n-1}}{2^{n-1}(n-1)!} + \sum_1^{\infty} (-1)^s \frac{x^{2s+n-1}}{2^{n+2s-1}} \left\{ \frac{1}{s!(n-1+s)!} - \frac{1}{(s-1)!(n+s)!} \right\}$$

$$= \frac{x^{n-1}}{2^{n-1}(n-1)!} + n \sum_1^{\infty} (-1)^s \frac{x^{2s+n-1}}{2^{n+2s-1}s!(n+s)!}$$

$$= \frac{n}{x} \sum_0^{\infty} (-1)^s \frac{x^{2s+n}}{2^{n+2s-1}s!(n+s)!}$$

$$= \frac{x^{2n}}{x} J_n(x).$$

3. We show next that

$$2J'_n(x) = J_{n-1}(x) - J_{n+1}(x) \quad n > 0.$$

For subtracting 7) from 6) gives

$$J_{n-1} - J_{n+1} = \frac{x^{n-1}}{2^{n-1}(n-1)!} + \sum_1^{\infty} (-1)^s \frac{x^{2s+n-1}}{2^{n+2s-1}} \cdot \frac{n+2s}{s!(n+s)!}$$

$$= \sum_0^{\infty} (-1)^s \frac{(n+2s)x^{2s+n-1}}{2^{n+2s-1}s!(n+s)!}$$

$$= 2J'_n.$$

From 8) we get, on replacing J_{n+1} by its value as given

$$J_{n-1} = 2J'_n + J_{n+1} = \frac{x^{2n}}{x} J_n(x) + J_{n+1}(x) \quad n > 0.$$

This may be shown by direct differentiation thus: Differentiating 9) gives

$$J_n'' = \frac{n}{x^2} J_n - \frac{n}{x} J_{n+1}'$$

Equation 10) gives

$$J_{n+1}' = \frac{n-1}{x} J_{n+1}$$

Replacing here J_{n+1} by its value as given by 9), we get

$$J_{n+1}' = \frac{n-1}{x} J_n + \left(\frac{n(n-1)}{x^2} J_n \right)$$

Putting this in 12) gives 11).

$$5. \quad e^{x^u} x^{\frac{u-1}{2}} = \sum_n u^n J_n$$

for any x , and for $u \neq 0$.

For

$$\begin{aligned} e^{x^u} x^{\frac{u-1}{2}} &= e^{\frac{1}{2}xu} e^{\frac{1}{2}xu} \\ &= \left\{ 1 + \frac{xu}{2} + \frac{x^2u^2}{2^2 \cdot 2!} + \frac{x^3u^3}{2^3 \cdot 3!} + \dots \right\} \end{aligned}$$

Now for any x and for any $u \neq 0$, the series are absolutely convergent. Their product is in the form

$$\begin{aligned} &\left(1 + \frac{x^2}{2^2} + \left(\frac{x}{2} \right)^4 \frac{1}{2! \cdot 2!} + \dots \right) \\ &+ u \left(\frac{x}{2} + \frac{1}{2!} \left(\frac{x}{2} \right)^3 + \frac{1}{3! \cdot 2} \left(\frac{x}{2} \right)^5 + \dots \right) \\ &- u^{-1} \left(\frac{x}{2} + \frac{1}{2!} \left(\frac{x}{2} \right)^3 + \frac{1}{3! \cdot 2} \left(\frac{x}{2} \right)^5 + \dots \right) \end{aligned}$$

194. 1. *Expression of $J_n(x)$ as an Integral.*

$$J_n(x) = \frac{x^n}{2^n \sqrt{\pi}} \cdot \frac{1}{\Gamma\left(\frac{2n+1}{2}\right)} \int_0^\pi \cos(x \cos \phi) \sin^{2n} \phi d\phi$$

For

$$\cos u = \sum_0^s (-1)^s \frac{u^{2s}}{(2s)!}.$$

Hence

$$\cos(x \cos \phi) = \sum_0^s \frac{(-1)^s}{(2s)!} x^{2s} \cos^{2s} \phi$$

and thus

$$\cos(x \cos \phi) \sin^{2n} \phi = \sum_0^s \frac{(-1)^s}{(2s)!} x^{2s} \cos^{2s} \phi \sin^{2n} \phi.$$

As this series converges uniformly in $(0, \pi)$ for any value of x , we may integrate termwise, getting

$$\begin{aligned} \int_0^\pi \cos(x \cos \phi) \sin^{2n} \phi d\phi &= \sum_0^s \frac{(-1)^s}{(2s)!} x^{2s} \int_0^\pi \cos^{2s} \phi \sin^{2n} \phi d\phi \\ &= \sum_0^s \frac{(-1)^s}{(2s)!} x^{2s} B\left(\frac{2s+1}{2}, \frac{2n+1}{2}\right) \text{ by } 225, \text{ a} \\ &= \sum_0^s \frac{(-1)^s}{(2s)!} x^{2s} \frac{\Gamma\left(\frac{2s+1}{2}\right) \Gamma\left(\frac{2n+1}{2}\right)}{\Gamma(s+n)} \text{ by } 225, \text{ b} \end{aligned}$$

We shall show in 225, a, that

$$\Gamma\left(\frac{2s+1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \cdots 2s-1}{2^s} \sqrt{\pi}.$$

Thus the last series above

$$= \sum_0^s \frac{(-1)^s}{(2s)!} x^{2s} \frac{1 \cdot 3 \cdot 5 \cdots (2s-1)}{2^s} \frac{\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma(s+n)} \sqrt{\pi}$$

CHAPTER VI

INFINITE PRODUCTS

195. 1. Let $\{a_{\iota_1, \dots, \iota_s}\}$ be an infinite system of indices $\iota = (\iota_1 \dots \iota_s)$ ranging over a linear space. The symbol

$$P = \prod_{\iota} a_{\iota_1, \dots, \iota_s} = \prod_{\iota} a_{\iota}$$

is called an *infinite product*. The numbers P_{μ} denote the product of all the factors R_{μ} . If

$$\lim_{\mu \rightarrow \infty} P_{\mu}$$

is finite or definitely infinite, we call it finite or infinite. It is customary to represent a product and its limit by P when no ambiguity will arise.

When the limit 2) is finite and $\neq 0$ or ∞ , we say P is *convergent*, otherwise P is *divergent*.

We shall denote by P_{μ} the product of all the factors $a_{\iota} = 1$, whose indices ι lie in the set μ . We call P_{μ} the *co-product* of P_{μ} .

The products most often occurring in the theory of

$$P = a_1 \cdot a_2 \cdot a_3 \cdot \dots$$

The factor P_{μ} is here replaced by

$$P_{\mu} = a_1 \cdot a_2 \cdot \dots \cdot a_{\mu}$$

GENERAL THEORY

2. Let

$$P = \frac{1}{1} \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \dots$$

Obviously the product $P = 0$, as

$$P_n = \frac{1}{n} \doteq 0.$$

Hence $P = 0$, although no factor is zero. Such products are called *zero products*. Now we saw in I, 77 that the product of a finite number of factors cannot vanish unless one of its factors vanishes. For this reason zero products hold an exception to the general rule and will not be considered in this work. We therefore have classified them among the divergent products. In the following theorems relative to convergence, we shall suppose, for simplicity, that *there are no zero factors*.

196. 1. For $P = \prod a_1, \dots, a_n$ to converge it is necessary that each a_n is convergent. If one of these P_μ converges, P is convergent

$$P = P_\mu \cdot \bar{P}_\mu.$$

The proof is obvious.

2. If the simple product $P = a_1 \cdot a_2 \cdot a_3 \dots$ is convergent, the factors finally remain positive.

For, when P is convergent, $|P_n| > \text{some positive number}$ $n > \text{some } m$. If now the factors after a_m were not all positive, and P_n could have opposite signs $n > m$, however large n is. Thus P_n has no limit.

197. 1. To investigate the convergence or divergence of an infinite product $P = \prod a_1, \dots, a_n$ when $a_i > 0$, it is often convenient to consider the series

$$L = \sum \log a_1, \dots, a_n = \sum l_1, \dots, l_n,$$

For let p be taken so large that $|x| < p$. We show that the co-product

$$\mathcal{G}_p = \prod_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^x}{1 + \frac{x}{n}}$$

converges for this x . The corresponding logarithmic series

$$\begin{aligned} L &= \sum_{n=1}^{\infty} \left\{ x \log \left(1 + \frac{1}{n} \right) - \log \left(1 + \frac{x}{n} \right) \right\} \\ &= \sum_{n=1}^{\infty} \left\{ \frac{x}{n} - \log \left(1 + \frac{x}{n} \right) \right\} = x \sum_{n=1}^{\infty} \left\{ \frac{1}{n} - \log \left(1 + \frac{1}{n} \right) \right\}. \end{aligned}$$

As each of the series on the right converges, so does L . \mathcal{G} converges for this value of x .

198. 1. When the associate logarithmic series

$$L = \sum \log a_{i_1, \dots, i_s}, \quad a_i > 0$$

is convergent,

$$\lim_{i_1, \dots, i_s \rightarrow \infty} \log a_{i_1, \dots, i_s} = 0, \quad \text{by 121, 1,}$$

and therefore

$$\lim_{i_1, \dots, i_s \rightarrow \infty} a_{i_1, \dots, i_s} = 1.$$

For this reason it is often convenient to write the a_{i_1, \dots, i_s} of an infinite product P in the form $1 + b_{i_1, \dots, i_s}$. We shall write P in the form

$$P = \Pi(1 + b_{i_1, \dots, i_s}),$$

we shall say it is written in its *normal form*. The series

$$\sum b_{i_1, \dots, i_s} = \sum b_i$$

For P and

$$L = \sum \log (1 + a_n)$$

converge or diverge simultaneously by 196, 2. P and L converge or diverge simultaneously by 123, 4.

3. If the simple product $P = a_1 \cdot a_2 \cdot a_3 \cdots$

For by 196, 2 the factors a_n finally become > 0 . Hence by 197, 1 the series

$$\sum_{n=m}^{\infty} \log a_n \quad (a_n > 0)$$

is convergent. Hence $\log a_n \rightarrow 0$. $\therefore a_n \rightarrow 1$

199. Let $R_{\lambda_1} < R_{\lambda_2} < \cdots$ $\lambda | \equiv r$ be r cells. Then if P is convergent,

$$P = P_{\lambda_1} + \sum_1^r (P_{\lambda_{n+1}} - P_{\lambda_n})$$

For P is a telescopic series and

$$P_{\lambda_{r+1}} = P_{\lambda_1} + \sum_1^r (P_{\lambda_{n+1}} - P_{\lambda_n})$$

200. 1. Let $P = \Pi (1 + a_{i_1} \cdots i_r)$.

We call $\mathfrak{P} = \Pi (1 + a_{i_1} \cdots i_r)$ $\quad a_{i_1} \cdots i_r$ the *adjoint* of P , and write

$$\mathfrak{P} = \text{Adj } P.$$

2. P converges, if its adjoint is convergent.

$$\epsilon > 0, \quad \lambda, \quad |P_\mu - P_\nu| < \epsilon$$

3. When the adjoint of P converges, we say P is *absolutely convergent*.

The reader will note that absolute convergence of infinite products is defined quite differently from that of infinite series. At first sight one would incline to define the adjoint of

$$P = \prod a_{i_1 \dots i_s}$$

to be

$$\mathfrak{P} = \prod |a_{i_1 \dots i_s}|.$$

With this definition the fundamental theorem 2 would be false or let

$$P = \prod (-1)^n;$$

its adjoint would be, by this definition,

$$\mathfrak{P} = 1 \cdot 1 \cdot 1 \cdot \dots$$

Now $\mathfrak{P}_n = 1$. $\therefore \mathfrak{P}$ is convergent. On the other hand $P_n = (-1)^n$ and this has no limit, as $n \rightarrow \infty$. Hence P is not convergent.

4. In order that $P = \prod (1 + a_{i_1 \dots i_s})$ converge absolutely, it is necessary and sufficient that $\sum a_{i_1 \dots i_s}$ converge absolutely.

Follows at once from 198, 2.

Example.

$$\prod_1^\infty \left(1 - \frac{x^2}{n^2}\right)$$

converges absolutely for every x .

For

$$\sum \frac{x^2}{n^2} = x^2 \sum \frac{1}{n^2}$$

We have now the following theorems:

2. *If an associate simple product Q is $P = Q$.*

For since Q is convergent, we may assume $\epsilon > 0$ by 196, 2. Then

$$\begin{aligned} Q &= e^{2 \log a_n} \\ &= e^{2 \log a_1} \\ &= P \end{aligned}$$

3. *If the associate simple product Q is P .*

For let

$$\begin{aligned} P &= \Pi(1 + a_{1, \dots, 1_n}) \\ Q &= \Pi(1 + a_n). \end{aligned}$$

Since Q is absolutely convergent,

$$\Pi(1 + a_n) = \prod_{n=1}^{\infty} (1 + a_n)$$

is convergent. Hence $\Pi(1 + a_{1, \dots, 1_n})$ is convergent.

4. *Let $P = \Pi(1 + a_{1, \dots, 1_n})$ be absolutely convergent. If the associate simple product $Q = \Pi(1 + a_n)$ is convergent, then $P = Q$.*

For since P is absolutely convergent,

$$\sum a_{1, \dots, 1_n}$$

converges by 200, 4. But then by 124, 5

$$\sum a_n$$

is convergent. Hence Q is absolutely convergent.

5. *If $P = \Pi(1 + a_{1, \dots, 1_n})$ is absolutely convergent,*

$$\text{For} \quad P = P_\mu \cdot \bar{P}_\mu;$$

and all the factors of P_μ are > 0 , if μ is sufficiently large. Then we shall feel at liberty to do, without further remark.

$$7. \quad A = \prod (1 + a_{i_1 \dots i_s}) \quad a_i > 0$$

$$L = \sum \log (1 + a_{i_1 \dots i_s})$$

converge or diverge simultaneously.

For if A is convergent,

$$\sum a_{i_1 \dots i_s}$$

is convergent by 200, 4. But then L is convergent by 123. The converse follows similarly.

202. 1. As in 124, 10 we may form from a given m -tuple product

$$A = \prod a_{i_1 \dots i_m}$$

an infinite number of conjugate n -tuple products

$$B = \prod b_{j_1 \dots j_n}$$

where $a_i = b_j$ if i and j are corresponding lattice points in the two systems.

We have now :

2. If A is absolutely convergent, so is B , and $A = B$.

For by 201, 6, without loss of generality, we may take all the factors > 0 .

Then

$$\begin{aligned} A &= e^{\sum \log a_{i_1 \dots i_m}} \\ &= e^{\sum \log a_\mu} \\ &= e^{\sum \log a_{j_1 \dots j_n}} \end{aligned}$$

For $\Sigma \log \alpha_i$ is convergent.
 Hence $\Sigma \log \beta_j$ is.

Arithmetical Operations

203. *Absolutely convergent products associate.*

For let $A = \Pi a_1 \dots a_m$
 be absolutely convergent. Then its associate
 $\mathfrak{A} = \Pi a_n$

is absolutely convergent and $A = \mathfrak{A}$, by rearranging the factors of A , getting the product which corresponds to a simple associate series \mathfrak{B} and $B = \mathfrak{B}$. \mathfrak{A} is absolutely convergent. Hence $A = B$.

Conversely, let A be commutative. Then the partial products finally become > 0 . For if not, let

$$R_1 < R_2 < \dots < R_n$$

be a sequence of rectangular cells such that R_n is in some cell. We may arrange the factors of A so that the products corresponding to R_n

$$A_1, A_2, A_3, \dots$$

have opposite signs alternately. Then A is not commutative. We may therefore assume A is commutative.

Then $A = e^{\Sigma \log a_1 \dots a_m}$

remains unaltered however the factors are rearranged.

Hence $\Sigma \log a_i$

204. 1. Let

$$A = \prod_{i_1, \dots, i_s} a_{i_1, \dots, i_s}$$

be absolutely convergent. Then the s -tuple iterated product

$$B = \prod_{i'_1} \prod_{i'_2} \dots \prod_{i'_s} a_{i_1, \dots, i_s}$$

is absolutely convergent and $A = B$ where $i'_1 \dots i'_s$ is a permutation of i_1, i_2, \dots, i_s .

For by 202, 3 all the products of the type

$$\prod_{i_s, i'_s} a_{i_1, \dots, i_s} \quad \prod_{i_s} a_{i_1, \dots, i_s}$$

are absolutely convergent, and by I, 324

$$\prod_{i_s, i'_s} = \prod_{i_s} \prod_{i'_s}.$$

Similarly the products of the type

$$\prod_{i_s, i'_s, i''_s, \dots, i^{(s)}_s} a_{i_1, \dots, i_s}$$

are absolutely convergent and hence

$$\prod_{i_s, i'_s, i''_s, \dots, i^{(s)}_s} = \prod_{i_s, i'_s} \prod_{i''_s} \dots \prod_{i^{(s)}_s}.$$

In this way we continue till we reach A and B .

2. We may obviously generalize 1 as follows :

Let

$$A = \prod_{i_1, \dots, i_s} a_{i_1, \dots, i_s}$$

be absolutely convergent. Let us establish a 1 to 1 correspondence between the lattice system \mathfrak{L} over which $i = (i_1 \dots i_s)$ ranges and the lattice system \mathfrak{M} over which

3. An important special case of 2 is the

Let $A = \prod a_n$, $n = 1, 2, \dots$

converge absolutely. Let us throw the a_r into

$$a_{11} \quad , \quad a_{12} \quad \cdots$$

$$a_{21} \quad , \quad a_{22} \quad \cdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{r1} \quad , \quad a_{r2} \quad \cdots$$

Then $B_1 = \prod a_{1n}$, $B_2 = \prod a_{2n}$,

converge absolutely, and

$$A = B_1 B_2 \cdots B_r.$$

4. The convergent infinite product

$$P = (1 + a_1)(1 + a_2) \cdots$$

is associative.

For let

$$m_1 < m_2 < \cdots < \infty$$

Let

$$1 + b_1 = (1 + a_1) \cdots (1 + a_{m_1})$$

$$1 + b_2 = (1 + a_{m_1+1}) \cdots (1 + a_{m_2})$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

We have to show that

$$Q = (1 + b_1)(1 + b_2) \cdots$$

is convergent and $P = Q$.

This, however, is obvious. For

205. Let $A = \prod a_1, \dots, a_n$, $B = \prod b_1, \dots, b_n$

convergent. Then

$$C = \prod a_i \cdot b_i, \quad D = \prod \frac{a_i}{b_i}$$

are convergent and

$$C = A \cdot B, \quad D = \frac{A}{B}.$$

Moreover if A, B are absolutely convergent, so are C, D .

Let us prove the theorem regarding C ; the rest follows similarly. We have

$$C_\mu = A_\mu \cdot B_\mu.$$

Now by hypothesis $A_\mu \rightarrow A, B_\mu \rightarrow B$ as $\mu \rightarrow \infty$.

Hence

$$C_\mu \rightarrow A \cdot B.$$

To show that C is absolutely convergent when A, B are, let us write $a_i = 1 + \alpha_i$, $b_i = 1 + \beta_i$ and set $|\alpha_i| = \alpha_i$, $|\beta_i| = \beta_i$. Since A, B converge absolutely,

$$\sum \log(1 + \alpha_i) \quad , \quad \sum \log(1 + \beta_i)$$

are convergent. Hence

$$\sum \{\log(1 + \alpha_i) + \log(1 + \beta_i)\} = \sum \log(1 + \alpha_i)(1 + \beta_i)$$

is absolutely convergent. Hence C is absolutely convergent by 201, 7.

206. Example. The following infinite products occur in the theory of elliptic functions:

$$Q_1 = \prod (1 + q^{2n})$$

$$Q_n = \prod (1 + q^{2n-1}) \quad n = 1, 2, \dots$$

For by 205,

$$\begin{aligned} P &= \Pi (1 + q^{2n})(1 + q^{2n}) \\ &= \Pi (1 + q^{2n})(1 + q^{2n}) \end{aligned}$$

Now all integers of the type $2n$, are of the type $4n$.
Hence by 204, 3,

$$\Pi (1 - q^{2n}) = \Pi (1 - q^{4n}) \Pi (1 - q^{4n-2})$$

or

$$\Pi (1 - q^{4n-2}) = \frac{\Pi (1 - q^{2n})}{\Pi (1 - q^{4n})}.$$

Thus

$$\begin{aligned} P &= \Pi \frac{(1 + q^{2n})(1 + q^{2n})}{(1 - q^{4n})} \\ &= 1. \end{aligned}$$

Uniform Convergence

207. In the limited or unlimited domain \mathfrak{A} ,

$$L = \sum \log f_{i_1, \dots, i_s}(x_1 \cdots x_m) \quad ,$$

be uniformly convergent and limited. Then

$$F = \Pi f_{i_1, \dots, i_s}$$

is uniformly convergent in \mathfrak{A} .

For

$$F_\lambda = e^{L_\lambda}.$$

Now $L_\lambda \doteq L$ uniformly. Hence by 144, 1, F_λ is uniformly convergent.

208. If the adjoint of

$$F = \Pi (1 + f_{i_1, \dots, i_s}(x_1 \cdots x_m))$$

UNIFORM CONVERGENCE

But as already noticed in 200, 2, 1)

$$|P_\mu - P_\nu| \leq |\mathfrak{P}_\mu - \mathfrak{P}_\nu|.$$

Hence P is uniformly convergent.

209. *The product*

$$F = \Pi(1 + f_{i_1 \dots i_s}(x_1 \dots x_m))$$

is uniformly convergent in the limited or unlimited domain \mathfrak{A} ,

$$\Phi = \Sigma \phi_{i_1 \dots i_s}(x_1 \dots x_m) \quad , \quad \phi_i = |f_i|$$

is limited and uniformly convergent in \mathfrak{A} .

For by 138, 2 the series

$$L = \Sigma \log(1 + \phi_i)$$

is uniformly convergent and limited in \mathfrak{A} . Then by 208, F is uniformly convergent, and hence by 208, F

210. *Let*

$$F(x_1 \dots x_m) = \Pi f_{i_1 \dots i_s}(x_1 \dots x_m)$$

be uniformly convergent at $x = a$. If each f_i is continuous at a , then F is also continuous at a .

This is a corollary of 147, 1.

211. 1. Let $G = \Sigma |f_{i_1 \dots i_s}(x_1 \dots x_m)|$ converge in the complete domain \mathfrak{A} having a as a limiting point. Let G and f_i be continuous at a . Then

$$F(x_1 \dots x_m) = \Pi(1 + f_{i_1 \dots i_s}(x_1 \dots x_m))$$

is continuous at a .

For by 149, 5, G is uniformly convergent near $x = a$. Thus by 209,

$$\Pi(1 + f_i)$$

is uniformly convergent at a . To establish this we apply 146, 1.

212. 1. Let $F = \Pi f_i(x)$,
converge in $\mathfrak{A} = (a, a + \delta)$. Then

$$\log F = L = \sum \log f_i.$$

If we can differentiate this series termwise

$$\frac{d}{dx} \log F = \sum \frac{f_i'}{f_i}.$$

Thus to each infinite product 1) of this chapter corresponds a finite series 3). Conditions for termwise differentiation 2) are given in 153, 155, 156. Other conditions are in Chapter XVI.

2. *Example.* Let us consider the infinite product

$$\theta(x) = 2q^{\frac{1}{2}}Q \sin \pi x \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2\pi x + q^{4n})$$

which occurs in the elliptic functions.

Let us set

$$1 - u_n = 1 - 2q^{2n} \cos 2\pi x + q^{4n}$$

Then

$$|u_n| \leq 2|q|^{2n} + |q|^{4n}$$

Thus if $|q| < 1$, the product 1) is absolutely convergent. It is uniformly convergent for any x and

Now the series Σa_n converges if $|q| < 1$. For setting the series Σb_n is convergent in this case. Moreover,

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1.$$

Thus we may differentiate termwise.

The Circular Functions

213. 1. *Sin x and cos x as Infinite Products.*

From the addition theorem

$$\sin (mx + x) = \sin (m + 1)x = \sin mx \cos x + \cos mx \sin x$$

$m = 1, 2, 3 \dots$ we see that for an odd n

$$\sin nx = a_0 \sin^n x + a_1 \sin^{n-1} x + \dots + a_{n-1} \sin x$$

where the coefficients a are integers. If we set $t = \sin x$

$$\sin nx = F_n(t) = a_0 t^n + a_1 t^{n-1} + \dots + a_{n-1} t$$

Now F_n being a polynomial of degree n , it has n roots

$$0, \quad \pm \sin \frac{\pi}{n}, \quad \pm \sin \frac{2\pi}{n}, \quad \dots \pm \sin \frac{n-1}{2} \frac{\pi}{n}$$

corresponding to the values of x which make $\sin nx = 0$

$$F_n(t) = a_0 t \left(t - \sin \frac{\pi}{n} \right) \left(t + \sin \frac{\pi}{n} \right) \dots$$

$$= a_0 t \left(t^2 - \sin^2 \frac{\pi}{n} \right) \dots \left(t^2 - \sin^2 \frac{n-1}{2} \frac{\pi}{n} \right)$$

To find α we observe that this equation gives

$$\frac{\sin nx}{\sin x} = \alpha \left[1 - \frac{\sin^2 x}{\sin^2 \frac{\pi}{n}} \right]$$

Letting $x \doteq 0$ we now get $\alpha = n$. Thus in 3), and replacing x by $\frac{x}{n}$, we have finally

$$\sin x = n \sin \frac{x}{n} P(x, n)$$

where

$$P(x, n) = \prod \left[1 - \frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{\pi}{n}} \right] \quad r = 1, 2, \dots$$

We note now that as $n \doteq \infty$,

$$n \sin \frac{x}{n} = x \frac{\sin \frac{x}{n}}{\frac{x}{n}} \doteq x.$$

Similarly

$$\frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{\pi}{n}} \doteq \frac{x^2}{\pi^2}.$$

It seems likely therefore that if we pass to 4), we shall get

$$\sin x = x P(x)$$

where

$$P(x) = \prod \left(1 - \frac{x^2}{\pi^2} \right).$$

We observe that

$$\lim_{n \rightarrow \infty} P(x, n) = \lim_{n \rightarrow \infty} e^{L(x, n)} = e^{L(x)} = P(x)$$

provided

$$\lim_{n \rightarrow \infty} L(x, n) = L(x).$$

We have thus only to prove 7). Let us denote the sum of the first m terms in (6) by $L_m(x, n)$ and the sum of the remaining terms by $\bar{L}_m(x, n)$. Then

$$|L(x, n) - L(x)| < |L_m(x, n) - L_m(x)| + |\bar{L}_m(x, n)| + |\bar{L}_m(x)|.$$

Since for $0 < x < \frac{\pi}{2}$,

$$\frac{x}{2} < \sin x < x,$$

we have

$$\frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{r\pi}{n}} > \frac{\frac{x^2}{n^2}}{\frac{\pi^2 r^2}{n^2}} = \frac{x^2}{\pi^2 r^2},$$

and hence for an m_1 so large that $\frac{|x|}{m_1} < 1$, we have,

$$\log \left[1 - \frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{r\pi}{n}} \right] < \log \left(1 - \frac{x^2}{\pi^2 r^2} \right), \quad r > m.$$

But the series

$$\sum_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 n^2} \right)$$

2. In algebra we learn that every polynomial

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

can be written as a product

$$a_n(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n),$$

where $\alpha_1, \alpha_2, \dots$ are its roots. Now

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

is the limit of a polynomial, viz. the first n terms. It is natural to ask, Can we not express $\sin x$ as the product of factors which vanishes at the zeros of $\sin x$? That this can be done we have just shown in 1.

3. If we set $x = \pi/2$ in 5), it gives,

$$1 = \frac{\pi}{2} \prod \left(1 - \frac{1}{4r^2}\right) = \frac{\pi}{2} \prod \frac{(2r-1)(2r+1)}{2r \cdot 2r}$$

Hence

$$\frac{\pi}{2} = \prod \frac{2r \cdot 2r}{(2r-1)(2r+1)} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdots}$$

a formula due to *Wallis*.

4. From 5) we can get another expression for

$$\sin x = x \prod \left(1 - \frac{x^2}{r^2\pi^2}\right) e^{\frac{x^2}{r^2\pi^2}} \quad r = 1, 2, 3, \dots$$

For the right side is convergent by 197, 2. If we group the factors in pairs, we have

14. We now show that

$$\cos x = \prod_1 \left(1 - \frac{4x^2}{(2n-1)^2\pi^2} \right).$$

To this end we use the relation

$$\sin 2x = 2 \sin x \cos x.$$

Hence

$$\begin{aligned} \cos x &= \frac{1}{2} \cdot \frac{2x}{x} \cdot \frac{\prod_1 \left(1 - \frac{4x^2}{n^2\pi^2} \right)}{\prod_1 \left(1 - \frac{x^2}{n^2\pi^2} \right)} = \frac{\prod_1 \left(1 - \frac{4x^2}{(2m)^2\pi^2} \right) \left(1 - \frac{4x^2}{(2m-1)^2\pi^2} \right)}{\prod_1 \left(1 - \frac{x^2}{n^2\pi^2} \right)} \\ &= \frac{\prod_1 \left(1 - \frac{x^2}{m^2\pi^2} \right)}{\prod_1 \left(1 - \frac{x^2}{n^2\pi^2} \right)} \cdot \prod_1 \left(1 - \frac{4x^2}{(2m-1)^2\pi^2} \right) \end{aligned}$$

in which 1) is immediate.

From 1) we have, as in 213, 4,

$$\cos x = \prod_1 \left(1 - \frac{2x}{(2n-1)\pi} \right) e^{\frac{2x}{(2n-1)\pi}} \quad n = 0, \pm 1, \pm 2, \dots$$

15. From the expression of $\sin x$, $\cos x$ as infinite products their periodicity is readily shown. Thus from 213, 12)

$$\sin x = \lim_{n \rightarrow \infty} P_n(x).$$

But

$$\frac{P_n(x + \pi)}{P_n(x)} = \frac{x + (n+1)\pi}{x - n\pi} = -1, \quad \text{as } n \rightarrow \infty.$$

Hence

$$\lim_{n \rightarrow \infty} P_n(x + \pi) = - \lim_{n \rightarrow \infty} P_n(x),$$

$$\sin(x + \pi) = -\sin x.$$

Similarly 214, 1) gives

$$\log \cos x = \sum_1^{\infty} \log \left(1 - \frac{4x^2}{(2s-1)^2\pi^2} \right), \quad (0)$$

To get formulæ having a wider range we have the products 213, 5) and 214, 1). We then get

$$\log \sin^2 x = \log x^2 + \sum_1^{\infty} \log \left(1 - \frac{x^2}{s^2\pi^2} \right)$$

valid for any x such that $\sin x \neq 0$; and

$$\log \cos^2 x = \sum_1^{\infty} \log \left(1 - \frac{4x^2}{(2s-1)^2\pi^2} \right)$$

valid for any x such that $\cos x \neq 0$.

If we differentiate 3), 4) we get

$$\cot x = \frac{1}{x} + 2 \sum_1^{\infty} \frac{x}{x^2 - s^2\pi^2},$$

$$\tan x = 2 \sum_1^{\infty} \frac{x}{\left(\frac{2s-1}{2} \right)^2 \pi^2 - x^2}$$

valid as in 3), 4).

Remark. The relations 5), 6) exhibit $\cot x$, $\tan x$ as rational functions whose poles are precisely the poles of the ζ -functions. They are analogous to the representation of a fraction as the sum of partial fractions.

2. To get developments of $\sec x$, $\operatorname{cosec} x$, we observe

$$\operatorname{cosec} x = \tan \frac{1}{2} x + \cot x.$$

Hence

3. To get $\sec x$, we observe that

$$\operatorname{cosec}\left(\frac{\pi}{2}-x\right)=\sec x.$$

$$\text{Now} \quad \operatorname{cosec} x = \frac{1}{x} + \sum_1^{\infty} (-1)^{s-1} \left\{ \frac{1}{s\pi-x} - \frac{1}{s\pi+x} \right\}.$$

Hence

$$\operatorname{cosec}\left(\frac{\pi}{2}-x\right)=\frac{1}{\frac{\pi}{2}-x}+\sum_1^{\infty}(-1)^{s-1}\left\{\frac{1}{s\pi-\frac{\pi}{2}+x}-\frac{1}{s\pi+\frac{\pi}{2}-x}\right\}.$$

Let us regroup the terms of S , forming the series

$$T=\left\{\frac{1}{\frac{\pi}{2}-x}+\frac{1}{\frac{\pi}{2}+x}\right\}-\left\{\frac{1}{\frac{3\pi}{2}-x}+\frac{1}{\frac{3\pi}{2}+x}\right\}+\dots$$

$$\text{As} \quad |S_n - T_n| = \frac{1}{\left|\frac{2n-1}{2}\pi - x\right|} \doteq 0,$$

we see that T is convergent and $=S$. Thus

$$\sec x = \sum_1^{\infty} (-1)^{s-1} \frac{\left(\frac{2s-1}{2}\pi\right)}{\left(\frac{2s-1}{2}\right)^2 \pi^2 - x^2},$$

valid for all x such that $\cos x \neq 0$.

217. As an exercise let us show the periodicity of $\cot x$ (216, 5). We have

$$\cot x = \lim_{n \rightarrow \infty} F_n(x) = \lim_{n \rightarrow \infty} \sum_1^n \frac{1}{x^2 - s^2 \pi^2} \quad x \neq s\pi.$$

218. *Development of $\log \sin x$, $\tan x$, etc., in power series*
 From 216, 1)

$$\log \frac{\sin x}{x} = \sum_1^{\infty} \log \left(1 - \frac{x^2}{s^2 \pi^2} \right).$$

If we give to $\frac{\sin x}{x}$ its limiting value 1 as $x \doteq 0$, the relation holds for $|x| < \pi$.

Now for $|x| < \pi$

$$-\log \left(1 - \frac{x^2}{s^2 \pi^2} \right) = \frac{x^2}{s^2 \pi^2} + \frac{1}{2} \frac{x^4}{s^4 \pi^4} + \dots$$

Thus

$$\begin{aligned} -\log \frac{\sin x}{x} &= \frac{x^2}{\pi^2} + \frac{1}{2} \frac{x^4}{\pi^4} + \frac{1}{3} \frac{x^6}{\pi^6} + \dots \\ &+ \frac{x^2}{2^2 \pi^2} + \frac{1}{2} \frac{x^4}{2^4 \pi^4} + \frac{1}{3} \frac{x^6}{2^6 \pi^6} + \dots \\ &+ \frac{x^2}{3^2 \pi^2} + \frac{1}{2} \frac{x^4}{3^4 \pi^4} + \frac{1}{3} \frac{x^6}{3^6 \pi^6} + \dots \\ &+ \dots \dots \dots \end{aligned}$$

provided we sum this double series by rows. But since it is a positive term series, we may sum by columns, too. Doing this we get

$$-\log \frac{\sin x}{x} = H_2 \frac{x^2}{\pi^2} + \frac{1}{2} H_4 \frac{x^4}{\pi^4} + \frac{1}{3} H_6 \frac{x^6}{\pi^6} + \dots$$

where

$$H_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \dots$$

The relation 2) is valid for $|x| < \pi$.

THE CIRCULAR FUNCTIONS

The terms of G_n are a part of H_n . Obviously

$$G_n = \frac{2^n - 1}{2^n} H_n.$$

These coefficients put in 3) give

$$-\log \cos x = (2^2 - 1) H_2 \frac{x^2}{\pi^2} + \frac{1}{2} (2^4 - 1) H_4 \frac{x^4}{\pi^4} + \frac{1}{3} (2^6 - 1) H_6 \frac{x^6}{\pi^6}$$

valid for $|x| < \frac{\pi}{2}$. If we differentiate 4) and 2), we get

$$\tan x = 2(2^2 - 1) H_2 \frac{x}{\pi^2} + 2(2^4 - 1) H_4 \frac{x^3}{\pi^4} + 2(2^6 - 1) H_6 \frac{x^5}{\pi^6}$$

valid for $|x| < \frac{\pi}{2}$;

$$\cot x = \frac{1}{x} - 2 H_2 \frac{x}{\pi^2} - 2 H_4 \frac{x^3}{\pi^4} - 2 H_6 \frac{x^5}{\pi^6} - \dots$$

valid for $0 < |x| < \pi$.

Comparing 5) with the development of $\tan x$ given gives

$$H_2 = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6} = \frac{1}{6} \cdot \frac{2 \pi^2}{2!} = B_1 \cdot \frac{2 \pi^2}{2!}.$$

$$H_4 = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90} = \frac{1}{30} \cdot \frac{2^3 \pi^4}{4!} = B_3 \cdot \frac{2^3 \pi^4}{4!}$$

$$H_6 = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \dots = \frac{\pi^6}{945} = \frac{1}{42} \cdot \frac{2^5 \pi^6}{6!} = B_5 \cdot \frac{2^5 \pi^6}{6!}$$

$$H_8 = \frac{1}{1^8} + \frac{1}{2^8} + \frac{1}{3^8} + \dots = \frac{\pi^8}{9450} = \frac{1}{30} \cdot \frac{2^7 \pi^8}{8!} = B_7 \cdot \frac{2^7 \pi^8}{8!}$$

.

Let us set

$$H_{2n} = \frac{2^{2n-1} \pi^{2n}}{(2n)!} B_{2n-1}.$$

From 6), 8) we get

$$\cotan x - \frac{1}{x} = - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} B_{2n-1} x^{2n-1}$$

valid for $0 < |x| < \pi$.

219. Recursion formula for the Bernoullian Numbers

If we set $f(x) = \tan x$,

we have by Taylor's development

$$f(x) = xf'(0) + x^3 \frac{f'''(0)}{3!} + x^5 \frac{f^{(5)}(0)}{5!} + \dots$$

where

$$\frac{f^{(2n-1)}(0)}{(2n-1)!} = \frac{2(2^{2n}-1)H_{2n}}{\pi^{2n}} = \frac{2^{2n}(2^{2n}-1)}{(2n)!}$$

Now by I, 408,

$$f^{(2n-1)}(0) - \binom{2n-1}{2} f^{(2n-3)}(0) + \binom{2n-1}{4} f^{(2n-5)}(0) - \dots$$

From 1), 2) we get

$$\begin{aligned} \frac{2^{2n-1}(2^{2n}-1)}{n} B_{2n-1} - \binom{2n-1}{2} \frac{2^{2n-3}(2^{2n-2}-1)}{n-1} B_{2n-3} \\ + \binom{2n-1}{4} \frac{2^{2n-5}(2^{2n-4}-1)}{n-2} B_{2n-5} - \dots \end{aligned}$$

We have already found B_1, B_3, B_5, B_7 ; it is now successively:

$$B_9 = \frac{5}{66}, \quad B_{11} = \frac{691}{270}, \quad B_{13} = \frac{7}{6}, \quad B_{15} = \frac{43867}{798}, \quad B_{17} = \frac{174611}{330},$$

Thus to calculate B_{19} , we have from 3)

$$\frac{2^9(2^{10}-1)}{19} B_{19} = \frac{9 \cdot 8 \cdot 2^7(2^8-1)}{19} \cdot \frac{1}{1} + \frac{9 \cdot 8 \cdot 7 \cdot 2^5(2^6-1)}{19 \cdot 2} B_{17} - \dots$$

The B and Γ Functions

220. In Volume I we defined the B and Γ functions by means of integrals:

$$B(u, v) = \int_0^1 \frac{x^{u-1} dx}{(1+x)^{u+v}}$$

$$\Gamma(u) = \int_0^\infty e^{-x} x^{u-1} dx$$

which converge only when $u, v > 0$. Under this condition we saw that

$$B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}.$$

We propose to show that $\Gamma(u)$ can be developed in the infinite product

$$G = \frac{1}{u} \prod_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^u}{1 + \frac{u}{n}}.$$

This product converges, as we saw, 197, 3, for any $u \neq 0, -1, -2, \dots$. From 201, 7 and 207 it is obvious that G converges absolutely and uniformly at any point u different from these singular points. Thus the expression 4) has a wider domain of definition than that of 2). Since $G = \Gamma$, as we said, for $u > 0$, we shall extend the definition of the Γ function in accordance with 4), to negative u .

It frequently happens that a function $f(x)$ can be represented by different analytic expressions whose domains of convergence are different. For example, we saw 218, 9), that $\tan x$ can be developed in a power series

and

$$\tan x = 2 \sum_1^{\infty} \frac{x}{\left(2^2 8^2 \dots 1^2\right)^2 \pi^2 - x^2} \quad \text{by}$$

are analytic expressions valid for every x for which $\tan x$ is defined.

221. 1. Before showing that G and Γ have the same properties for $u > 0$, let us develop some of the properties of the function $G(u)$ (see 220, 4). In the first place, we have, by 210:

The function $G(u)$ is continuous, except at the points $u = -2, -4, \dots$

Since the factors of 4) are all positive for $u > 0$,

$G(u)$ is positive for $u > 0$.

2. In the vicinity of the point $x = -m$, where $m = 0, 1, 2, \dots$

$$G(u) = \frac{H(u)}{x + m}$$

where $H(u)$ is continuous near this point, and $H(-m) \neq 0$ at this point.

For

$$G(u) = \frac{\left(1 + \frac{1}{m}\right)^u}{1 + \frac{u}{m}} H(u)$$

where H is the infinite product G with one factor removed. If we may reason on H as we did on G , we see H converges for $x = -m$. Hence $H \neq 0$ at this point. But H also converges uniformly about this point; hence H is continuous at

Also

$$(u+1)(u+2)\cdots(u+n-1) = (n-1)! \left(1 + \frac{u}{1}\right) \left(1 + \frac{u}{2}\right) \cdots \left(1 + \frac{u}{n-1}\right)$$

Thus $P_n = G_n$. But $G_n \doteq G$, hence P_n is convergent, and $\lim P_n = G$.

223. Euler's Constant. This is defined by the convergent series

$$G = \sum_{n=1}^{\infty} \left\{ \frac{1}{n} - \log \left(1 + \frac{1}{n} \right) \right\}.$$

It is easy to see at once that

$$G < \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2} H_2 = \frac{\pi^2}{12} = .82 \dots$$

by 218, 7). By calculation it is found that

$$G = .577215 \dots$$

224. Another expression of G is

$$G = \lim_{n \rightarrow \infty} \left\{ \frac{e^{eu}}{u \Pi \left(1 + \frac{u}{n} \right)^e} - \frac{e^u}{u} \right\}, \quad n = 1, 2, \dots$$

where G is the Eulerian constant.

For when $a \rightarrow 0$, $a^u = e^{u \log a}$.

Hence

$$G = \lim_{n \rightarrow \infty} \left\{ \frac{1}{u} \frac{e^{u \log \left(1 + \frac{1}{n} \right)}}{1 + \frac{u}{n}} - \frac{e^u}{u} \right\}$$

are convergent. Hence

$$G = \frac{He^{u[\log(1 + \frac{1}{n}) - \frac{1}{n}]}}{uH\left(1 + \frac{u}{n}\right)e^{-\frac{u}{n}}}$$

from which 1) follows at once, using 223.

225. Further Properties of G .

$$1. \quad G(u+1) = uG(u).$$

Let us use the product

$$P_n(u) = \frac{1}{u} \cdot \frac{(n-1)!}{(u+1) \cdots (u+n-1)}$$

employed in 222. Then

$$P_n(u+1) = \frac{nuP_n(u)}{u+n}.$$

As

$$\frac{nu}{u+n} \doteq u \quad \text{as } n \doteq \infty$$

we get 1) from 2) at once on passing to the limit.

$$2. \quad G(u+n) = u(u+1) \cdots (u+n-1)G(u).$$

This follows from 1) by repeated application.

$$3. \quad G(n) = 1 \cdot 2 \cdot \cdots \cdot n-1 = (n-1)!$$

where n is a positive integer.

$$4. \quad G(u)G(1-u) = \frac{\pi}{\sin \pi u}.$$

For

$$G(1-u) = -uG(-u) \quad \text{by 1)}$$

$$= \frac{e^{u\pi i}}{e^{u\pi i}} \quad \text{by 2)}$$

We now use 213, 5).

Let us note that by virtue of 1, 2 the value of G is known for all $u > 0$, when it is known in the interval $(0, 1)$. By virtue of 5) G is known for $u < 0$ when its value is known for $u > 0$. Moreover the relation 5) shows the value of G is known in $(-\frac{1}{2}, \frac{1}{2})$ when its value is known in $(0, \frac{1}{2})$.

As a result of this we see G is known when its values in any interval $(0, \frac{1}{2})$ are known; or indeed in any interval of length $\frac{1}{2}$.

Gauss has given a table of $\log G(u)$ for $1 \leq u \leq 1.5$ calculated to 20 decimal places. A four-place table is given in "A Table of Integrals" by B. O. Peirce, for $1 \leq u \leq 2$.

$$5. \quad G\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

For in 5) set $u = \frac{1}{2}$. Then

$$G^2\left(\frac{1}{2}\right) = \pi.$$

$$\text{Hence} \quad G\left(\frac{1}{2}\right) = \pm \sqrt{\pi}.$$

We must take the plus sign here, since $G > 0$ when $u > 0$, by 1).

$$6. \quad G\left(\frac{2n+1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \cdots 2n-1}{2^n} \cdot \sqrt{\pi}$$

where n is a positive integer.

$$\text{For} \quad G\left(\frac{2n+1}{2}\right) = G\left(1 + \frac{2n-1}{2}\right) = \frac{2n-1}{2} G\left(\frac{2n-1}{2}\right),$$

$$\text{Similarly} \quad G\left(\frac{2n-1}{2}\right) = \frac{2n-3}{2} G\left(\frac{2n-3}{2}\right), \text{ etc.}$$

$$\text{Thus} \quad G\left(\frac{2n+1}{2}\right) = \frac{2n-1}{2} \cdot \frac{2n-3}{2} \cdots \frac{3}{2} \cdot \frac{1}{2} G\left(\frac{1}{2}\right)$$

We may write 2)

$$L' = -C + \sum_1^{\infty} \left\{ \frac{1}{n} - \frac{1}{u+n-1} \right\}$$

That the relations 2), 3) hold for any $u \neq 0$, — by reasoning similar to that employed in 216. In

$$L^{(r)} = (-1)^r (r-1)! \sum_1^{\infty} \frac{1}{(u+n-1)^r}$$

In particular,

$$L'(1) = -C.$$

$$L^{(r)}(1) = (-1)^r (r-1)! \sum \frac{1}{n^r} = (-1)^r (r-1)! \zeta(r)$$

227. *Development of $\log G(u)$ in a Power Series* — development is valid about the point $u=1$, we have

$$\log G(u) = L(u) = L(1) + \frac{u-1}{1!} L'(1) + \frac{(u-1)^2}{2!} L''(1) + \dots$$

or using 226, 5), and setting $u = 1+x$,

$$\log G(1+x) = -Cx + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} H_n x^n$$

We show now *this relation is valid for $-\frac{1}{2} \leq x$* that

$$R_s = \frac{x^s}{s!} L^{(s)}(1+\theta x), \quad 0 < \theta < 1$$

converges to 0, as $s \rightarrow \infty$.

For, if $0 \leq x \leq 1$, then

$$|R_s| \leq \frac{1}{s!} \sum_{n=0}^{\infty} \frac{1}{n!} \rightarrow 0.$$

is shown how the series 1) may be made to converge more rapidly. We have for any x in \mathfrak{A}

$$\log(1+x) = x - \sum_2^{\infty} (-1)^n \frac{x^n}{n}.$$

This on adding and subtracting from 1) gives

$$\log G(1+x) = -\log(1+x) + (1-C)x + \sum_2^{\infty} (-1)^n (H_n - 1) \frac{x^n}{n}.$$

(Changing here x into $-x$ gives

$$\log G(1-x) = -\log(1-x) - (1-C)x + \sum (H_n - 1) \frac{x^n}{n}.$$

Subtracting this from the foregoing gives

$$\begin{aligned} \log G(1+x) - \log G(1-x) \\ = -\log \frac{1+x}{1-x} + 2(1-C)x - \sum_1^{\infty} \frac{x^{2m+1}}{2m+1} (H_{2m+1} - 1). \end{aligned}$$

From 225, 4

$$\log G(1+x) + \log G(1-x) = \log \frac{\pi x}{\sin \pi x}.$$

This with the preceding relation gives

$$\begin{aligned} \log G(1+x) \\ = (1-C)x - \frac{1}{2} \log \frac{1+x}{1-x} + \log \frac{\pi x}{\sin \pi x} - \frac{1}{2} \sum_1^{\infty} (H_{2m+1} - 1) \frac{x^{2m+1}}{2m+1} \end{aligned}$$

valid in \mathfrak{A} .

This series converges rapidly for $0 \leq x \leq \frac{1}{2}$, and enables us to compute $G(u)$ in the interval $1 \leq u \leq \frac{3}{2}$. The other values of $G(u)$ may be readily obtained as already observed.

Now for any x in the interval $(0, n)$,

$$x^u \leq n^u \quad , \quad x^u > x n^{u-1}$$

since $u \geq 0$ and $u - 1 \leq 0$.

Also for any x in the interval (n, ∞)

$$x^u < x n^{u-1} \quad , \quad x^u > n^u.$$

Hence

$$\begin{aligned} n^{u-1} \int_0^n e^{-x} x^n dx + n^u \int_n^\infty e^{-x} x^{n-1} dx &< \Gamma(u+n) \\ &< n^u \int_0^n e^{-x} x^{n-1} dx \end{aligned}$$

Thus

$$\begin{aligned} \frac{\Gamma(u+n)}{n^u} &< \int_0^n e^{-x} x^{n-1} dx + \frac{1}{n} \int_n^\infty e^{-x} x^n dx \\ &< \int_0^n e^{-x} x^{n-1} dx + \frac{1}{n} \int_0^\infty e^{-x} x^n dx \end{aligned}$$

Let us call these integrals A , B , C respectively.

We see at once that

$$B = \frac{\Gamma(n+1)}{n} = \frac{n!}{n} = (n-1)!$$

Also, integrating by parts,

$$A = \left[\frac{e^{-x} x^n}{n} \right]_0^n + \frac{1}{n} \int_0^n e^{-x} x^n dx = \frac{n^n}{n e^n} -$$

Thus

$$\frac{\Gamma(u+n)}{n^u} < (n-1)! + \frac{n^{n-1}}{e^n}$$

Similarly

$$\frac{\Gamma(u+n)}{n^u} > (n-1)! - \frac{n^{n-1}}{e^n}$$

But

$$\nu_n > 1 + \frac{n}{n+1} + \dots + \frac{n^m}{(n+1) \dots (n+m)} \quad , \quad \text{for any } m$$

$$> \frac{mn^m}{(n+1) \dots (n+m)} = \frac{m}{\left(1 + \frac{1}{n}\right) \dots \left(1 + \frac{m}{n}\right)} > \frac{m}{\left(1 + \frac{m}{n}\right)^m}.$$

Let us take

$$n > m^2 \quad \text{or} \quad \frac{m}{n} < \frac{1}{m}.$$

Then

$$\nu_n > \frac{m}{\left(1 + \frac{1}{m}\right)^m} > \frac{m}{e}.$$

Since m may be taken large at pleasure,

$$\lim \nu_n = \infty$$

and hence

$$\lim q_n = 0.$$

Thus

$$\lim_{n \rightarrow \infty} \frac{\Gamma(u+n)}{(n-1)! n^u} = 1 \quad , \quad 0 \leq u \leq 1.$$

But from $\Gamma(u+1) = u\Gamma(u)$ we have

$$\frac{\Gamma(u+1+n)}{n^{u+1}(n-1)!} = \frac{u+n}{n} \frac{\Gamma(u+n)}{n^u(n-1)!} = 1$$

so, as $n \rightarrow \infty$. Thus the relation 1) holds for $1 \leq u \leq 2$, and so on, for any $u > 0$.

As

$$\Gamma(u+n) = u(u+1) \dots (u+n-1)\Gamma(u),$$

we have

$$\Gamma(u) = \frac{\Gamma(u+n)}{u(u+1) \dots (u+n-1)}.$$

Hence using 1),

$$\Gamma(u) = \frac{(n-1)! n^u}{u(u+1) \dots (u+n-1)} \cdot \frac{\Gamma(u+n)}{(n-1)! n^u}.$$

Letting $n \rightarrow \infty$, we get $\Gamma(u) = G(u)$ for any $u > 0$, making use of 1) and 222 1).

CHAPTER VIII

AGGREGATES

Equivalence

229. 1. Up to the present the aggregates have been point aggregates. We now consider aggregates in general. Any collection of well-determined objects is distinguishable one from another, and thought of as an *aggregate* or *set*.

Thus the class of prime numbers, the collections of nations, the inhabitants of the United States,

Some of the definitions given for point aggregates apply to aggregates in general, and we shall not repeat them here, as it is only necessary to replace the word object or element.

As in point sets, $\mathfrak{A} = 0$ shall mean that \mathfrak{A} is empty.

Let \mathfrak{A} , \mathfrak{B} be two aggregates such that each element a of \mathfrak{A} is associated with some one element b of \mathfrak{B} , and vice versa, that \mathfrak{A} is *equivalent* to \mathfrak{B} and write

$$\mathfrak{A} \sim \mathfrak{B}.$$

We also say \mathfrak{A} and \mathfrak{B} are in *one to one correspondence*. To indicate that a corresponds to b in this correspondence we write

$$a \sim b.$$

EQUIVALENCE

For we can associate the elements of \mathfrak{A} with \mathfrak{B} keeping precisely the correspondence which exists between the elements of \mathfrak{B} and B , of \mathfrak{C} and C , etc.

Example 1. $\mathfrak{A} = 1, 2, 3, \dots$

$$\mathfrak{B} = a_1, a_2, a_3, \dots$$

If we set $a_n \sim n$, \mathfrak{A} and \mathfrak{B} will stand in 1, 1 correspondence.

Example 2. $\mathfrak{A} = 1, 2, 3, 4, \dots$

$$\mathfrak{B} = 2, 4, 6, 8, \dots$$

If we set n of \mathfrak{A} in correspondence with $2n$ of \mathfrak{B} , we have uniform correspondence.

We note that \mathfrak{B} is a part of \mathfrak{A} ; we have thus a partial aggregate of an infinite aggregate may be put in uniform correspondence with a partial aggregate of itself.

This is obviously impossible if \mathfrak{A} is finite.

Example 3. $\mathfrak{A} = 1, 2, 3, 4, \dots$

$$\mathfrak{B} = 10^1, 10^2, 10^3, 10^4, \dots$$

If we set $n \sim 10^n$, we establish a uniform correspondence between \mathfrak{A} and \mathfrak{B} . We note again that $\mathfrak{A} \sim \mathfrak{B}$ although \mathfrak{B} is a part of \mathfrak{A} .

Example 4. Let $\mathfrak{C} = \{\xi\}$, where, using the triadic notation,

$$\xi = \cdot \xi_1 \xi_2 \xi_3 \dots \quad \xi_n = 0, 2$$

denote the Cantor set of 1, 272. Let us associate

$$x = \cdot x_1 x_2 x_3 \dots$$

where $x_n = 0$ when $\xi_n = 0$, and $= 1$ when $\xi_n = 2$, thus establishing a 1, 1 correspondence between the elements of \mathfrak{C} and \mathfrak{A} .

viously we can select enough of these del to make their lower content is as near 1 as we choose.

$$\text{Cont } U = 1.$$

As $\text{Cont } U \leq 1$, U is metric and its content is discrete.

230. 1. Let $\mathfrak{A} = \alpha + A$, $\mathfrak{B} = \beta + B$, where α, β are elements of $\mathfrak{A}, \mathfrak{B}$ respectively. If $\mathfrak{A} \sim \mathfrak{B}$, then $A \sim B$.

For, since $\mathfrak{A} \sim \mathfrak{B}$, each element a of \mathfrak{A} is one element b of \mathfrak{B} , and the same holds for β that $\alpha \sim \beta$, the uniform correspondence of A with B is established. On the contrary $\alpha \sim \beta'$ and $\beta \sim \alpha'$, the uniform correspondence between A, B can be established by setting α correspond to β' and the other elements in A, B correspond as in $\mathfrak{A} \sim \mathfrak{B}$.

2. We state as obvious the theorems:

No part \mathfrak{B} of a finite set \mathfrak{A} can be $\sim \mathfrak{A}$.

No finite part \mathfrak{B} of an infinite set \mathfrak{A} can be $\sim \mathfrak{A}$.

Cardinal Numbers

231. 1. We attach now to each aggregate \mathfrak{A} a number, called its *cardinal number*, which is defined as follows:

1° Equivalent aggregates have the same cardinal number.

2° If \mathfrak{A} is \sim to a part of \mathfrak{B} , but \mathfrak{B} is not \sim to a part of \mathfrak{A} , the cardinal number of \mathfrak{A} is less than that of \mathfrak{B} . If \mathfrak{B} is \sim to a part of \mathfrak{A} , but \mathfrak{A} is not \sim to a part of \mathfrak{B} , the cardinal number of \mathfrak{B} is greater than that of \mathfrak{A} .

2. It is a property of any two *finite* cardinal numbers either

$$a = b \quad , \quad \text{or } a > b \quad , \quad \text{or } a < b.$$

This property has not yet been established for cardinal numbers. There is in fact a fourth alternative $\mathfrak{A}, \mathfrak{B}$, besides the three involved in 1). For until it has been shown, there is the possibility that:

No part of \mathfrak{A} is $\sim \mathfrak{B}$, and no part of \mathfrak{B} is $\sim \mathfrak{A}$.

The reader should thus guard against expressing himself as assuming that one of the three relations 1) must hold between two cardinal numbers.

3. We note here another difference. If $\mathfrak{A}, \mathfrak{B}$ have no common element,

$$\text{Card}(\mathfrak{A} + \mathfrak{B}) > \text{Card } \mathfrak{A}.$$

Let now \mathfrak{A} denote the positive even and \mathfrak{B} the positive odd numbers. Obviously

$$\text{Card}(\mathfrak{A} + \mathfrak{B}) = \text{Card } \mathfrak{A} = \text{Card } \mathfrak{B}$$

and the relation 2) does not hold for these transfinite numbers.

4. We have, however, the following:

Let $\mathfrak{A} > \mathfrak{B}$, then

$$\text{Card } \mathfrak{A} \geq \text{Card } \mathfrak{B}.$$

For obviously \mathfrak{B} is \sim to a part of \mathfrak{A} , viz. \mathfrak{B} itself.

5. This may be generalized as follows:

Let

$$\mathfrak{A} = \mathfrak{B} + \mathfrak{C} + \mathfrak{D} + \dots$$

$$A = B + C + D + \dots$$

Enumerable Sets

232. 1. An aggregate which is equivalent to a finite set of positive integers \mathfrak{F} or to a part of \mathfrak{F} is

finite. Thus all finite aggregates are enumerable. A number is attached to an infinite enumerable set in the following manner.

At times we shall also denote this cardinal number by

$$\epsilon = \aleph_0.$$

2. *Every infinite aggregate \mathfrak{A} contains an infinite subset.*

For let a_1 be an element of \mathfrak{A} and

$$\mathfrak{A} = a_1 + \mathfrak{A}_1.$$

Then \mathfrak{A}_1 is infinite; let a_2 be one of its elements and

$$\mathfrak{A}_1 = a_2 + \mathfrak{A}_2.$$

Then \mathfrak{A}_2 is infinite, etc.

Then

$$\mathfrak{B} = a_1, a_2, \dots$$

is a part of \mathfrak{A} and forms an infinite enumerable set.

3. From this follows that

\aleph_0 is the least transfinite cardinal number.

233. *The rational numbers are enumerable.*

For any rational number may be written

$$r = \frac{m}{n}$$

where, as usual, m is relatively prime to n .

The equation

$$|m| + |n| = p$$

admits but a finite number of solutions for

Let us now arrange these solutions in a sequence, putting those corresponding to $p = q$ before those corresponding to $p = q + 1$. We get

$$r_1, r_2, r_3 \dots$$

which is obviously enumerable.

234. *Let the indices $\iota_1, \iota_2, \dots, \iota_p$ range over enumerable sets. The*

$$\mathfrak{A} = \{a_{\iota_1 \dots \iota_p}\}$$

is

For the equation

$$\nu_1 + \nu_2 + \dots + \nu_p = n,$$

where the ν 's are positive integers, admits but a finite number of solutions for each $n = p, p + 1, p + 2, p + 3 \dots$. Thus the elements of

$$\mathfrak{B} = \{b_{\nu_1 \dots \nu_p}\}$$

may be arranged in a sequence

$$b_1, b_2, b_3 \dots$$

by giving to n successively the values $p, p + 1, \dots$ and putting the elements $b_{\nu_1 \dots \nu_p}$ corresponding to $n = q + 1$ after those corresponding to $n = q$.

Thus the set \mathfrak{B} is enumerable. Consider now \mathfrak{A} . Since each index ι_m ranges over an enumerable set, each value of ι_m as ι'_m is associated with some positive integer as m' and conversely. We may now establish a 1, 1 correspondence between \mathfrak{A} and \mathfrak{B} by putting

$$b_{m'_1 m'_2 \dots m'_p} \sim a_{\iota'_1 \iota'_2 \dots \iota'_p}.$$

Hence \mathfrak{A} is enumerable.

235. 1. *An enumerable set of enumerable aggregates form*

Thus the α -elements in 1) form a set

$$\{\alpha_{mn} \} \quad m, n, = 1, 2,$$

which is enumerable by 234.

2. *The real algebraic numbers form an enumerable aggregate.*

For each algebraic number is a root of an irreducible equation of the form

$$x^n + a_1 x^{n-1} + \dots + a_n = 0$$

the a 's being rational numbers. Thus the algebraic numbers may be represented by

$$\{\rho_n, a_1 a_2 \dots a_n\}$$

where the index n runs over the positive integers and the a 's over the rational numbers.

3. *Let $\mathfrak{A}, \mathfrak{B}$ be two enumerable sets. The*

$$\text{Card } \mathfrak{A} = \text{Card } \mathfrak{B} = \aleph_0$$

$$\text{Card } (\mathfrak{A} + \mathfrak{B}) = \aleph_0$$

And in general if $\mathfrak{A}_1, \mathfrak{A}_2, \dots$ are an enumerable aggregate,

$$\text{Card } (\mathfrak{A}_1, \mathfrak{A}_2, \dots) = \aleph_0.$$

This follows from 1.

236. *Every isolated aggregate \mathfrak{A} , limited or unlimited, is enumerable.*

For let us divide \mathfrak{A}_m into cubes of side 1. Let \mathfrak{U} be an enumerable set $\mathcal{U}_1, \mathcal{U}_2, \dots$. About each point of \mathfrak{U} as center we describe a cube of side σ , so small that no other point of \mathfrak{A} lies within it. This is possible since \mathfrak{A} is isolated.

where \mathfrak{A}_i denotes the isolated points of \mathfrak{A} and \mathfrak{A}'_p the limit points of \mathfrak{A} .

Similarly,

$$\mathfrak{A}'_p = \mathfrak{A}'_{p,i} + \mathfrak{A}''_p$$

$$\mathfrak{A}''_p = \mathfrak{A}''_{p,i} + \mathfrak{A}'''_p$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Thus,

$$\mathfrak{A} = \mathfrak{A}_i + \mathfrak{A}'_{p,i} + \mathfrak{A}''_{p,i} + \dots + \mathfrak{A}^{(n)}_p$$

But $\mathfrak{A}^{(n)}$ is finite and $\mathfrak{A}^{(n)}_p \leq \mathfrak{A}^{(n)}$.

Thus \mathfrak{A} being the sum of $n + 1$ enumerable

2. *If \mathfrak{A}' is enumerable, so is \mathfrak{A} .*

For as in 1,

$$\mathfrak{A} = \mathfrak{A}_i + \mathfrak{A}'_p$$

and

$$\mathfrak{A}'_p \leq \mathfrak{A}'$$

238. 1. *Every infinite aggregate \mathfrak{A} contains $\mathfrak{B} \sim \mathfrak{A}$.*

For let $\mathfrak{E} = (a_1, a_2, a_3 \dots)$ be an infinite enumerable aggregate so that

$$\mathfrak{A} = \mathfrak{E} + \mathfrak{F}.$$

Let

$$\mathfrak{E} = a_1 + E.$$

To establish a uniform correspondence between \mathfrak{A} and \mathfrak{B} we associate a_n in \mathfrak{E} with a_{n+1} in E . Thus $\mathfrak{E} \sim E$.

We now set

$$\mathfrak{B} = E + \mathfrak{F}.$$

239. 1. A theorem of great importance is that whether two aggregates are equivalent is the converse of 238, 2.

Let $\mathfrak{A}_1 < \mathfrak{A}$, $\mathfrak{B}_1 < \mathfrak{B}$. If $\mathfrak{A}_1 \sim \mathfrak{B}$ and $\mathfrak{A} \sim \mathfrak{B}_1$ then

$$\mathfrak{A} \sim \mathfrak{B}.$$

In the correspondence $\mathfrak{A}_1 \sim \mathfrak{B}$, let \mathfrak{A}_2 be associated with \mathfrak{B}_1 . Then

$$\mathfrak{A}_2 \sim \mathfrak{B}_1 \sim \mathfrak{A}$$

and hence

$$\mathfrak{A} \sim \mathfrak{A}_2.$$

But as $\mathfrak{A}_1 > \mathfrak{A}_2$, we would infer from 1)

$$\mathfrak{A} \sim \mathfrak{A}_1.$$

As $\mathfrak{A}_1 \sim \mathfrak{B}$ by hypothesis, the truth of 1) follows once from 2).

To establish 2) we proceed thus. In the correspondence $\mathfrak{A}_1 \sim \mathfrak{B}$, let \mathfrak{A}_2 be that part of \mathfrak{A}_1 which $\sim \mathfrak{B}_1$ in \mathfrak{A} . In the correspondence $\mathfrak{A}_2 \sim \mathfrak{B}_1$, let \mathfrak{A}_3 be that part of \mathfrak{A}_2 which $\sim \mathfrak{B}_2$ in \mathfrak{A}_1 . Continuing in this way, we get the indefinite sequence

$$\mathfrak{A} > \mathfrak{A}_1 > \mathfrak{A}_2 > \mathfrak{A}_3 > \dots$$

such that

$$\mathfrak{A} \sim \mathfrak{A}_2 \sim \mathfrak{A}_4 \sim \dots$$

$$\mathfrak{A}_1 \sim \mathfrak{A}_3 \sim \mathfrak{A}_5 \sim \dots$$

Let now

$$\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{C}_1, \quad \mathfrak{A}_1 = \mathfrak{A}_2 + \mathfrak{C}_2$$

$$\mathfrak{D} = \text{Dev}(\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}_2, \dots)$$

Then

$$\mathfrak{A} = \mathfrak{D} + \mathfrak{C}_1 + \mathfrak{C}_2 + \mathfrak{C}_3 + \dots$$

and similarly

$$\mathfrak{A}_1 = \mathfrak{D} + \mathfrak{C}_2 + \mathfrak{C}_3 + \mathfrak{C}_4 + \dots$$

2. In connection with the foregoing proof, *v. Bernstein*, the reader must guard against the fallacy that does not *in general* follow from

$$\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{C}_1 \quad , \quad \mathfrak{A}_2 = \mathfrak{A}_3 + \mathfrak{C}_3 \quad , \quad \mathfrak{A} \sim \mathfrak{A}_2$$

that

$$\mathfrak{C}_1 \sim \mathfrak{C}_3$$

which is the first relation in 5).

Example. Let $\mathfrak{A} = (1, 2, 3, 4, \dots)$.

$$\mathfrak{A}_1 = (2, 3, 4, 5, \dots) \quad , \quad \mathfrak{A}_2 = (3, 4, 5, 6, \dots)$$

$$\mathfrak{A}_3 = (5, 6, 7, 8, \dots).$$

Then

$$\mathfrak{C}_1 = 1 \quad \mathfrak{C}_3 = (3, 4).$$

Now \mathfrak{A} , \mathfrak{A}_1 , \mathfrak{A}_2 , \mathfrak{A}_3 are all enumerable sets; hence

$$\mathfrak{A} \sim \mathfrak{A}_2 \quad , \quad \mathfrak{A}_1 \sim \mathfrak{A}_3.$$

But obviously \mathfrak{C}_1 is not equivalent to \mathfrak{C}_3 , since only one element cannot be put in 1 to 1 correspondence with a set consisting of two elements.

240. 1. If $\mathfrak{A} > \mathfrak{B} > \mathfrak{C}$, and $\mathfrak{A} \sim \mathfrak{C}$, then $\mathfrak{A} \sim \mathfrak{B}$.

For by hypothesis a part of \mathfrak{B} , viz. \mathfrak{C} , is $\sim \mathfrak{A}$.
is $\sim \mathfrak{B}$, viz. \mathfrak{B} itself. We apply now 239.

2. Let α be any cardinal number. If

$$\alpha \leq \text{Card } \mathfrak{B} \leq \alpha,$$

then

$$\alpha = \text{Card } \mathfrak{B}.$$

For let $\text{Card } \mathfrak{A} = \alpha$. Then from

$$\alpha \leq \text{Card } \mathfrak{B}$$

it follows that $\mathfrak{A} \sim$ a part or the whole of \mathfrak{B} ; which

4. *Two infinite enumerable sets are equivalent.*

For both are equivalent to \mathfrak{N} , the set of positive integers.

241. 1. *Let \mathfrak{C} be any enumerable set in \mathfrak{A} ; set $\mathfrak{A} - \mathfrak{C}$ is infinite, $\mathfrak{A} \sim \mathfrak{A} - \mathfrak{C}$.*

For $\mathfrak{A} - \mathfrak{C}$ being infinite, contains an infinite enumerable set \mathfrak{B} . Let $\mathfrak{B} = \mathfrak{C} + \mathfrak{D}$. Then

$$\mathfrak{A} = \mathfrak{C} + \mathfrak{B} + \mathfrak{D},$$

$$\mathfrak{B} = \mathfrak{C} + \mathfrak{D}.$$

But $\mathfrak{C} + \mathfrak{B} \sim \mathfrak{B}$. Hence $\mathfrak{A} \sim \mathfrak{A} - \mathfrak{C}$.

2. We may state 1 thus:

$$\text{Card}(\mathfrak{A} - \mathfrak{C}) = \text{Card } \mathfrak{A}$$

provided $\mathfrak{A} - \mathfrak{C}$ is infinite.

3. From 1 follows at once the theorem:

Let \mathfrak{A} be any infinite set and \mathfrak{C} an enumerable set.

$$\text{Card}(\mathfrak{A} + \mathfrak{C}) = \text{Card } \mathfrak{A}.$$

Some Space Transformations

242. 1. Let T be a transformation of space such that to each point x corresponds a single point x_T , and conversely.

Moreover, let x, y be *any* two points of space. At the transformation they go over into x_T, y_T . If

$$\text{Dist}(x, y) = \text{Dist}(x_T, y_T)$$

If \mathfrak{R} denotes the original space, and \mathfrak{R}_t the space after displacement, we have, obviously,

$$\mathfrak{R} \sim \mathfrak{R}_t.$$

2. Let $y_1 = tx_1, \dots, y_m = tx_m, \dots, t$

Then when x ranges over the m -way space \mathfrak{X} , y ranges over the m -way space \mathfrak{Y} . If we set $x \sim y$ as defined by

$$x \sim y.$$

Also

$$\text{Dist}(0, y) = t \text{Dist}(0, x)$$

We call 1) a transformation of *similitude*. The space is *dilated*; if $t < 1$, it is *contracted*.

3. Let Q be any point in space. About it describe a sphere S of radius R . Let P be any point in space. Join P, Q ; let us take a point P' such that

$$\text{Dist}(P', Q) = \frac{R^2}{\text{Dist}(P, Q)}.$$

Then P' is called the *inverse of P with respect to S*. The transformation of space is called *inversion*. Q is the center of inversion.

Obviously points without S go over into points within S and conversely. As $P \rightarrow \infty$, $P' \rightarrow Q$.

The correspondence between the old and new spaces is one-to-one except there is no point corresponding to Q .

The Cardinal c

243. 1. All or any part of space \mathfrak{S} may be transformed into space \mathfrak{S}' by a transformation of similitude.

correspondence between \mathfrak{S} and the resulting aggregate is uniform since all the transformations employed are uniform.

As a result of this and 240, 1 we see that the set of real numbers is \sim to those lying in the interval $(0, 1)$, the aggregate of all points of \mathfrak{R}_m is \sim to the points of a unit sphere, etc.

244. 1. *The points lying in the unit interval are not enumerable.*

For if they were, they could be arranged in a sequence

$$a_1, a_2, a_3 \dots$$

Let us express the a 's as decimals in the normal form

$$a_n = . a_{n1} a_{n2} a_{n3} \dots$$

Consider the decimal

$$b = . b_1 b_2 b_3 \dots$$

also written in the normal form, where

$$b_1 \neq a_{1,1} \quad , \quad b_2 \neq a_{2,2} \quad , \quad b_3 \neq a_{3,3} \quad , \quad \dots$$

Then b lies in \mathfrak{A} and is yet different from any number in the sequence.

2. We have $(0^*, 1^*) \sim (0, 1)$, by 241, 3.
 $\sim (a, b)$, by 243,

where a, b are finite or infinite.

Thus the cardinal number of any interval, finite or infinite, with or without its end points is the same.

We denote it by c and call it *the cardinal number of the linear continuum*, or of the real number system \mathfrak{R} .

4. The cardinal number of the Cantor set \mathfrak{C} of I, 272 is

For each point α of \mathfrak{C} has the representation in the system

$$\alpha = \cdot a_1 a_2 a_3 \cdots, \quad a = 0, 2.$$

But if we read these numbers in the dyadic system, each $a_n = 2$ by the value 1, we get all the points in the $(0, 1)$. As there is a uniform correspondence between sets of points, the theorem is established.

245. An enumerable set \mathfrak{A} is not perfect, and conversely set is not enumerable.

For suppose the enumerable set

$$\mathfrak{A} = a_1, a_2 \cdots$$

were perfect. In $D_{r_1}^*(a_1)$ lies an infinite partial set since by hypothesis \mathfrak{A} is perfect. Let a_{m_2} be the point of index in \mathfrak{A}_1 . Let us take $r_2 < r_1$ such that $D_{r_2}(a_{m_2}) \subset D_{r_1}^*(a_1)$. In $D_{r_2}^*(a_{m_2})$ lies an infinite partial set \mathfrak{A}_2 of a_{m_2} be the point of lowest index in \mathfrak{A}_2 , etc.

Consider now the sequence

$$a_1, a_{m_2}, a_{m_3}, \cdots$$

It converges to a point α by I, 127, 2. But α lies in \mathfrak{A} , is perfect. Thus α is some point of 1), say $\alpha = a_s$. leads to a contradiction. For a_s lies in every $D_{r_{m_n}}^*(a_{m_n})$ other hand, no point in this domain has an index as 1 which $\doteq \infty$, as $n \doteq \infty$. Thus \mathfrak{A} cannot be perfect.

Conversely, suppose the perfect set \mathfrak{A} were enumerable is impossible, for we have just seen that when \mathfrak{A} is enumerable cannot be perfect.

Let us now complete \mathfrak{C} by adding its faces, ob

By a transformation of similitude T we can b

Hence

$$\text{Card } \mathfrak{C} \geq \text{Card } C.$$

On the other hand, \mathfrak{C} is a part of C , hence

$$\text{Card } \mathfrak{C} \leq \text{Card } C.$$

Thus $\text{Card } C = c$. The rest of the theorem fo

248. Let $\mathfrak{F} = \{f\}$ denote the aggregate of one-
functions over a unit cube \mathfrak{C} in \mathfrak{R}_n .

Then

$$\text{Card } \mathfrak{F} = c.$$

Let C denote the rational points of \mathfrak{C} , i.e. whose coördinates are rational. Then any f is values over C are known. For if α is an irrational we can approach it over a sequence of rational points. But f being continuous, $f(\alpha) = \lim f(a_n)$, and . On the other hand, C being enumerable, we can in a sequence

$$C = c_1, c_2, \dots$$

Let now \mathfrak{R}_∞ be a space of an infinite enumerable dimensions, and let $y = (y_1, y_2, \dots)$ denote any

Let f have the value η_1 at c_1 , the value η_2 at the points of C . Then the complex η_1, η_2, \dots determines f in \mathfrak{C} . But this complex also determines $\eta = (\eta_1, \eta_2, \dots)$ in \mathfrak{R}_∞ . We now associate f with

$$\text{Card } \mathfrak{F} \leq \text{Card } \mathfrak{R} = c.$$

But obviously $\text{Card } \mathfrak{F} \geq c$, for among the ele

aggregate formed of all possible a 's of this cardinal number.

Let β be an arbitrary element of \mathfrak{B} . Let a be that a which has the value 1 for $b = \beta$ and 0 for all other b 's. This establishes a correspondence between β and a , a part of \mathfrak{A} . Hence

$$a \geq b.$$

Suppose $a = b$. Then there exists a a which associates with each b some one a and no other. This is impossible.

For call a_b that element of \mathfrak{A} which is associated with b . a_b has the value 1 or 2 for each β of \mathfrak{B} . In \mathfrak{A} an element a' which for each β of \mathfrak{B} has a different determination than the one a_b has. But a' is associated with some element of \mathfrak{B} , say b' . Then

$$a' = a_{b'}.$$

Then for $b = b'$, a' must have that one of the two values which $a_{b'}$ has. But it has not, hence the contradiction.

250. *The aggregate of limited integrable functions f in $\mathfrak{A} = (0, 1)$ has a cardinal number $\aleph > \mathfrak{c}$.*

For let $f(x) = 0$ in \mathfrak{A} except at the points of a Cantor set of I, 272, and 229, Ex. 4. At each point of this set f may have the value 1 or 2 at pleasure. The aggregate of all possible such functions has a cardinal number \mathfrak{c} . A reasoning of 249 shows. But each f is continuous except at a set which is discrete. Hence f is integrable.

We have now the following obvious relations :

$$\aleph_0 + n = \aleph_0 \quad , \quad n \text{ a positive integer}$$

$$\aleph_0 + \aleph_0 + \cdots + \aleph_0 = \aleph_0 \quad , \quad n \text{ terms.}$$

$$\aleph_0 + \aleph_0 + \cdots = \aleph_0 \quad , \quad \text{an infinite enumeration}$$

If the cardinal numbers of \mathfrak{A} , \mathfrak{B} , \mathfrak{C} are a , b , c

$$a + (b + c) = (a + b) + c,$$

$$a + b = b + a.$$

The first relation states that addition is *associative*.
that it is *commutative*.

252. Multiplication.

1. Let $\mathfrak{A} = \{a\}$, $\mathfrak{B} = \{b\}$ have the cardinal numbers a and b . The union of all the pairs (a, b) forms a set called $\mathfrak{A} \cdot \mathfrak{B}$. It is denoted by $\mathfrak{A} \cdot \mathfrak{B}$. We agree that $\mathfrak{A} \cdot \mathfrak{B}$ has the same cardinal number as $\mathfrak{B} \cdot \mathfrak{A}$. Then

$$\mathfrak{A} \cdot \mathfrak{B} = \mathfrak{B} \cdot \mathfrak{A}.$$

We define the product of a and b to be

$$\text{Card } \mathfrak{A} \cdot \mathfrak{B} = \text{Card } \mathfrak{B} \cdot \mathfrak{A} = a \cdot b =$$

2. We have obviously the following formal properties of cardinal numbers :

$$a(b \cdot c) = (a \cdot b)c,$$

$$a \cdot b = b \cdot a,$$

$$a(b + c) = ab + ac,$$

which express respectively the associative, commutative, and distributive properties of cardinal numbers.

Example 2. Let $\mathfrak{A} = \{a\}$ denote the family

$$x^2 + y^2 = a^2.$$

Let $\mathfrak{B} = \{b\}$ denote a set of segments of length a . Interpret (a, b) to be the points on a cylinder of radius a and whose height is b . Then $\mathfrak{A} \cdot \mathfrak{B}$ is the set of all cylinders.

253. 1. $\aleph_0 = n \cdot \aleph_0$, or $nc = c$.

For let

$$\mathfrak{A} = (a_1, a_2, \dots, a_n),$$

$$\mathfrak{C} = (c_1, c_2, \dots \text{ in inf. })$$

Then

$$\mathfrak{A} \cdot \mathfrak{C} = (a_1, c_1) \quad , \quad (a_1, c_2) \quad ,$$

$$(a_2, c_1) \quad , \quad (a_2, c_2) \quad ,$$

$$\dots \dots \dots$$

$$= \mathfrak{C}_1 + \mathfrak{C}_2 + \dots + \mathfrak{C}_n.$$

The cardinal number of the set on the left is \aleph_0 . The cardinal number of the set on the right is \aleph_0 .

2.

$$ec = c.$$

For let $\mathfrak{C} = \{c\}$ denote the points on a right line. Interpret (n, c) to be the points on a right line of length n . Then $\mathfrak{C} \mathfrak{C} = \{(n, c)\}$.

Then

$$\mathfrak{C} \mathfrak{C} = \{(n, c)\}$$

may be regarded as the points on a right line of length c .

$$\text{Card } \{L_n\} = c.$$

Hence

$$ec = \text{Card } \mathfrak{C} \mathfrak{C} = c.$$

object from any one of these classes in $C_2 \dots$ compartment. The result is a certain *distribution* from these k classes K , among the γ compartments

The number of distributions of objects from compartments is k^γ .

For in C_1 we may put an object from any Thus C_1 may be filled in k ways. Similarly k ways. Thus the compartments C_1, C_2 may Similarly C_1, C_2, C_3 may be filled in k^3 ways,

255. 1. The totality of distributions of objects K among the γ compartments C form an aggregate denoted by

$$K^C.$$

We call it the *distribution of K over C* . The distribution of this kind may be called the cardinal and we have then

$$\text{Card } K^C = k^\gamma.$$

2. What we have here set forth for finite extended to any aggregates, $\mathfrak{A} = \{a\}$, $\mathfrak{B} = \{b\}$ where a and b are members we call α, β . Thus the totality of distributions of \mathfrak{A} among the b 's, or the *distribution of \mathfrak{A} over \mathfrak{B}* ,

$$\mathfrak{A}^\mathfrak{B},$$

and its cardinal number is taken to be the definition of α^β . Thus,

$$\text{Card } \mathfrak{A}^\mathfrak{B} = \alpha^\beta.$$

256. *Example 1.* Let

$$x^n + a_1 x^{n-1} + \dots + a_n = 0$$

have rational number coefficients. Each coefficient a_i is chosen over the enumerable set of elements in the rational number system $R = \{r\}$, whose cardinal number is \aleph_0 . The a_i form a set $\mathfrak{A} = (a_i, \dots a_n) = \{a\}$. To the totality of such sets corresponds a distribution of the r 's among the

As $\text{Card } R^{\aleph} = \aleph_0 = e$
 we have the relation : $\aleph_0^n = \aleph_0$, or $e^n = e$
 for any integer n .

On the other hand, the equations 1) make the complex

$$(a_1, \dots a_n),$$

and the totality of equations 1) is associated

$$\mathfrak{E} = \{(a_1, \dots a_n)\}.$$

But

$$\{(a_1, a_2)\} = \{a_1\} \cdot \{a_2\},$$

$$\{(a_1, a_2, a_3)\} = \{(a_1, a_2)\} \cdot \{a_3\}$$

Hence

$$\mathfrak{E} = \{a_1\} \cdot \{a_2\} \cdots \{a_n\}$$

Thus

$$\text{Card } \mathfrak{E} = e \cdot e \cdots e \quad , \quad n$$

But

$$\text{Card } \mathfrak{E} = \text{Card } R^{\aleph},$$

since each of these sets is associated uniformly with 1). Thus

$$e^n = e \cdot e \cdots e \quad , \quad n$$

257. Example 2. Any point x in m -way space has m coördinates $x_1, x_2, \dots x_m$, each of which may be any real number of \aleph , whose cardinal number is \aleph . The coördinates $x_1 \cdots x_m$ form a finite set

$$\mathfrak{X} = (x_1, \dots x_m).$$

Thus to $\aleph_m = \{x\}$ corresponds the distribution \aleph , among the m elements of \mathfrak{X} , or the set

$$258. \quad a^{b+c} = a^b \cdot a^c.$$

To prove this we have only to show that

$$\mathfrak{A}^{\mathfrak{B}+\mathfrak{C}} \text{ and } \mathfrak{A}^{\mathfrak{B}} \cdot \mathfrak{A}^{\mathfrak{C}}$$

can be put in 1-1 correspondence. But this is so, for the set on the left is the totality of all the distributions of the elements of \mathfrak{A} among the sets formed of \mathfrak{B} and \mathfrak{C} . On the other hand, the set on the right is formed of a combination of the elements of \mathfrak{A} among the \mathfrak{B} , and among the \mathfrak{C} ; such a distribution may be regarded as the distribution considered.

$$259. \quad (a^b)^c = a^{b \cdot c}.$$

We have only to show that we can put in 1-1 correspondence the elements of

$$(\mathfrak{A}^{\mathfrak{B}})^{\mathfrak{C}} \text{ and } \mathfrak{A}^{\mathfrak{B} \cdot \mathfrak{C}}.$$

Let $\mathfrak{A} = \{a\}$, $\mathfrak{B} = \{b\}$, $\mathfrak{C} = \{c\}$. We note that $\mathfrak{A}^{\mathfrak{B}}$ is the set of all distributions of the a 's among the b 's, and that the set $(\mathfrak{A}^{\mathfrak{B}})^{\mathfrak{C}}$ is formed of the distributions of these sets among the c 's. On the other hand, $\mathfrak{A}^{\mathfrak{B} \cdot \mathfrak{C}}$ is formed of the distributions of the a 's among the elements of $\mathfrak{B} \cdot \mathfrak{C}$. These two sets are obviously associated uniformly with the distributions of the a 's among the elements of $\mathfrak{B} \cdot \mathfrak{C}$.

$$260. \quad 1. \quad c^n = (m^c)^n = m^{nc} = m^c = c$$

where m, n are positive integers.

For each number in the interval $\mathfrak{C} = (0, 1^*)$ can be put in normal form once and once only by

$\cdot a_1 a_2 a_3 \dots$ in the m -adic system,

Hence, $m^e = c$.

As $n^e = e$, we have 1), using 1) in 257.

2. *The result obtained in 247, 2 may be stated:*

$$c^e = c.$$

3. $2^e = c$.

For obviously $n^e \leq e^e \leq c^e$.

But by 3), $c^e = c$ and by 1) $n^e = c$.

261. 1. *The cardinal number t of all functions f take on but two values in the domain of definition \mathfrak{A} , of f is $2^{\mathfrak{A}}$.*

Moreover, $2^{\mathfrak{A}} > \mathfrak{a}$.

This follows at once from the reasoning of 249.

2. *Let \mathfrak{f} be the cardinal number of the class of f defined over a domain \mathfrak{A} whose cardinal number is c .*

$$\mathfrak{f} = c^e = 2^c > c.$$

For the class of functions which have but two values, 2^c .

On the other hand, obviously

$$\mathfrak{f} = c^e.$$

But

$$c^e = (2^e)^e, \quad \text{by 260, 1)}$$

$$= 2^{ec}, \quad \text{by 259, 1)}$$

$$= 2^c, \quad \text{by 253, 2).}$$

Thus,

$$c^e = 2^c.$$

For let \mathfrak{D} be a Cantor set in \mathfrak{A} [I, 272]. Being a limited function defined over \mathfrak{D} is integrable. But the points of \mathfrak{A} may be set in uniform correspondence with the points of \mathfrak{D} .

4. *The set of all functions*

$$f(x) = f_1(x) + f_2(x) + \dots$$

which are the sum of convergent series, and whose terms are continuous in \mathfrak{A} , has the cardinal number c .

For the set \mathfrak{F} of continuous functions in \mathfrak{A} has the cardinal number c by 248. These functions are to be distinguished from the enumerable set \mathfrak{E} of terms in 2). Hence the set of all functions is

$$\mathfrak{F}^{\mathfrak{E}},$$

whose cardinal number is

$$c^c = c.$$

Remark. Not every integrable function can be expressed as the sum of the series 2).

For the class of integrable functions has a cardinal number c by 250.

5. *The cardinal number of all enumerable sets in \mathfrak{R}_m is c .*

For it is obviously the cardinal number of the set \mathfrak{R}_m over an enumerable set \mathfrak{E} , or

$$\text{Card } \mathfrak{R}_m^{\mathfrak{E}} = c^c = c.$$

Numbers of Liouville

262. In I, 200 we have defined algebraic numbers

The first to actually show the existence of such numbers was *Liouville*. He showed how to form transcendental numbers. At present we have practical methods for determining whether a given number is algebraic or not. The signal achievements of *Hermite* to have shown that e is transcendental.

Shortly after *Lindemann*, adapting *Hermite's* method, showed that $\pi = 3.14159 \dots$ is also transcendental. The problem the *Quadrature of the Circle* was answered. The researches of *Hermite* and *Lindemann* establish the existence of an infinity of transcendental numbers. It is, however, impossible to give an account of these famous results. We shall restrict our considerations to certain numbers which are algebraic, as of *Liouville*.

In passing let us note that the existence of transcendental numbers follows at once from 235, 2 and 244, 2.

For the cardinal number of the set of real algebraic numbers is \aleph_1 , and that of the set of all real numbers is c ,

263. In algebra it is shown that any algebraic number is a root of an *irreducible* equation,

$$f(x) = a_0 x^m + a_1 x^{m-1} + \dots + a_m$$

whose coefficients are integers without common factor, the *order of α* is m .

We prove now the theorem

Let

$$r_n = \frac{p_n}{q_n} \quad , \quad p_n, q_n \text{ relatively prime}$$

$\doteq \alpha$, an algebraic number of order m , as $n \doteq \infty$.

for $n > s$, since the numerator of the middle member is an integer and hence ≥ 1 .

On the other hand, by the Law of the Mean [I, 397],

$$f(r_n) - f(\alpha) = (r_n - \alpha)f'(\beta)$$

where β lies in $D_\delta(\alpha)$. Now $f(\alpha) = 0$ and $f'(\beta) < \infty$ so that
Hence

$$|r_n - \alpha| > \frac{f(r_n)}{M} \geq \frac{1}{Mq_n^m},$$

on using 3). But however large M is, there exists a ν , such that $q_n > M$, for any $n > \nu$. This in 4) gives 2).

264. 1. *The numbers*

$$L = \frac{a_1}{10^{1!}} + \frac{a_2}{10^{2!}} + \frac{a_3}{10^{3!}} + \dots$$

where $a_n \leq 10^n$, and not all of them vanish after a certain index, are transcendental.

For if L is algebraic, let its order be m . Then if L_n is the sum of the first n terms of 1), there exists a ν such that

$$\eta = |L - L_n| > \frac{1}{10^{(m+1)n!}}, \quad \text{for } n > \nu.$$

But

$$\eta = \frac{a_{n+1}}{10^{(n+1)!}} + \dots < \frac{1}{10^{(m+1)n!}}, \quad n > \nu',$$

ν' being taken sufficiently large. But 3) contradicts 2).

The numbers 1) we call the *numbers of Liouville*.

2. *The set of Liouville numbers has the cardinal number of the continuum*

For all real numbers in the interval $(0, 1)$ can be represented by

$$\beta = \frac{b_1}{10^1} + \frac{b_2}{10^2} + \frac{b_3}{10^3} + \dots, \quad 0 \leq b_n \leq 9,$$

CHAPTER IX

ORDINAL NUMBERS

Ordered Sets

265. An aggregate \mathfrak{A} is *ordered*, when a , its elements, either a precedes b , or a succeeds b ; such that if a precedes b , and b precedes c . The fact that a precedes b may be in-

$$a < b.$$

Then

$$a > b$$

states that a succeeds b .

Example 1. The aggregates

$$1, 2, 3, \dots$$

$$2, 4, 6, \dots$$

$$a_1, a_2, a_3, \dots$$

$$\dots -3, -2, -1, 0, 1, 2, 3,$$

$$\dots a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3,$$

are ordered.

Example 2. The rational number system can be arranged in an infinite variety of ways. For, being enumerable, it can be arranged in a sequence $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \dots$

Example 4. The positive integers \mathfrak{J} may be nite variety of ways besides their natural order. may write them in the order

$$1, 3, 5, \dots 2, 4, 6, \dots$$

so that the odd numbers precede the even. Or

$$1, 4, 7, 10, \dots 2, 5, 8, 11, \dots 3, 6, 9, \dots$$

and so on. We may go farther and arrange the of sets. Thus in the first set put all primes; the products of two primes; in the third set three primes; etc., allowing repetitions of the number in set m precede all the numbers in set n if $m < n$. The numbers in each set may be arranged in order of magnitude.

Example 5. The points of the plane \mathfrak{R}_2 may be arranged in infinite variety of ways. Let L_y denote the right line parallel to the x -axis at a distance y from this axis, taking $y > 0$ for y above the x -axis and $y < 0$ for y below the x -axis. We order now the points of \mathfrak{R}_2 by stating that a point on $L_{y'}$ precedes the points on any $L_{y''}$ where $y' < y''$. The points on any L_y shall have the order they already have on the line due to their position.

266. Similar Sets. Let $\mathfrak{A}, \mathfrak{B}$ be ordered aggregates. Let $a \sim b, \alpha \sim \beta$. If when $a < \alpha$ in \mathfrak{A} , $b < \beta$ in \mathfrak{B} , then $a < b$ in \mathfrak{A} , and write

$$\mathfrak{A} \simeq \mathfrak{B}.$$

Thus the two ordered aggregates are similar when corresponding elements in the two sets have the same relative order.

In the correspondence $\mathfrak{A} \sim \mathfrak{B}$, let $a_r \sim r$ for $r = 1, 2, \dots$ and let $b_n \sim m + n$, $n = 1, 2, \dots$. Then $\mathfrak{A} \sim \mathfrak{B}$.

Example 3. Let $\mathfrak{A} = 1, 2, 3, \dots$

$$\mathfrak{B} = b_1, b_2, \dots, a_1, a_2, \dots$$

Let the correspondence between \mathfrak{A} and \mathfrak{B} be as in Ex. 2. Then \mathfrak{A} is not similar to \mathfrak{B} . For 1 is the first element of \mathfrak{A} while its associated element a_1 is not first in \mathfrak{B} .

Example 4. Let $\mathfrak{A} = 1, 2, 3, \dots$

$$\mathfrak{B} = a_1, a_2, \dots, b_1, b_2, \dots$$

Let $a_n \sim 2n$, $b_n \sim 2n - 1$. Then $\mathfrak{A} \sim \mathfrak{B}$ but \mathfrak{A} is not similar to \mathfrak{B} .

267. Let $\mathfrak{A} \cong \mathfrak{B}$, $\mathfrak{B} \cong \mathfrak{C}$. Then $\mathfrak{A} \cong \mathfrak{C}$.

For let $a \sim b$, $a' \sim b'$ in $\mathfrak{A} \sim \mathfrak{B}$. Let $b \sim c$, $b' \sim c'$ in $\mathfrak{B} \sim \mathfrak{C}$. Then we can establish a correspondence $\mathfrak{A} \sim \mathfrak{C}$ by setting $a \sim c$ if $a < a'$ in \mathfrak{A} , $c < c'$ in \mathfrak{C} . Hence $\mathfrak{A} \cong \mathfrak{C}$.

Eutactic Sets

268. Let \mathfrak{A} be any ordered aggregate, and \mathfrak{B} a part of \mathfrak{A} . If the elements of \mathfrak{B} being kept in the same relative order in \mathfrak{A} and each \mathfrak{B} both have a first element, we say that \mathfrak{A} is *eutactic* or *eutactic*.

Example 1. $\mathfrak{A} = 2, 3, \dots, 500$ is well ordered. Its first element is 2. Moreover any part of \mathfrak{A} as $6, 15, 25, \dots$ has a first element.

Example 3. Let $\mathfrak{A} =$ rational numbers arranged in their order of magnitude. \mathfrak{A} also has a first element, viz. 0. If \mathfrak{B} is any part of \mathfrak{A} consisting of positive numbers, \mathfrak{B} has no first element.

Example 4. An ordered set which is not well ordered can sometimes be made so by ordering its elements in a different law.

Thus in Ex. 3, let us arrange the positive numbers in descending order. Obviously \mathfrak{A} is now well ordered.

Example 5. $\mathfrak{A} = a_1, a_2 \cdots b_1, b_2 \cdots$ where a_1 is the first element of \mathfrak{A} ; and any part of \mathfrak{A} is well ordered.

$$a_1, a_2 \cdots$$

$$b_1, b_2 \cdots$$

$$a_1, a_2 \cdots b_1, b_2 \cdots$$

has a first element.

269. 1. Every partial set \mathfrak{B} of a well ordered set \mathfrak{A} is well ordered.

For \mathfrak{B} has a first element, since \mathfrak{A} has one. If \mathfrak{C} is a part of \mathfrak{B} , it is also a part of \mathfrak{A} and has a first element.

2. If a is not the last element of \mathfrak{A} , then the part of \mathfrak{A} immediately following a is well ordered.

For let \mathfrak{B} be the part of \mathfrak{A} following a . \mathfrak{B} has a first element b since \mathfrak{A} is well ordered.

ORDINAL NUMBERS

For example, let

$$\mathfrak{A} = a_1 a_2 \dots b_1 b_2 \dots$$

Then

$$a_n + 1 = a_{n+1} \quad , \quad b_m + 1 = b_{m+1}$$

$$a_n - 1 = a_{n-1} \quad , \quad b_m - 1 = b_{m-1}.$$

There is, however, no $b_1 - 1$.

70. 1. If \mathfrak{A} is well ordered, it is impossible to pick out an infinite sequence of the type

$$a_1 > a_2 > a_3 > \dots$$

For

$$\mathfrak{B} = \dots a_3, a_2, a_1$$

part of \mathfrak{A} whose elements occur in the same relative order in \mathfrak{A} , and \mathfrak{B} has no first element.

2. A sequence as 1) may be called a *decreasing sequence*, while a sequence as 2) may be called *increasing*.

$$a_1 < a_2 < a_3 < \dots$$

may be called *increasing*.

In every infinite well ordered aggregate there exist increasing and decreasing sequences.

3. Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \dots$ be a well ordered set. Let $\mathfrak{A} = \{a\}$ be ordered in the a 's, $\mathfrak{B} = \{b\}$ be well ordered in the b 's, etc. Then

$$\mathfrak{U} = \mathfrak{A}, \mathfrak{B}, \mathfrak{C} \dots$$

is well ordered with regard to the little letters $a, b \dots$

For \mathfrak{U} has a first element in the little letters, viz. the first element of \mathfrak{A} . Moreover, any part of \mathfrak{U} , as \mathfrak{B} , has a first element in the little letters. For if it has not, there exists in \mathfrak{B} an infinite decreasing sequence

$$t > s > r > \dots$$

Let each $\mathfrak{A}, \mathfrak{B}, \dots$ be well ordered.

Let

$$\mathfrak{B} = \mathfrak{A} + B, \quad \mathfrak{C} = \mathfrak{B} +$$

Then

$$\mathfrak{S} = \mathfrak{A} + B + C + \dots$$

is a well ordered set, \mathfrak{S} preserving the relative order intact.

For \mathfrak{S} has a first element, viz. the first element of the first part \mathfrak{A} of \mathfrak{S} has a first element. For, if not, we have an infinite decreasing sequence

$$r > q > p > \dots$$

Now r lies in some set of 1) as \mathfrak{A} . Hence \mathfrak{A} has no infinite decreasing sequence. But in \mathfrak{A} there is no sequence as 2).

5. Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \dots$ be an ordered set of well ordered sets, no two of which have an element in common. We must guard against assuming that $\mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \dots$ preserving relative order intact, is necessarily well ordered.

For let us modify Ex. 5 in 265 by taking points on each L_v only a well ordered set which is not well ordered.

Then the sum

$$\mathfrak{A} = \Sigma \mathfrak{A}_v$$

has a definite meaning. The elements of \mathfrak{A} will be ordered as in Ex. 5 of 265.

Obviously \mathfrak{A} is not well ordered.

Example 1. Let $\mathfrak{A} = 1, 2, 3, \dots$

Then

$$S100 = 1, 2, \dots 99$$

is the section of \mathfrak{A} generated by the element 100.

Example 2. Let

$$\mathfrak{A} = a_1, a_2 \dots b_1, b_2 \dots$$

Then

$$Sb_5 = a_1 a_2 \dots b_1 b_2 b_3 b_4$$

is the section generated by b_5 .

$$Sb_1 = a_1 a_2 \dots$$

that generated by b_1 , etc.

272. 1. *Every section of a well ordered aggregate*

For each section of \mathfrak{A} is a partial aggregate of \mathfrak{A} well ordered by 269, 1.

2. *In the well ordered set \mathfrak{A} , let $a < b$. Then $Sa < Sb$ of Sb .*

3. *Let \mathfrak{S} denote the aggregate of sections of a well ordered set \mathfrak{A} . If we order \mathfrak{S} such that $Sa < Sb$ in \mathfrak{S} if $a < b$ in \mathfrak{A} , \mathfrak{S} is well ordered.*

For the correspondence between \mathfrak{A} and \mathfrak{S} is uniform.

273. *Let $\mathfrak{A}, \mathfrak{B}$ be well ordered and $\mathfrak{A} \simeq \mathfrak{B}$. Then $Sa \simeq Sb$.*

For in \mathfrak{A} let $a'' < a' > a$. Let $b' \sim a'$ and $b'' \sim a''$. If $\mathfrak{A} \simeq \mathfrak{B}$, we have

$$b'' < b' < b;$$

hence the theorem.

SECTIONS

275. Let $\mathfrak{A}, \mathfrak{B}$ be well ordered and $\mathfrak{A} \simeq \mathfrak{B}$. Then to Sa in \mathfrak{A} correspond two sections $Sb, S\beta$ each $\simeq Sa$.

For let $b < \beta$, and $Sa \simeq Sb, Sa \simeq S\beta$. Then

$$Sb \simeq S\beta, \quad \text{by 267.}$$

But 1) contradicts 274.

276. Let $\mathfrak{A}, \mathfrak{B}$ be two well ordered aggregates. It is impossible to establish a uniform and similar correspondence between \mathfrak{A} and \mathfrak{B} in more than one way.

For say $Sa \simeq Sb$ in one correspondence, and $Sa \simeq S\beta$ in another, b, β being different elements of \mathfrak{B} . Then

$$Sb \simeq S\beta, \quad \text{by 267.}$$

This contradicts 275.

277. 1. We can now prove the following theorem, which is the converse of 273.

Let $\mathfrak{A}, \mathfrak{B}$ be well ordered. If to each section of \mathfrak{A} corresponds a similar section of \mathfrak{B} , and conversely, then $\mathfrak{B} \simeq \mathfrak{A}$.

Let us first show that $\mathfrak{A} \sim \mathfrak{B}$. Since to any Sa of \mathfrak{A} corresponds a similar section Sb in \mathfrak{B} , let us set $a \sim b$. No other $a' \sim b$, and no other $b' \sim a$, as then $Sa' \simeq Sb$ or $Sb' \simeq Sa$, which contradicts 274. Let the first element of \mathfrak{A} correspond to the first element of \mathfrak{B} . Thus the correspondence we have set up between \mathfrak{A} and \mathfrak{B} is uniform and $\mathfrak{A} \sim \mathfrak{B}$.

We show now that this correspondence is *similar*. For let

$$a \sim b \text{ and } a' \sim b', \quad a' < a.$$

Then $b' < b$. For a' lies in $Sa \simeq Sb$ and $b' \sim a'$ lies in Sb .

Let us begin by ordering the sections of \mathfrak{A} and \mathfrak{B} . Let B denote the aggregate of sections of \mathfrak{B} to which sections of \mathfrak{A} do not correspond. Then B is well ordered. Let Sb be the first section of B , say Sb . Let $\beta < b$. Then to $S\beta$ in \mathfrak{B} by hypothesis a similar section $S\alpha$ in \mathfrak{A} . On the other hand to any section Sa' of \mathfrak{A} corresponds a similar section Sb' in \mathfrak{B} previously $b' < b$. Thus to any section of \mathfrak{A} corresponds a section of Sb and conversely. Hence $\mathfrak{A} \simeq Sb$ by 277.

279. *Let \mathfrak{A} , \mathfrak{B} be well ordered. Either \mathfrak{A} is similar to a section of the other.*

For either :

- 1° To each section of \mathfrak{A} corresponds a similar section in \mathfrak{B} and conversely ;
- or 2° To each section of one corresponds a similar section in the other but not conversely ;
- or 3° There is at least one section in both \mathfrak{A} and \mathfrak{B} to which no similar section corresponds in the other.

If 1° holds, $\mathfrak{A} \simeq \mathfrak{B}$ by 277, 1. If 2° holds, either \mathfrak{A} is similar to a section of the other.

We conclude by showing 3° is impossible.

For let A be the set of sections of \mathfrak{A} to which no section in \mathfrak{B} corresponds. Let B have the same meaning for \mathfrak{B} . Suppose \mathfrak{A} , \mathfrak{B} ordered as in 272, 3, A will have a first section $S\alpha$, and B a first section $S\beta$.

Let $a < \alpha$. Then to Sa in \mathfrak{A} corresponds by hypothesis a section Sb of $S\beta$ as in 278. Similarly if $b' < \beta$, to Sb' corresponds a section Sa' of $S\alpha$. But then $S\alpha \simeq S\beta$ by 277, which contradicts the hypothesis.

3° If a section of \mathfrak{A} is $\simeq \mathfrak{B}$, the ordinal number of \mathfrak{A} is *greater* than that of \mathfrak{B} .

The ordinal number of \mathfrak{A} may be denoted by

$$\text{Ord } \mathfrak{A},$$

when no ambiguity can arise, by the corresponding small letter a . Any two well ordered aggregates \mathfrak{A} , \mathfrak{B} fall under one and only one of the three preceding cases, any two ordinal numbers satisfy one of the three following relations, and only one, viz.

$$a = b \quad , \quad a < b \quad , \quad a > b,$$

if $a < b$, it follows that $b > a$.

Obviously they enjoy also the following properties.

1. If $a = b$, $b = c$, then $a = c$.

For if $c = \text{Ord } \mathfrak{C}$, the first two relations state that

$$\mathfrak{A} \simeq \mathfrak{B} \quad , \quad \mathfrak{B} \simeq \mathfrak{C}.$$

But then

$$\mathfrak{A} \simeq \mathfrak{C} \quad , \quad \text{by 267.}$$

Hence

$$a = c.$$

2. If $a > b$, $b > c$, then $a > c$.

81. 1. Let \mathfrak{A} be a finite aggregate, embracing say n elements. Then we set

$$\text{Ord } \mathfrak{A} = n.$$

Thus the ordinal number of a finite aggregate has exactly the same properties to those of finite cardinal numbers. The ordinal number of a finite aggregate is called *finite*, otherwise *transfinite*.

The ordinal number belonging to the well ordered set for

ORDINAL NUMBERS

. The cardinal number of a set \mathfrak{A} is independent of the order in which the elements of \mathfrak{A} occur. This is not so in general for ordinal numbers.

For example, let

$$\mathfrak{A} = 1, 2, 3, \dots$$

$$\mathfrak{B} = 1, 3, 5, \dots 2, 4, 6, \dots$$

Here

$$\text{Card } \mathfrak{A} = \text{Card } \mathfrak{B} = \aleph_0.$$

But

$$\text{Ord } \mathfrak{A} < \text{Ord } \mathfrak{B},$$

because \mathfrak{A} is similar to a section of \mathfrak{B} , viz. the set of odd numbers $1, 3, 5, \dots$

82. 1. Addition of Ordinals. Let $\mathfrak{A}, \mathfrak{B}$ be well ordered sets without common elements. Let \mathfrak{C} be the aggregate formed by placing the elements of \mathfrak{B} after those of \mathfrak{A} , leaving the order in each set otherwise unchanged. Then the ordinal number of \mathfrak{C} is called the sum of the ordinal numbers of \mathfrak{A} and \mathfrak{B} , or

$$\text{Ord } \mathfrak{C} = \text{Ord } \mathfrak{A} + \text{Ord } \mathfrak{B},$$

$$c = a + b.$$

The extension of this definition to any set of well-ordered aggregates such that the result is well ordered is obvious.

. We note that

$$a + b > a, \quad a + b \geq b.$$

For \mathfrak{A} is similar to a section of \mathfrak{C} , and \mathfrak{B} is equivalent to a section of \mathfrak{C} .

. The addition of ordinal numbers is associative.

This is an immediate consequence of the definition of addition.

But $\mathfrak{A} \simeq$ a section of \mathfrak{C} , viz.: $\simeq S\mathfrak{b}_1$, while $\mathfrak{D} \simeq \mathfrak{A}$. Hence

$$\omega \leq c, \quad \omega = d,$$

$$\omega + n > \omega, \quad y + \omega = \omega.$$

. If $a > b$, then $c + a > c + b$, and $a + c = b + c$.

For let $a = \text{Ord } \mathfrak{A}$, $b = \text{Ord } \mathfrak{B}$, $c = \text{Ord } \mathfrak{C}$.

Since $a > b$, we can take for \mathfrak{B} a section $S\mathfrak{b}$ of \mathfrak{A} . Then $c + a$ is the ordinal number of $\mathfrak{C} + \mathfrak{A}$,

and $c + b$ is the ordinal number of

$$\mathfrak{C} + S\mathfrak{b},$$

preserving the relative order of the elements.

But 2) is a section of 1), and hence $c + a > c + b$.

The proof of the rest of the theorem is obvious.

83. 1. The ordinal number immediately following a is $a + 1$.

For let $a = \text{Ord } \mathfrak{A}$. Let \mathfrak{B} be a set formed by adding after the elements of \mathfrak{A} another element b . Then

$$a + 1 = \text{Ord } \mathfrak{B} = b.$$

Suppose now

$$a < c < b, \quad c = \text{Ord } \mathfrak{C}.$$

Then \mathfrak{C} is similar to a section of \mathfrak{B} . But the greatest section of \mathfrak{B} is $S\mathfrak{b} = \mathfrak{A}$. Hence

$$c \leq a,$$

which contradicts 1).

. Let $a \geq b$. Then there is one and only one ordinal number d such that

$$a = b + d.$$

For let $a = \text{Ord } \mathfrak{A}$, $b = \text{Ord } \mathfrak{B}$.

284. 1. *Multiplication of Ordinals.* Let \mathfrak{A} , \mathfrak{B} be aggregates having a , b as ordinal numbers. Let us multiply the first element of \mathfrak{A} by an aggregate $\simeq \mathfrak{B}$. The resulting aggregate we denote by

$$\mathfrak{B} \cdot \mathfrak{A}.$$

As \mathfrak{C} is a well-ordered set by 270, \mathfrak{B} has an ordinal number c . We define now the *product* $b \cdot a$ to be c , and write

$$b \cdot a = c.$$

We say c is the result of *multiplying* a *by* b , and call c the *product* of a and b . We write

$$a \cdot a = a^2 \quad , \quad a \cdot a \cdot a = a^3 \quad , \quad \text{etc.}$$

2. *Multiplication is associative.*

This is an immediate consequence of the definition.

3. *Multiplication is not always commutative.*

For example, let

$$\mathfrak{A} = (a_1 a_2),$$

$$\mathfrak{B} = (1, 2, 3 \dots \text{in inf.}).$$

Then

$$\mathfrak{B} \cdot \mathfrak{A} = (b_1 b_2 b_3 \dots, \quad c_1 c_2 c_3 \dots).$$

$$\mathfrak{A} \cdot \mathfrak{B} = (b_1, c_1, \quad b_2, c_2, \quad b_3, c_3, \dots)$$

Hence

$$\text{Ord}(\mathfrak{B} \cdot \mathfrak{A}) = \omega \cdot 2 > \omega,$$

$$\text{Ord}(\mathfrak{A} \cdot \mathfrak{B}) = 2 \omega = \omega.$$

4. *If* $a < b$, *then* $ca < cb$.

For $\mathfrak{C} \cdot \mathfrak{A}$ is a section of $\mathfrak{C} \cdot \mathfrak{B}$.

Since $\alpha_{n-1} < \alpha_n$, \mathfrak{A}_{n-1} is similar to a section of \mathfrak{A}_n . For simplicity may take \mathfrak{A}_{n-1} to be a section of \mathfrak{A}_n . Let, therefore,

$$\mathfrak{A}_n = \mathfrak{A}_{n-1} + \mathfrak{B}_n.$$

Consider now

$$\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{B}_2 + \mathfrak{B}_3 + \dots$$

keeping the relative order of the elements intact. Then \mathfrak{A} is ordered and has an ordinal number α .

As any \mathfrak{A}_n is a section of \mathfrak{A} ,

$$\alpha_n < \alpha.$$

Moreover any number $\beta < \alpha$ is also $<$ some α_n . For if \mathfrak{B} an ordinal number β , \mathfrak{B} must be similar to a section of \mathfrak{A} . There is no last section of \mathfrak{A} .

1. The number α we have just determined is called the *limit* (sequence 1). We write

$$\alpha = \lim \alpha_n, \quad \text{or } \alpha_n \doteq \alpha.$$

We also say that α *corresponds to the sequence* 1).

All numbers corresponding to infinite enumerable increasing sequences of ordinal numbers are called *limitary*.

2. If every α_n in 1) is $< \beta$, then $\alpha \leq \beta$.

For if $\beta < \alpha$, α is not the least ordinal number greater than every α_n .

3. If $\beta < \alpha$, β is $<$ some α_n .

86. In order that

$$\alpha_1 < \alpha_2 < \dots$$

$$\beta_1 < \beta_2 < \dots$$

287. *Cantor's Principles of Generating Ordinals.* two methods of generating ordinal numbers. First to any ordinal number α . In this way we get

$$\alpha, \alpha + 1, \alpha + 2, \dots$$

Secondly, by taking the limit of an infinite enumerating sequence of ordinal numbers, as

$$\alpha_1 < \alpha_2 < \alpha_3 < \dots$$

Cantor calls these two methods the *first* and *second* of generating ordinal numbers.

Starting with the ordinal number 1, we get by successions of the first principle the numbers

$$1, 2, 3, 4, \dots$$

The limit of this sequence is ω by 285, 1. Using principle alone, this number would not be attained; to get the application of the second principle. Making use of principle again, we obtain

$$\omega + 1, \quad \omega + 2, \quad \omega + 3, \dots$$

The second principle gives now the limitary number by 285, 1. From this we get, using the first principle

$$\omega 2 + 1, \quad \omega 2 + 2, \quad \omega 2 + 3, \dots$$

whose limit is $\omega 3$. In this way we may obtain the numbers

$$\omega m + n, \quad m, n \text{ finite.}$$

The limit of any increasing sequence of these numbers

$$\omega, \quad \omega 2, \quad \omega 3, \quad \omega 4, \dots$$

is ω^2 by 285, 1.

But here the process does not end. For the

$$\omega < \omega^2 < \omega^3 < \dots$$

has a limit which we denote by ω^ω .

Continuing we obtain

$$\omega^{\omega^\omega}, \quad \omega^{\omega^{\omega^\omega}}, \quad \text{etc.}$$

288. It is interesting to see how we may sets of points whose ordinal numbers are the sidered.

In the unit interval $\mathfrak{I} = (0, 1)$, let us take

$$\frac{1}{2}, \quad \frac{3}{4}, \quad \frac{7}{8}, \quad \frac{15}{16}, \dots$$

These form a well ordered set whose ordinal n

The points 1) divided \mathfrak{I} into a set of interv

$$\mathfrak{I}_1, \quad \mathfrak{I}_2, \quad \mathfrak{I}_3, \dots$$

In m of these intervals, let us take a set gives us a set whose ordinal number is ωm .

In each interval 2), let us take a set similar us a set whose ordinal number is ω^2 . The divide \mathfrak{I} into a set of ω^2 intervals. In each let us take a set of points similar to 1). points whose ordinal number is ω^3 , etc.

Let us now put in \mathfrak{I}_1 a set of points \mathfrak{P}_1 wh is ω . In \mathfrak{I}_2 let us put a set \mathfrak{P}_2 whose ordina so on, for the other intervals of 2).

We thus get in \mathfrak{I} the well ordered set

$$\mathfrak{P} = \mathfrak{P}_1 + \mathfrak{P}_2 + \mathfrak{P}_3 + \dots$$

Classes of Ordinals

289. Cantor has divided the ordinal numbers

Class 1, denoted by Z_1 , embraces all finite ord

Class 2, denoted by Z_2 , embraces all transfinite ordinals corresponding to well ordered enumerable sets whose cardinal number is \aleph_0 . For this reason we

$$Z_2 = Z(\aleph_0).$$

It will be shown in 293, 1 that Z_2 is not enumerable if we set

$$\aleph_1 = \text{Card } Z_2,$$

there is no cardinal number between \aleph_0 and \aleph_1 as 294. We are thus justified in saying that Class 3, denoted by Z_3 or $Z(\aleph_1)$, embraces all ordinal numbers corresponding to well ordered sets whose cardinal number is \aleph_1 , etc.

Let $\beta = \text{Ord } \mathfrak{B}$ be any ordinal number. The ordinals $\alpha < \beta$ correspond to sections of \mathfrak{B} . These sections are well ordered by 272, 3. Therefore if we arrange the ordinals $\alpha < \beta$ in an order such that α' precedes α when α' is a section of a section of \mathfrak{B} , we have a well ordered set. We shall call this the natural order of the ordinals less than β . The first number in Z_1 is 1, the first number of Z_2 is ω , the first number in Z_3 is denoted by Ω .

290. As the numbers in Class 1 are the positive integers, they need no comment here. Let us therefore turn to

If α is in Z_2 , so is $\alpha + 1$.

For let $\alpha = \text{Ord } \mathfrak{A}$. Let \mathfrak{B} be the well ordered set obtained by placing an element b after all the elements of \mathfrak{A} .

Using the notation employed in the proof of 285, 1, α is a number of

$$\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{B}_1 + \mathfrak{B}_2 + \dots$$

that $\mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{B}_2 \dots$ are each enumerable.

Hence \mathfrak{A} is enumerable by 235, 1, and α lies in Z_2 .

2. We prove now the converse of 290 and 291.

Each number α in Z_2 , except ω , is obtained by adding 1 to some number in Z_2 ; or it is the limit of an infinite enumerable increasing sequence of numbers in Z_2 .

Let $\alpha = \text{Ord } \mathfrak{A}$. Suppose first, that \mathfrak{A} has a last element.

Since \mathfrak{A} is enumerable, so is Sa . Hence

$$\beta = \text{Ord} \cdot Sa$$

Z_2 . Then

$$\alpha = \beta + 1.$$

Propose secondly, that \mathfrak{A} has no last element. All the numbers in Z_2 belong to sections of \mathfrak{A} . Since \mathfrak{A} is enumerable, the β 's are enumerable. Let them be arranged in a sequence

$$\beta_1, \beta_2, \beta_3 \dots$$

since they have no greatest, let β'_1 be the first number in it, let β'_2 be the first number in it $> \beta'_1$, etc. We get thus a sequence

$$\beta_1 < \beta'_1 < \beta'_2 < \dots$$

whose limit is λ , say.

Here $\lambda = \alpha$. For λ is $>$ any number in 1), which embraces all numbers of $Z_2 < \alpha$. Moreover it is the least number which has this property.

3. 1. The numbers of Z_2 are not enumerable

Let $\alpha'_n \doteq \alpha'$. Then α' lies in Z_2 by 291. On 285, α' is $>$ any number in 2), and therefore 1). But 1) embraces all the numbers of Z_2 , by 285, and are thus led to a contradiction.

2. We set

$$\aleph_1 = \text{Card } Z_2.$$

294. *There is no cardinal number between \aleph_0 and \aleph_1 .*

For let $\alpha = \text{Card } \mathfrak{A}$ be such a number. Then \mathfrak{A} is a partial aggregate of Z_2 , which without loss of generality may be taken to be a section of Z_2 . But every such \mathfrak{A} is enumerable. Hence \mathfrak{A} is enumerable and $\alpha = \aleph_0$, which contradicts the assumption.

295. We have just seen that the numbers in Z_2 are well ordered. Let us order them so that each number is preceded by a succeeding number. We shall call this the *natural order*.

1. *The numbers of Z_2 when arranged in their natural order form a well ordered set.*

For Z_2 has a first element ω . Moreover any section of Z_2 , relative order being preserved, has a first element. If not, there exists an infinite enumerable decreasing sequence.

$$\alpha > \beta > \gamma > \dots$$

This, however, is not possible. For β, γ, \dots is a section of Z_2 which is well ordered.

There is thus one well ordered set having \aleph_1 elements. Let

$$\Omega = \text{Ord } Z_2.$$

Let now \mathfrak{A} be an enumerable well ordered set. The number of elements of \mathfrak{A} is α . The set \mathfrak{A} is well ordered.

296. 1. *An aggregate formed of an \aleph_1 set of \aleph_1 sets*
 Consider the set

$$A = \begin{array}{cccc|cccc|cccc|cccc|} a_{11}, & a_{12}, & a_{13} & \cdots & a_{1\omega} & \cdots & a_{1a} & \cdots & & & & & & & \\ a_{21}, & a_{22}, & a_{23} & \cdots & a_{2\omega} & \cdots & a_{2a} & \cdots & & & & & & & \\ \hline a_{\omega 1}, & a_{\omega 2}, & a_{\omega 3} & \cdots & a_{\omega\omega} & \cdots & a_{\omega a} & \cdots & & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{a1}, & a_{a2}, & a_{a3} & \cdots & a_{a\omega} & \cdots & a_{aa} & \cdots & & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Here each row is an \aleph_1 set. As there are an \aleph_1 sets is an \aleph_1 set of \aleph_1 sets. To show that A is an \aleph_1 set each $a_{\iota\kappa}$ with some number in the first two number

In the first place the elements $a_{\iota\kappa}$ where $\iota, \kappa < \omega$ associated with the numbers $1, 2, 3, \dots < \omega$. The elements lying just inside the ω^{th} square and which are associated by the condition that $\iota = 1, 2, \dots \omega; \kappa = 1, 2, \dots \omega$ is an enumerable set and may therefore be associated with the numbers $\omega, \omega + 1, \dots < \omega^2$. For the same reason the elements of the $\omega + 1^{\text{st}}$ square may be associated with the ordinals $\omega^2, \omega^2 + 1, \dots < \omega^3$. In this way we may continue. For when we arrived at the α^{th} row and column (edge of the square) we have only used up an enumerable set of numbers in

$$1, 2, \dots \omega \dots < \Omega$$

in our process of association. There are thus still Ω elements left in 1) to continue the process of association.

2. As a corollary of 1 we have :

The ordinal numbers

$$\Omega^2, \quad \Omega^3, \quad \Omega^4, \dots$$

lie in \mathbb{Z}

Since $\alpha < \beta$ we may take \mathfrak{A} to be a section we may suppose \mathfrak{B} is a section of \mathfrak{C} , etc.

Let now

$$\mathfrak{B} = \mathfrak{A} + B, \quad \mathfrak{C} = \mathfrak{B} + C,$$

Consider now

$$\mathfrak{L} = \mathfrak{A} + B + C + \dots$$

keeping the relative order intact. Then \mathfrak{L} is 270, 4. Let

$$\lambda = \text{Ord } \mathfrak{L}.$$

Since $\text{Card } \mathfrak{L} = \aleph_1$, by 296, 1, λ lies in Z_3 .

As any $\mathfrak{A}, \mathfrak{B}, \dots$ is a section of \mathfrak{L} ,

$$\alpha < \beta < \dots < \lambda.$$

Moreover, any number $\mu < \lambda$ is also $<$ some \mathfrak{M} has ordinal number μ , \mathfrak{M} must be similar. But there is no last section in \mathfrak{L} .

2. We shall call sequences of the type 1) The number λ whose existence we have just established call the *limit of* 1). We shall also write

$$\alpha < \beta < \gamma \dots \doteq \lambda$$

to indicate that α, β, \dots is an \aleph_1 sequence whose

298. 1. The preceding theorem gives us a generating ordinal numbers. We call it the *th*

We have seen that the first and second principles generate the numbers of the first two classes of ordinals do not suffice to generate even the first number, prove now the following fundamental theorem :

2. The three principles already described are

where β lies in Z_2 . In the latter case, reason-
shows that we can pick out an \aleph_1 increasing so

$$\beta_1 < \beta_2' < \beta_3' \dots < \alpha.$$

299. 1. *The numbers of Z_3 form a set whose
is $> \aleph_1$.*

The proof is entirely similar to 293, 1. Set
 $\alpha = \aleph_1$. Let us arrange the elements of Z_3 in

$$\alpha_1, \alpha_2 \dots \alpha_\omega \dots \alpha_\Omega \dots$$

As in 292, there exists in this sequence an \aleph_1 i

$$\alpha'_1 < \alpha'_2 < \dots < \alpha'.$$

Then α' lies in Z_3 by 297, 1. (On the other hand
any number in 2) and hence greater than α .
But 1) embraces all the numbers in Z_3 by hypothesis,
thus led to a contradiction.

2. We set

$$\aleph_2 = \text{Card } Z_3.$$

3. *There is no cardinal number between \aleph_1 and*

For let $\alpha = \text{Card } \mathfrak{A}$ be such a number. Then
a section of Z_3 . But every such section has
ber \aleph_1 .

300. The reasoning of the preceding para-
once generalized. The ordinal numbers of Z_n
well ordered sets of cardinal number \aleph_{n-2} form
having a greater cardinal number α than \aleph_{n-2} .
no cardinal lying between \aleph_{n-2} and α . We
appropriately denote α by \aleph_n . The \aleph_n is

CHAPTER X

POINT SETS

Pantaxis

301. 1. (Borel.) *Let each point of the limited \mathfrak{A} lie at the center of a cube \mathfrak{C} . Then there exists a set of non-overlapping cubes $\{c\}$ such that each c lies within \mathfrak{A} and each point of \mathfrak{A} lies in some c . If \mathfrak{A} is limited and \mathfrak{A} is a finite set $\{c\}$ having this property.*

For let $D_1, D_2 \dots$ be a sequence of superposed divisions of norms $\doteq 0$. Any cell of D_1 which lies within \mathfrak{A} and contains a point of \mathfrak{A} we call a black cell; any cell of D which does not contain a point of \mathfrak{A} we call white. The black cells are not further divided. The division D_2 divides each white cell. Any of the cells which lies within some \mathfrak{C} and contains a point of \mathfrak{A} is a black cell, the others are white. Continuing we obtain a set of non-overlapping cubical cells $\{c\}$.

Each point a of \mathfrak{A} lies within some c . For a given a and some \mathfrak{C} . But when n is taken sufficiently large, a lies within D_n , which cell lies within \mathfrak{C} .

Let now \mathfrak{A} be limited and complete. Each a lies on the faces of a finite number of these c . We take the diagonal δ of the smallest of these cubes. $\text{Min } \delta = 0$ in \mathfrak{A} . As \mathfrak{A} is complete there is a point

cells of some D_n , n sufficiently large, which surcube c , lying within \mathfrak{C} . Thus the points of \mathfrak{A} enumerable set of cells $\{c\}$, each c lying within cells c of course will in general overlap. Obviously complete, the points of \mathfrak{A} will lie within a finite number of these c 's.

302. *If \mathfrak{A} is dense, \mathfrak{A}' is perfect.*

For, in the first place, \mathfrak{A}' is dense. In fact, let α be a point of \mathfrak{A}' . Then in any $D^*(\alpha)$ there are points of \mathfrak{A} . If α is not a point of \mathfrak{A} , since \mathfrak{A} is dense, it is a limiting point of \mathfrak{A} , and hence a point of \mathfrak{A}' . Thus in any $D^*(\alpha)$ there are points of \mathfrak{A}' .

Secondly, \mathfrak{A}' is complete, by I, 266.

303. *Let \mathfrak{B} be a complete partial set of the perfect set \mathfrak{A} . Then $\mathfrak{C} = \mathfrak{A} - \mathfrak{B}$ is dense.*

For if \mathfrak{C} contains the isolated point c , all the points of \mathfrak{C} lie in \mathfrak{B} , if r is taken sufficiently small. But \mathfrak{C} is complete, c must then lie in \mathfrak{B} .

Remark. We take this occasion to note that a finite set is regarded as complete.

304. 1. *If \mathfrak{A} does not embrace all \mathfrak{R}_n , it has at least one frontier point in \mathfrak{R}_n .*

For let a be a point of \mathfrak{A} , and b a point of \mathfrak{R}_n not in \mathfrak{A} . Points on the join of a , b have coördinates

$$x_i = a_i + \theta(b_i - a_i) = x_i(\theta), \quad 0 \leq \theta \leq 1, \quad i = 1, \dots, n.$$

Let θ' be the maximum of those θ 's such that $x(\theta)$ is in \mathfrak{A} if $\theta \leq \theta'$. Then $x(\theta')$ is a frontier point of \mathfrak{A} .

Example 1. Let \mathfrak{A} be the unit interval (containing rational points in \mathfrak{A}). Then \mathfrak{B} is pantactic in \mathfrak{A} .

Example 2. Let \mathfrak{A} be the interval $(0, 1)$, a domain of I, 272. Then \mathfrak{B} is apantactic in \mathfrak{A} .

2. If $\mathfrak{B} < \mathfrak{A}$ is pantactic in \mathfrak{A} , \mathfrak{A} contains points not in \mathfrak{B} .

For let a be a point of \mathfrak{A} not in \mathfrak{B} . Then $D_\delta(a)$ there is a point of \mathfrak{B} . Hence there are points of \mathfrak{B} in this domain. Hence a is a limiting point of \mathfrak{B} .

306. Let \mathfrak{A} be complete. We say $\mathfrak{B} \leq \mathfrak{A}$ in \mathfrak{A} , if \mathfrak{B} is the union of an enumerable set of points in \mathfrak{A} .

If \mathfrak{B} is not of the 1° category, we say it is not of the 1° category. Sets of the 1° category may be called *Baire* sets.

Example. Let \mathfrak{A} be the unit interval, containing rational points in it. Then \mathfrak{B} is of the 1° category.

For \mathfrak{B} being enumerable, let $\mathfrak{B} = \{b_n\}$. But b_n is a point and is thus apantactic in \mathfrak{A} .

The same reasoning shows that if \mathfrak{B} is an enumerable set in \mathfrak{A} , then \mathfrak{B} is of the 1° category.

307. 1. If \mathfrak{B} is of the 1° category in \mathfrak{A} , $\mathfrak{A} - \mathfrak{B}$ is of the 1° category.

For since \mathfrak{B} is of the 1° category in \mathfrak{A} , it is the union of an enumerable set of apantactic sets $\{\mathfrak{B}_n\}$. Then $\mathfrak{A} - \mathfrak{B}$ is the union of an enumerable set of points a_1, a_2, \dots in \mathfrak{A} such that

$$D_{\delta_1}(a_1) > D_{\delta_2}(a_2) > \dots, \quad \delta_n > 0$$

3. Let \mathfrak{B} be of the 1° category in \mathfrak{A} . Then $\mathfrak{A} - \mathfrak{B}$ is of the 2° category in \mathfrak{A} .

For otherwise $\mathfrak{B} + B$ would be of the 1° category in \mathfrak{A} .

$$\mathfrak{A} - (\mathfrak{B} + B) = 0,$$

and this violates 1.

4. It is now easy to give examples of sets of the 2° category. For instance, the irrational points in the interval $[0, 1]$ form a set of the 2° category.

308. Let \mathfrak{A} be a set of the 1° category in \mathfrak{Q} . Then $\mathfrak{A} = \mathfrak{Q} - \mathfrak{A}$ has the cardinal number c .

If A has an inner point, $D_\delta(a)$, for sufficient δ , then $\text{Card } D_\delta = c$, the theorem is proved.

Suppose that A has no inner point. Let \mathfrak{A} be a sequence of apantactic sets $\mathfrak{A}_1 < \mathfrak{A}_2 < \dots$ in \mathfrak{Q} . Let A_n be the maximum of the sides of the cubes lying in \mathfrak{A}_n . Obviously $q_n \doteq 0$, since by hypothesis A has no inner point. There is no cube lying in A_1 . As $q_n \doteq 0$, there exists an $n_2 > n_1$ such that Q_0, Q_1 each have two cubes lying in A_{n_1} ; call them $Q_{0,0}, Q_{0,1}; Q_{1,0}, Q_{1,1}$.

or more shortly Q_{i_1, i_2} .

Each of these gives rise similarly to two more cubes, which may be denoted by Q_{i_1, i_2, i_3} , where the i_3 has the values 0, 1. In this way we may continue

$$Q_{i_1}, Q_{i_1 i_2}, Q_{i_1 i_2 i_3} \dots$$

Let α be a point lying in a sequence of the cubes $Q_{i_1, i_2, i_3, \dots}$. Obviously α does not lie in \mathfrak{A} , if the indices are not all 0 or all 1. This point α is characterized by the sequence of indices i_1, i_2, i_3, \dots .

For let $D_1 > D_2 > \dots$ be a sequence of superimposed cells of \mathfrak{D} , whose norms $\delta_n \doteq 0$. Let

$$d_{11}, d_{12}, d_{13} \dots$$

be the cells of D_1 containing no point of \mathfrak{A} with

$$d_{21}, d_{22}, d_{23} \dots$$

denote those cells of D_2 containing no point of \mathfrak{A} not lying in a cell of 1). In this way we may construct a sequence of cells $\mathfrak{D} = \{d_{mn}\}$, where for each m , the set of n is finite, and $m \doteq \infty$. Each point α of \mathfrak{A} lies in some cell of \mathfrak{D} , being complete, $\text{Dist}(\alpha, \mathfrak{A}) > 0$. As the norm δ_n in some cell of D_n , for a sufficiently large n , is small, the theorem is now obvious.

310. *Let \mathfrak{B} be pantactic in \mathfrak{A} . Then there exists a set $\mathfrak{C} \leq \mathfrak{B}$ which is pantactic in \mathfrak{A} .*

For let $D_1 > D_2 > \dots$ be a set of superimposed cells of norms $\delta_n \doteq 0$. In any cell of D_1 containing a point of \mathfrak{A} , there is at least one point of \mathfrak{B} . If the point belongs to the face of two or more cells, the foregoing statement holds for at least one of the cells. Let us now take a point of \mathfrak{B} in each of these cells; this gives an enumerable set \mathfrak{C}_1 . The same holds for the cells of D_2 . Let us take a point of \mathfrak{B} in each of these cells, taking when possible points of \mathfrak{C}_1 . Let \mathfrak{C}_2 be the set of these points of this set not in \mathfrak{C}_1 . Continuing in this

$$\mathfrak{C} = \mathfrak{C}_1 + \mathfrak{C}_2 + \dots$$

Then \mathfrak{C} is pantactic in \mathfrak{A} , and is enumerable,

Corollary. *In any set \mathfrak{A} finite or infinite, the*

As f is continuous,

$$f(c_n) \doteq f(c).$$

Hence

$$f(c) = g,$$

and c lies in \mathfrak{C} .

2. Let $f(x_1 \dots x_m)$ be continuous in the limits. If the value of f is known in an enumerable point set \mathfrak{A} which contains all the isolated points of \mathfrak{A} , in \mathfrak{C} the value of f is known at every point of \mathfrak{A} .

For let a be a limiting point of \mathfrak{A} not in \mathfrak{C} . In \mathfrak{A} , there exists a sequence of points e_1, e_2, \dots . Since f is continuous, $f(e_n) \doteq f(a)$. As f is continuous, it is known at a .

3. Let $\mathfrak{F} = \{f\}$ be the class of one-valued functions defined over a limited point set \mathfrak{A} . Then

$$\mathfrak{f} = \text{Card } \mathfrak{F} = c.$$

For let \mathfrak{R}_∞ be a space of an infinite number of dimensions, and let

$$y = (y_1, y_2, \dots)$$

denote one of its points. Let f have the value η_1 at $e_1 \dots$ for the points of \mathfrak{C} defined in 2. Then

$$(\eta_1, \eta_2, \dots)$$

completely determines f . But this complex point η in \mathfrak{R}_∞ whose coördinates are η_n . We have η . Hence

$$\mathfrak{f} \leq \text{Card } \mathfrak{R}_\infty = c.$$

On the other hand, $\mathfrak{f} \geq c$, since in \mathfrak{F} the functions $f(x_1 \dots x_m) = g$ in \mathfrak{A} , where g is any real number.

Obviously the cardinal number of the class 1) is $\epsilon^e = c$. But $(a, a_{i_1}, a_{i_2}, a_{i_3} \dots)$ is a complete set in \mathfrak{A} . Hence $b \geq c$. On the other hand, for let

$$D_1 > D_2 > \dots$$

be a sequence of superimposed cubical divisions. Each D_n embraces an enumerable set of cells. Each division gives an enumerable set of cells. Each cell is assigned to it, for a given set in \mathfrak{B} , the sign $+$ or $-$, \mathfrak{B} is exterior to this cell or not. This determines a distribution of two things over an enumerable set of comparisons.

The cardinal number of the class of these distributions is c . But each \mathfrak{B} determines a distribution. Hence

Transfinite Derivatives

313. 1. We have seen, I, 266, that

$$\mathfrak{A}' \geq \mathfrak{A}'' \geq \mathfrak{A}''' \geq \dots$$

Thus

$$\mathfrak{A}^{(n)} = Dv(\mathfrak{A}', \mathfrak{A}'' \dots \mathfrak{A}^{(n)}).$$

Let now \mathfrak{A} be a limited point aggregate of the first kind. It has then derivatives of every finite order.

$$Dv(\mathfrak{A}', \mathfrak{A}'', \mathfrak{A}''', \dots)$$

contains at least one point, and in analogy with set 2) the derivative of order ω of \mathfrak{A} , and denote it by

$$\mathfrak{A}^{(\omega)},$$

or more shortly by

$$\mathfrak{A}^\omega.$$

Now we may reason on \mathfrak{A}^ω as on any point set

In order that the point set \mathfrak{A} is of the first species and sufficient that $\mathfrak{A}^{(\omega)} = 0$.

2. We have seen in 18 that \mathfrak{A}^ω is complete. I, 266 shows that $\mathfrak{A}^{\omega+1}, \mathfrak{A}^{\omega+2}, \dots$, when they exist. Then 18 shows that, if $\mathfrak{A}^{\omega+n}$ $n = 1, 2, \dots$ exists

$$Dv(\mathfrak{A}^\omega \supseteq \mathfrak{A}^{\omega+1} \supseteq \mathfrak{A}^{\omega+2} \supseteq \dots)$$

exists and is complete. The set \mathfrak{B}) is called the $\omega \cdot 2$ and is denoted by

$$\mathfrak{A}^{(\omega^2)} \text{ or } \mathfrak{A}^{\omega^2}.$$

Obviously we may continue in this way in order to reach a derivative of order α containing only α points. Then

$$\mathfrak{A}^{\alpha+1} = 0.$$

That this process of derivation may never stop taking for \mathfrak{A} any limited perfect set, for then

$$\mathfrak{A} = \mathfrak{A}' = \mathfrak{A}'' = \dots = \mathfrak{A}^\omega = \mathfrak{A}^{\omega^2} = \dots$$

3. We may generalize as follows: Let α denote an ordinal number. If each $\mathfrak{A}^\beta > 0$, $\beta < \alpha$, we set

$$\mathfrak{A}^{(\alpha)} = \mathfrak{A}^\alpha = Dv\{\mathfrak{A}^\beta\}$$

when it exists.

4. If $\mathfrak{A}^\alpha > 0$, while $\mathfrak{A}^{\alpha+1} = 0$, we say \mathfrak{A} is of order α .

314. 1. Let a be a limiting point of \mathfrak{A} . Let

$$\alpha_\delta = \text{Card } V_\delta(a).$$

Obviously α_δ is monotone decreasing with δ . There exists an α and a $\delta_0 > 0$, such that for all $\delta > \delta_0$

2. *Let \mathfrak{A} be a limited aggregate of cardinal number α which has at least one limiting point of \mathfrak{A} , of rank α .*

The demonstration is entirely similar to I, 2. $\delta_2 > \dots \doteq 0$. Let us effect a cubical division of \mathfrak{A} such that at least one cell lies an aggregate \mathfrak{A}_1 having the cardinal number α . Let us effect a cubical division of \mathfrak{A}_1 of rank α such that at least one cell lies an aggregate \mathfrak{A}_2 having the cardinal number α , etc. These cells converge to a point a , such that

$$\text{Card } V_\delta(a) = \alpha,$$

however small δ is taken.

3. *If $\text{Card } \mathfrak{A} > e$, there exists a limiting point of \mathfrak{A} of rank $> e$.*

The demonstration is similar to that of 2.

4. *If there is no limiting point of \mathfrak{A} of rank $> e$, $\text{Card } \mathfrak{A} \leq e$.*

This follows from 3.

5. *Let $\text{Card } \mathfrak{A} > e$. Let \mathfrak{B} denote the limit set of \mathfrak{A} whose ranks are $> e$. Then \mathfrak{B} is perfect.*

For obviously \mathfrak{B} is complete. We need therefore only show that it is dense. To this end let b be a point of \mathfrak{B} . We can describe a sequence of concentric spheres of radii r_1, r_2, \dots such that the spheres determine a sequence of spherical shells $V_{r_1}^*(b) - V_{r_2}^*(b), V_{r_2}^*(b) - V_{r_3}^*(b), \dots$ which have a point in common. If \mathfrak{A}_n denote the points of \mathfrak{A} in $V_{r_n}^*(b)$ we have

$$V = V_{r_1}^*(b) = \mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{A}_3 + \dots$$

Thus if each \mathfrak{A}_n were enumerable, V is enumerable. But $\text{Rank } b$ is not $> e$. Thus there is one set \mathfrak{A}_m which is not enumerable, and hence by 3 there exists a point of \mathfrak{B} in $V_{r_m}^*(b)$. Thus there are points of \mathfrak{B} in any $V_r^*(b)$, and b is not isolated.

6. *A set \mathfrak{A} which contains no dense component is enumerable.*

complete. The derivatives of \mathfrak{A} of order $\leq \beta$ are complete, form a well ordered set and have their least point \mathfrak{A}^β , where β is necessarily a limitary number.

$$\mathfrak{A}^\beta = Dv(\mathfrak{A}^\gamma) \quad , \quad \gamma < \beta.$$

But every point of \mathfrak{A}^β lies in each \mathfrak{A}^γ . If \mathfrak{A}^β is not complete, then \mathfrak{A}^β is a limiting point of each \mathfrak{A}^γ and hence \mathfrak{A}^β is complete, which is a contradiction.

316. Let α be a limitary number in Z_n .
 $\beta < \alpha$, \mathfrak{A}^α exists.

For there exists an \aleph_m , $m \leq n - 2$, such that

$$\gamma < \delta < \epsilon < \eta < \dots \doteq \alpha.$$

Let c be a point of \mathfrak{A}^γ , d a point of \mathfrak{A}^δ , e a point of \mathfrak{A}^ϵ .
 Then the set

$$(c, d, e, f, \dots)$$

has at least one limiting point l of rank \aleph_m .
 (in 1). Then l is a limiting point of rank \aleph_m .

$$(e, f, \dots).$$

Thus l is a limiting point of every \mathfrak{A}^β , $\beta < \alpha$.

317. Let us show how we may form points of rank \aleph_m .
 is any number in Z_1 or Z_2 .

We take the unit interval $\mathfrak{A} = (0, 1)$ as our starting point for our considerations.

In \mathfrak{A} , take the points

$$\mathfrak{B}_1 = \frac{1}{2} \quad , \quad \frac{3}{4} \quad , \quad \frac{7}{8} \quad , \quad \frac{15}{16}$$

Obviously $\mathfrak{B}'_2 = \frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$,

Hence $\mathfrak{B}''_2 = \mathfrak{B}'_1 = 1$ and $\mathfrak{B}'''_2 =$

Thus \mathfrak{B}_2 is of order 2.

In each of the intervals 2) we may place a point to \mathfrak{B}_2 , such that the right-hand end point is a limiting point of the set. The resulting set is of order 2.

This shows that we may form sets of order ω .

Let us now place a set of order 1 in each of the intervals etc. The resulting set \mathfrak{B}_ω is of order ω . The point 1 lies in $\mathfrak{A}_1, \mathfrak{A}_2 \dots \mathfrak{A}_{n-1}$, while the point 1 lies in \mathfrak{A}_n .

Thus $\mathfrak{B}^{(\omega)}_\omega = 1$.

Hence $\mathfrak{B}^{(\omega+1)}_\omega = 0$,
and \mathfrak{B}_ω is of order ω .

Let us now place in each \mathfrak{A}_n a set of order 1, such that the right-hand end point of \mathfrak{A}_n as limiting point. The resulting set $\mathfrak{B}_{\omega+1}$ is of order $\omega + 1$. In this way we may form sets of order $\omega + 2, \omega + 3, \dots$ just as we did for finite orders. We may also form now a set of order ω^2 , and hence of order ω .

Thus we may form sets of order

$$\omega, \omega \cdot 2, \omega \cdot 3, \dots$$

and hence of order ω^2 , etc.

318. 1. Let \mathfrak{A} be limited or not, and let \mathfrak{A}^β be the set of points of \mathfrak{A}^β . Then

$$\mathfrak{A}' = \sum \mathfrak{A}^{(\beta)} + \mathfrak{A}^\omega, \quad \beta = 1, 2, 3, \dots$$

On \mathfrak{A}^ω we can reason as on \mathfrak{A}' , and in general we have

$$\mathfrak{A}' = \sum_{\beta < \alpha} \mathfrak{A}_i^{(\beta)} + \mathfrak{A}^{(\alpha)},$$

which gives 1).

2. *If $\mathfrak{A}^\Omega = 0$, \mathfrak{A} and \mathfrak{A}' are enumerable.*

For not every $\mathfrak{A}^{(\alpha)} > 0 \quad \alpha < \Omega$, by 316.

Hence there is a first α , call it γ , such that $\mathfrak{A}^{(\gamma)} = 0$. This reduces to

$$\mathfrak{A}' = \sum_{\beta} \mathfrak{A}_i^{(\beta)}, \quad \beta = 1, 2, \dots < \gamma.$$

But the summation extends over an enumerable set, each of which is enumerable by 289. Hence \mathfrak{A}' is enumerable. But then \mathfrak{A} is also enumerable by 237, 2.

3. *Conversely, if \mathfrak{A}' is enumerable, $\mathfrak{A}^\Omega = 0$.*

For if $\mathfrak{A}^\Omega > 0$, there is a non-enumerable set $\mathfrak{A}^{(\beta)}$ no $\mathfrak{A}^{(\beta)}$ is perfect; and as each term contains a non-perfect partial set and is therefore not enumerable. If some $\mathfrak{A}^{(\beta)}$ is perfect, \mathfrak{A}' is not enumerable.

4. From 2, 3, we have :

For \mathfrak{A}' to be enumerable, it is necessary and sufficient that there exists a number α in Z_1 or Z_2 such that $\mathfrak{A}^\alpha = 0$.

5. *If \mathfrak{A} is complete, it is necessary and sufficient that there exists an α in Z_1 or Z_2 such that $\mathfrak{A}^\alpha = 0$.*

For

$$\mathfrak{A} = \mathfrak{A}_i + \mathfrak{A}',$$

and the first term is enumerable.

6. If $\mathfrak{A}^\beta = 0$ for some $\beta < \Omega$, we say \mathfrak{A} is *reducible*. If $\mathfrak{A}^\alpha > 0$ for all $\alpha < \Omega$, \mathfrak{A} is *irreducible*.

Obviously $\mathfrak{B}'_2 = \frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$, $\dots = \mathfrak{B}_1$.

Hence $\mathfrak{B}''_2 = \mathfrak{B}'_1 = 1$ and $\mathfrak{B}'''_2 = 0$

Thus \mathfrak{B}_2 is of order 2.

In each of the intervals 2) we may place a set of to \mathfrak{B}_2 , such that the right-hand end point of each is limiting point of the set. The resulting set \mathfrak{B}_3 is of

This shows that we may form sets of every finite

Let us now place a set of order 1 in \mathfrak{A}'_1 , a set of etc. The resulting set \mathfrak{B}_ω is of order ω . For $\mathfrak{B}_\omega^{(n)}$ in $\mathfrak{A}_1, \mathfrak{A}_2 \dots \mathfrak{A}_{n-1}$, while the point 1 lies in every \mathfrak{B}_ω .

Thus

$$\mathfrak{B}_\omega^{(\omega)} = 1.$$

Hence

$$\mathfrak{B}_\omega^{(\omega+1)} = 0,$$

and \mathfrak{B}_ω is of order ω .

Let us now place in each \mathfrak{A}_n a set similar to \mathfrak{B}_ω right-hand end point of \mathfrak{A}_n as limiting point. The $\mathfrak{B}_{\omega+1}$ is of order $\omega + 1$. In this way we may proceed of order $\omega + 2, \omega + 3, \dots$ just as we did for orders ω may also form now a set of order ω^2 , as we before of order ω .

Thus we may form sets of order

$$\omega \quad , \quad \omega \cdot 2 \quad , \quad \omega \cdot 3 \quad , \quad \omega \cdot 4 \quad \dots$$

and hence of order ω^2 , etc.

318. 1. *Let \mathfrak{A} be limited or not, and let $\mathfrak{A}_i^{(\beta)}$ denote points of \mathfrak{A}^β . Then*

$$\mathfrak{A}' = \sum \mathfrak{A}_i^{(\beta)} + \mathfrak{A}^\omega \quad , \quad \beta = 1, 2, \dots < \omega$$

On \mathfrak{A}^ω we can reason as on \mathfrak{A}' , and in general we have

$$\mathfrak{A}' = \sum_{\beta < \alpha} \mathfrak{A}_i^{(\beta)} + \mathfrak{A}^{(\alpha)},$$

which gives 1).

2. If $\mathfrak{A}^\alpha = 0$, \mathfrak{A} and \mathfrak{A}' are enumerable.

For not every $\mathfrak{A}^{(\alpha)} > 0 \quad \alpha < \Omega$, by 316.

Hence there is a first α , call it γ , such that \mathfrak{A}^γ reduces to

$$\mathfrak{A}' = \sum_{\beta} \mathfrak{A}_i^{(\beta)} \quad , \quad \beta = 1, 2, \dots < \gamma.$$

But the summation extends over an enumerable set, each of which is enumerable by 289. Hence \mathfrak{A}' is enumerable. But then \mathfrak{A} is also enumerable by 237, 2.

3. Conversely, if \mathfrak{A}' is enumerable, $\mathfrak{A}^\alpha = 0$.

For if $\mathfrak{A}^\alpha > 0$, there is a non-enumerable set $\mathfrak{A}^{(\beta)}$ no $\mathfrak{A}^{(\beta)}$ is perfect; and as each term contains a perfect partial set and is therefore not enumerable. If some $\mathfrak{A}^{(\beta)}$ is perfect, \mathfrak{A}' is not enumerable.

4. From 2, 3, we have :

For \mathfrak{A}' to be enumerable, it is necessary and sufficient that there exists a number α in Z_1 or Z_2 such that $\mathfrak{A}^\alpha = 0$.

5. If \mathfrak{A} is complete, it is necessary and sufficient that there exists an α in Z_1 or Z_2 such that $\mathfrak{A}^\alpha = 0$.

For

$$\mathfrak{A} = \mathfrak{A}_i + \mathfrak{A}',$$

and the first term is enumerable.

6. If $\mathfrak{A}^\beta = 0$ for some $\beta < \Omega$, we say \mathfrak{A} is reducible. If $\mathfrak{A}^\alpha > 0$ for all $\alpha < \Omega$, we say \mathfrak{A} is irreducible.

Let S_n denote a sphere about a of radius r_n . Let \mathfrak{B} denote the set of points of \mathfrak{B} lying between S_{n-1} , S_n , including the boundary S_{n-1} . Then

$$\mathfrak{B} = \mathfrak{B}_1 + \mathfrak{B}_2 + \mathfrak{B}_3 + \cdots + a.$$

Each \mathfrak{B}_m is enumerable. For any point of \mathfrak{B} there is a unique m such that $\mathfrak{B}_m = a$. Hence $\mathfrak{B}_m^a = 0$ and \mathfrak{B}_m is enumerable by \mathfrak{B}_m^a .

Thus \mathfrak{B} is enumerable. This, however, is not true. $\mathfrak{B}^a = a$, and is thus > 0 .

320. 1. In the relation

$$\mathfrak{A}' = \sum_{\beta} \mathfrak{A}_t^{(\beta)} + \mathfrak{A}^a \quad \beta = 1, 2, \dots < \aleph_0$$

the first term on the right is enumerable.

For let us set

$$\mathfrak{B} = \sum_{\beta} \mathfrak{A}_t^{(\beta)};$$

also let

$$r_1 > r_2 > \cdots \rightarrow 0.$$

Let \mathfrak{B}_n denote the points of \mathfrak{B} whose distance from a is less than r_n . \mathfrak{B}_n satisfies the relation

$$r_n \geq \delta > r_{n+1}.$$

Then the distance of any point of \mathfrak{B}'_n from a is less than r_n . \mathfrak{B}'_n includes all points of \mathfrak{B} whose distance from a is less than r_n .

$$\mathfrak{B} = \mathfrak{B}_0 + \mathfrak{B}_1 + \mathfrak{B}_2 + \cdots$$

Each \mathfrak{B}_n is enumerable. For if not, $\mathfrak{B}_n^a > 0$. Hence \mathfrak{B}_n^a as b lies in \mathfrak{A}^a . Hence

$$\text{Dist}(b, \mathfrak{A}^a) = 0.$$

On the other hand, as b lies in \mathfrak{B}'_n , its distance from a is $\geq r_{n+1}$, which is a contradiction.

2. If \mathfrak{A}' is not enumerable, there exists a first n such that \mathfrak{A}^a is perfect.

Complete Sets

321. Let us study now some of the properties of sets. We begin by considering limited perfect sets. Let \mathfrak{A} be such a set. It has a first point a and a last point b , and therefore lies in the interval $I = (a, b)$. If \mathfrak{A} is part of a partial interval $J = (\alpha, \beta)$ of I , \mathfrak{A} embraces all the points of J since \mathfrak{A} is perfect. Let us therefore suppose that \mathfrak{A} lies in I . An example of such sets is the Cantor set of points.

Let $D = \{\delta\}$ be a set of intervals no two of which have a point in common. We say D is *pantactic* in an interval I if it contains no interval which does not contain some interval of I , or at least a part of some δ .

It is *separated* when no two of its intervals have a point in common.

322. 1. *Every limited rectilinear apantactic perfect set contains an enumerable pantactic set of separated intervals whose end points alone lie in \mathfrak{A} .*

For let \mathfrak{A} lie in $I = (a, \beta)$, where a, β are the endpoints of I . Let $\mathfrak{B} = I - \mathfrak{A}$. Each point b of \mathfrak{B} is the endpoint of an interval δ whose end points lie in \mathfrak{A} . For other points of \mathfrak{B} we can approach b as near as we chose, ranging over a set of points of \mathfrak{B} . But then b is a point of \mathfrak{A} , as this is perfect. Let us take these intervals as large as possible and call them δ .

The intervals δ are pantactic in I , for otherwise \mathfrak{A} would be apantactic. They are enumerable, for but a finite number of intervals have lengths $> I/n + 1$ and $\leq I/n$, $n = 1, 2, \dots$

It is separated, since \mathfrak{A} contains no isolated points.

2. The set of intervals $D = \{\delta\}$ just considered

\mathfrak{A} contains no other points. For let a be a point of E' . Let α be another point of \mathfrak{A} . In the interval (a, α) there is an end point e of some interval of D . In the interval (e, α) there is another end point e_1 . In the interval (a, e_1) lies another end point e_2 , etc. The set of points $e, e_1, e_2 \dots \doteq a$. Hence $a \in \mathfrak{A}$, which is a contradiction.

324. *Conversely, the end points $E = \{e\}$ and the limit points L of the end points of a pantactic enumerable set of separate intervals $D = \{\delta\}$ form a perfect apantactic set \mathfrak{A} .*

For in the first place, \mathfrak{A} is complete, since $\mathfrak{A} = (E, L)$ contains no isolated points, since the intervals δ are dense. Hence \mathfrak{A} is perfect. It is apantactic, since otherwise it would contain all the points of some interval, which is impossible for a pantactic set.

325. Since the adjoint set of intervals $D = \{\delta\}$ is enumerable, it can be arranged in a 1, 2, 3, ... order according to size.

Let δ be the largest interval, or if several are equal, let δ be one of them. The interval δ causes I to fall into two other intervals. The interval to the left of δ , call I_0 , that to the right of δ , call I_1 . The largest interval in I_0 , call δ_0 , that in I_1 , call δ_1 . In I_0 we may continue without end, getting a sequence of

$$\delta, \delta_0, \delta_1, \delta_{00}, \delta_{01}, \delta_{10}, \delta_{11} \dots$$

and a similar series of intervals

$$I, I_0, I_1, I_{00}, I_{01} \dots$$

The lengths of the intervals in 1) form a monotonic decreasing sequence which $\doteq 0$.

If ν denote a complex of indices $ij\kappa \dots$

Let $D = \{\delta_\nu\}$ be its adjoint set of intervals, a
 Let \mathfrak{C} be the Cantor set of I, 272. Let its adjoint
 be $H = \{\eta_\nu\}$, arranged also as in 325. If we set
 $D \simeq H$. Hence $\text{Card } \mathfrak{A} = \text{Card } \mathfrak{C}$.

But $\text{Card } \mathfrak{C} = c$ by 244, 4.

2. *The cardinal number of every limited rectifiable set is either e or c .*

For we have seen, 320, 4, that

$$\mathfrak{A} = \mathfrak{C} + \mathfrak{B}, \quad \mathfrak{B} \geq 0,$$

where \mathfrak{C} is enumerable and \mathfrak{B} is perfect,

$$\text{If } \mathfrak{B} = 0, \quad \text{Card } \mathfrak{A} = e.$$

$$\text{If } \mathfrak{B} > 0, \quad \text{Card } \mathfrak{A} = c.$$

* For $\text{Card } \mathfrak{A} = \text{Card } \mathfrak{C} + \text{Card } \mathfrak{B} = e + c = c$.

327. *The cardinal number of every limited complete set is either e or c . It is c , if \mathfrak{A} has a perfect component.*

The proof may be made by induction.

For simplicity take $m = 2$. By a transformation we may bring \mathfrak{A} into a unit square S . Let us suppose \mathfrak{A} were in S originally. Then $\text{Card } \mathfrak{A} \leq c$ by 244.

Let \mathfrak{C} be the projection of \mathfrak{A} on one of the sides of S . Let \mathfrak{B} be the set of points of \mathfrak{A} lying on a parallel to the other side of S not containing any point of \mathfrak{C} . If \mathfrak{B} has a perfect component, $\text{Card } \mathfrak{B} = c$. If \mathfrak{B} does not have a perfect component, $\text{Card } \mathfrak{B} = e$. Now \mathfrak{C} is complete by 327. If \mathfrak{C} contains a perfect component, $\text{Card } \mathfrak{C} = c$.

complete, any point a of A is an inner point of A and lies in A , for some ρ sufficiently small. Hence a is an inner point of A .

We have thus the result:

Any limited complete set is uniquely determined by the set of cubes $\{d_n\}$, each of which is exterior to it.

We may call $\mathfrak{B} = \{d_n\}$ the *border* of \mathfrak{A} , and the d_n the *border cells*.

2. *The totality of all limited perfect or complete sets has cardinal number c .*

For any limited complete set \mathfrak{C} is completely determined by its border $\{d_n\}$. The totality of such sets has a cardinal number $\leq c^c = c$. Hence $\text{Card } \{\mathfrak{C}\} \leq c$. Since among the d_n there are segments, $\text{Card } \mathfrak{C} \geq c$.

329. If \mathfrak{A}_i denote the isolated points of \mathfrak{A} , and \mathfrak{A}_λ the limiting points, we may write

$$\mathfrak{A} = \mathfrak{A}_i + \mathfrak{A}_\lambda.$$

Similarly we have

$$\mathfrak{A}_\lambda = \mathfrak{A}_{\lambda_1} + \mathfrak{A}_{\lambda_2},$$

$$\mathfrak{A}_{\lambda_2} = \mathfrak{A}_{\lambda_2_1} + \mathfrak{A}_{\lambda_2_2}, \text{ etc.}$$

We thus have

$$\mathfrak{A} = \mathfrak{A}_i + \mathfrak{A}_{\lambda_1} + \mathfrak{A}_{\lambda_2_1} + \cdots + \mathfrak{A}_{\lambda^{n-1}_1} + \mathfrak{A}_{\lambda^n}.$$

At the end of each step, certain points of \mathfrak{A} are separated from the rest and may be considered as *adhering* loosely to \mathfrak{A} , while the remainder remains may be regarded as *cohering* more closely to \mathfrak{A} . We may call $\mathfrak{A}_{\lambda^{n-1}_1}$ the n^{th} *adherent*, and \mathfrak{A}_{λ^n} the n^{th} *coherent*.

If α is a limitary number, defined by

$$\alpha_1 < \alpha_2 < \alpha_3 \cdots \doteq \alpha,$$

we set

$$\mathfrak{A}_\alpha = Dv \{ \mathfrak{A}_\lambda^{\alpha n} \}$$

and call it, when it exists, the *coherent of order*
write

$$\mathfrak{A} = \sum_{\alpha} \mathfrak{A}_{\lambda^{\alpha_i}} + \mathfrak{A}_{\lambda^{\beta}} \quad \alpha = 1, 2, \dots < \beta$$

where β is a number in Z_2 .

330. 1. When \mathfrak{A} is enumerable,

$$\mathfrak{A} = \sum_{\alpha} \mathfrak{A}_{\lambda^{\alpha_i}} + \mathfrak{A}_{\lambda^{\beta}} \quad \alpha = 1, 2,$$

$$= \mathfrak{J} + \mathfrak{D};$$

where \mathfrak{J} is the sum of an enumerable set of isolated points, if it exists, is dense.

For the adherences of different orders have no relation with those of any other order. They are thus independent. The sum \mathfrak{J} can contain but an enumerable set of isolated points. If \mathfrak{A} is not enumerable, \mathfrak{A} could not be enumerable. Thus there is no number β for which

$$\mathfrak{A}_{\lambda^{\beta_i}} = 0.$$

As now in general

$$\mathfrak{A}_{\lambda^{\beta}} = \mathfrak{A}_{\lambda^{\beta_i}} + \mathfrak{A}_{\lambda^{\beta+1}},$$

we have

$$\mathfrak{A}_{\lambda^{\beta}} = \mathfrak{A}_{\lambda^{\beta+1}} = \mathfrak{A}_{\lambda^{\beta+2}} = \dots$$

As $\mathfrak{A}_{\lambda^{\beta}}$ thus contains no isolated points, it is dense by I, 270.

2. When \mathfrak{A} is not enumerable, $\mathfrak{D} > 0$. For if

these may be consolidated with the cells of D in division Δ of norm δ which in general will not be

$$\overline{\mathfrak{A}}_{\Delta} = \overline{\mathfrak{A}}'_{\Delta} + \overline{\mathfrak{A}}_{\Delta}^*.$$

The last term is formed of cells that contain only one of points of \mathfrak{A} . These cells may be subdivided in division E such that in

$$\overline{\mathfrak{A}}_E = \overline{\mathfrak{A}}'_E + \overline{\mathfrak{A}}_E^*$$

the last term is $< \epsilon/3$. Now if δ is sufficiently

$$\overline{\mathfrak{A}}_{\Delta} - \overline{\mathfrak{A}} < \frac{\epsilon}{3}, \quad \overline{\mathfrak{A}}'_{\Delta} - \overline{\mathfrak{A}}' < \frac{\epsilon}{3}$$

Hence from 2), 3) we have 1).

332. If $\overline{\mathfrak{A}} > 0$, $\text{Card } \mathfrak{A} = c$.

For let \mathfrak{B} denote the sifted set of \mathfrak{A} [I, 71]. By hypothesis \mathfrak{B} is perfect. Hence $\text{Card } \mathfrak{B} = c$, hence $\text{Card } \mathfrak{A} = c$.

333. Let $\mathfrak{A} = \{\alpha\}$, where each α is metric and no two of the α 's have more than their frontiers in common. \mathfrak{A} may be unlimited.

Let us first suppose that \mathfrak{A} lies in a cube Ω . Let \mathfrak{A}' be the set of points of \mathfrak{A} removing its proper frontier points. Then no two of the α 's have a point in common. Let

$$q_1 > q_2 > \dots \doteq 0,$$

where the first term $q_1 = \widehat{\Omega}$. There can be but a finite number of sets α , such that their contents lie between q_{s-1} and q_s . For if

$$\widehat{\alpha}_{s-1}, \widehat{\alpha}_s, \dots \geq q_s$$

CHAPTER XI

MEASURE

Upper Measure

334. 1. Let \mathfrak{A} be a limited point set. An metric sets $D = \{d_i\}$, such that each point of \mathfrak{A} called an *enclosure* of \mathfrak{A} . If each point of \mathfrak{A} lies is called an *outer* enclosure. The sets d_i are called enclosure corresponds the finite or infinite series

$$\sum d_i$$

which may or may not converge. In any case the the numbers 1) is finite and ≤ 0 . For let Δ be of space, $\bar{\mathfrak{A}}_\Delta$ is obviously an enclosure and the 1) is also $\bar{\mathfrak{A}}_\Delta$, since we have agreed to read this as a point set or as its content.

We call

$$\text{Min } \sum d_i,$$

with respect to the class of all possible enclosures *measure* of \mathfrak{A} , and write

$$\bar{\mathfrak{A}} = \text{Meas } \mathfrak{A} = \text{Min}_D \sum d_i.$$

2. The minimum of the sums 1) is the same w

As ϵ is small at pleasure, $\text{Min } \Sigma d_i$ of closures = $\text{Min } \Sigma \bar{d}_i$ over the class of all

3. Two metric sets whose common points are called *non-overlapping*. The enclosure is called *non-overlapping*, when any two of its cells are non-overlapping.

Any enclosure D may be replaced by a non-overlapping enclosure.

For let $U(d_1, d_2) = d_1 + e_1$

$$U(d_1, d_2, d_3) = d_1 + e_2$$

$$U(d_1, d_2, d_3, d_4) = d_1 + e_3$$

Obviously each e_n is metric. For uncountable sets. Then $E = \{e_n\}$ is a non-overlapping enclosure.

$$\Sigma \bar{e}_n \leq \Sigma \bar{d}_n$$

we see that the minimum of the sums 1) is attained for ourselves to the class of non-overlapping enclosures.

Obviously we may adjoin to any enclosure improper limiting points.

4. In the enclosure $E = \{e_n\}$ found above, any two cells have a point in common. Such enclosure is called *non-overlapping*.

335. 1. Let $D = \{d_i\}$, $E = \{e_k\}$ be two enclosures of \mathfrak{A} . Let

$$\delta_{\iota\kappa} = Dv(d_i, e_k)$$

Then

$$\Delta = \{\delta_{\iota\kappa}\}, \quad \iota, \kappa = 1, 2, \dots$$

is a non-overlapping enclosure of \mathfrak{A} .

For $\delta_{\iota\kappa}$ is metric by 22, 2. Two of its cells are non-overlapping. Each point of \mathfrak{A} lies in one of its cells, hence α lies in δ .

As $\Sigma \widehat{d}_i \leq \Sigma \widehat{e}_i$,
we have

$$\text{Min}_{\Delta} \Sigma \widehat{\delta}_i \leq \text{Min}_D \Sigma \widehat{d}_i \leq \text{Min}_E \Sigma \widehat{e}_i$$

from which 1) follows at once.

337. *If \mathfrak{A} is metric,* $\overline{\mathfrak{A}} = \mathfrak{A}$.

For let D be a cubical division of space such

$$\overline{\mathfrak{A}}_D - \mathfrak{A} < \epsilon, \quad \mathfrak{A} - \underline{\mathfrak{A}}_D < \epsilon.$$

Let us set $\mathfrak{B} = \mathfrak{A}_D$. Let $\mathcal{E} = \{e_i\}$ be an outer measure. Since \mathfrak{B} is complete, there exists a finite set of points which contain all the points of \mathfrak{B} by 301. The volume of \mathcal{E} obviously $> \mathfrak{B}$; hence *a fortiori*

$$\Sigma \widehat{e}_i > \mathfrak{B}.$$

Hence

$$\overline{\mathfrak{B}} > \mathfrak{B}.$$

But

$$\overline{\mathfrak{A}} > \mathfrak{B}, \text{ by 336,}$$

$$\geq \mathfrak{B} = \underline{\mathfrak{A}}_D$$

$$> \mathfrak{A} - \epsilon, \text{ by 2).}$$

On the other hand,

$$\overline{\mathfrak{A}} < \overline{\mathfrak{A}}_D < \mathfrak{A} + \epsilon, \text{ by 2).}$$

From 3), 4) we have 1), since ϵ is arbitrarily

338. *If \mathfrak{A} is complete,* $\overline{\mathfrak{A}} = \mathfrak{A}$.

For by definition

$$\overline{\mathfrak{A}} = \text{Min} \Sigma \widehat{d}_i,$$

339. *Let the limited set $\mathfrak{A} = \{\mathfrak{A}_n\}$ be the union of an enumerable set of sets \mathfrak{A}_n . Then*

$$\overline{\mathfrak{A}} \leq \Sigma \overline{\mathfrak{A}}_n.$$

For to each \mathfrak{A}_n corresponds an enclosure D_n ,

$$\Sigma \hat{d}_{n_i} < \overline{\mathfrak{A}}_n + \frac{\epsilon}{2^n}, \quad \epsilon > 0, \text{ arbitrary.}$$

But the cells of all the enclosures D_n , also
Hence

$$\begin{aligned} \overline{\mathfrak{A}} &\leq \Sigma_{i,n} \hat{d}_{n_i} < \Sigma \left(\overline{\mathfrak{A}}_n + \frac{\epsilon}{2^n} \right) \\ &\leq \Sigma \overline{\mathfrak{A}}_n + \epsilon. \end{aligned}$$

This gives 1), as ϵ is small at pleasure.

340. *Let \mathfrak{A} lie in the metric set \mathfrak{M} . Let \mathfrak{A}' be the complementary set. Then*

$$\overline{\mathfrak{A}} + \overline{\mathfrak{A}'} \geq \widehat{\mathfrak{M}}.$$

For from

$$\mathfrak{M} = \mathfrak{A} + \mathfrak{A}',$$

follows

$$\overline{\mathfrak{M}} \leq \overline{\mathfrak{A}} + \overline{\mathfrak{A}'}, \quad \text{by 339.}$$

But

$$\overline{\mathfrak{M}} = \widehat{\mathfrak{M}}, \quad \text{by 337.}$$

341. *If $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$, and \mathfrak{B} , \mathfrak{C} are exterior to each other,*

$$\overline{\mathfrak{A}} = \overline{\mathfrak{B}} + \overline{\mathfrak{C}}.$$

For, if any enclosure $D = \{d_i\}$ of \mathfrak{A} embraces a point of \mathfrak{B} and \mathfrak{C} , it may be split up into two, each containing points of \mathfrak{B} only, or of \mathfrak{C} only.

By properly choosing D , we may crowd the down toward its minimum. Now the class \mathfrak{A} is included in the class of all enclosures of \mathfrak{B} , and holds for D' .

Thus from 2) follows that

$$\overline{\mathfrak{A}} > \overline{\mathfrak{B}} + \overline{\mathfrak{C}}.$$

This with 339 gives 1).

342. If $\mathfrak{A} = \mathfrak{B} + \mathfrak{M}$, \mathfrak{M} being metric,

$$\mathfrak{A} = \mathfrak{B} + \widehat{\mathfrak{M}}.$$

For let D be a cubical division of norm d . Let \mathfrak{M} of \mathfrak{M} in the cells containing points of Front \mathfrak{M} and the other points of \mathfrak{M} . Then \mathfrak{m} and \mathfrak{B} are extended and by 337 and 341,

$$\text{Meas } (\mathfrak{B} + \mathfrak{m}) = \overline{\mathfrak{B}} + \widehat{\mathfrak{m}}.$$

As

$$\mathfrak{A} = \mathfrak{B} + \mathfrak{m} + \mathfrak{n},$$

$$\text{Meas } (\mathfrak{B} + \mathfrak{m}) < \mathfrak{A} \quad \text{by 1)}$$

Also

$$\overline{\mathfrak{A}} < \overline{\mathfrak{B}} + \widehat{\mathfrak{m}} + \widehat{\mathfrak{n}} \quad \text{by 341}$$

Thus

$$\overline{\mathfrak{B}} + \widehat{\mathfrak{m}} < \overline{\mathfrak{A}} < \overline{\mathfrak{B}} + \widehat{\mathfrak{m}} + \widehat{\mathfrak{n}}.$$

Now if d is sufficiently small,

$$\widehat{\mathfrak{M}} - \epsilon < \widehat{\mathfrak{m}} \quad ; \quad \mathfrak{n} < \epsilon.$$

Thus 2) gives, as $\widehat{\mathfrak{m}} < \widehat{\mathfrak{M}}$,

$$\overline{\mathfrak{B}} + \widehat{\mathfrak{M}} - \epsilon < \overline{\mathfrak{A}} < \overline{\mathfrak{B}} + \widehat{\mathfrak{M}} + \epsilon,$$

which gives 1), as $\epsilon > 0$ is arbitrarily small.

Thus

$$\widehat{\mathfrak{B}} - \overline{\overline{B}} = \widehat{\mathfrak{D}} + \widehat{\mathfrak{B}}_1 - (\widehat{\mathfrak{B}}_1 + \overline{\overline{D}}) =$$

$$\widehat{\mathfrak{C}} - \overline{\overline{C}} = \widehat{\mathfrak{D}} + \widehat{\mathfrak{C}}_1 - (\widehat{\mathfrak{C}}_1 + \overline{\overline{D}}) =$$

2. If $\mathfrak{A} < \mathfrak{B}$, the complement of \mathfrak{A} with frequently be denoted by the corresponding E

$$A = C(\mathfrak{A}), \quad \text{Me}$$

$$= \mathfrak{B} - \mathfrak{A}.$$

Lower Measure

344. 1. We are now in position to define measure. Let \mathfrak{A} lie in a metric set \mathfrak{M} . The $A = \mathfrak{M} - \mathfrak{A}$ has an upper measure $\overline{\overline{A}}$. We s is the *lower measure* of \mathfrak{A} , and write

$$\underline{\underline{\mathfrak{A}}} = \underline{\text{Meas}} \mathfrak{A} = \widehat{\mathfrak{M}} - \overline{\overline{A}}.$$

By 343 this definition is independent of the

When

$$\underline{\underline{\mathfrak{A}}} = \underline{\overline{\overline{\mathfrak{A}}}}$$

we say \mathfrak{A} is *measurable*, and write

$$\underline{\underline{\mathfrak{A}}} = \underline{\overline{\overline{\mathfrak{A}}}} = \mathfrak{A}.$$

A set whose measure is 0 is called a *null set*

2. Let $E = \{e_i\}$ be an enclosure of A .

Then

$$\underline{\underline{\mathfrak{A}}} = \text{Max} (\widehat{\mathfrak{M}} - \Sigma \bar{e}_i),$$

$$345. \quad 1. \quad \mathfrak{N} > 0.$$

For let \mathfrak{N} lie in the metric set \mathfrak{M} .

$$\text{Then} \quad \underline{\mathfrak{N}} = \widehat{\mathfrak{M}} - \overline{A}.$$

$$\text{But by 336,} \quad A < \widehat{\mathfrak{M}},$$

$$\text{hence} \quad \widehat{\mathfrak{M}} - A \geq 0.$$

$$2. \quad \underline{\mathfrak{N}} < \mathfrak{N}.$$

For let \mathfrak{N} lie in the metric set \mathfrak{M} .

$$\text{Then} \quad \mathfrak{N} + A > \widehat{\mathfrak{M}} \quad \text{by 340.}$$

$$\text{Hence} \quad \mathfrak{N} = \widehat{\mathfrak{M}} - A < \mathfrak{N}.$$

346. 1. For any limited set \mathfrak{N} ,

$$\underline{\mathfrak{N}} \leq \underline{\mathfrak{N}} \leq \overline{\mathfrak{N}} \leq \overline{\mathfrak{N}}.$$

For let $D = \{d_i\}$ be an enclosure of \mathfrak{N} . Then

$$\overline{\mathfrak{N}} = \text{Min}_D \Sigma \widehat{d}_i,$$

when D ranges over the class \mathcal{F} of all *finite* enclosures. On the other hand,

$$\overline{\mathfrak{N}} = \text{Min}_D \Sigma \widehat{d}_i,$$

when D ranges over the class \mathcal{E} of all *enumerable* enclosures. But the class \mathcal{E} includes the class \mathcal{F} . Hence $\underline{\mathfrak{N}} \leq \overline{\mathfrak{N}}$.

2. If \mathfrak{A} is metric, it is measurable, and

$$\widehat{\mathfrak{A}} = \widehat{\mathfrak{A}}.$$

This follows at once from 1).

347. Let \mathfrak{A} be measurable and lie in the metric \mathfrak{M} is measurable, and

$$\widehat{\mathfrak{A}} + \widehat{A} = \widehat{\mathfrak{M}}.$$

For

$$\underline{A} = \underline{\mathfrak{M}} - \underline{\mathfrak{A}}.$$

$$\underline{\mathfrak{A}} = \underline{\mathfrak{M}} - \underline{A} = \underline{\mathfrak{A}},$$

since \mathfrak{A} is measurable. This last gives

$$\overline{A} = \overline{\mathfrak{M}} - \overline{\mathfrak{A}}.$$

This with 2) shows that $\overline{A} = \underline{A}$; hence A is metric.
2) now follows 1).

348. If $\mathfrak{A} < \mathfrak{B}$, then $\underline{\mathfrak{A}} \leq \underline{\mathfrak{B}}.$

For as usual let A, B be the complements of $\mathfrak{A}, \mathfrak{B}$ respectively in a metric set \mathfrak{M} . Since $\mathfrak{A} < \mathfrak{B}$, $A > B$.

Hence, by 336,

$$\overline{A} \geq \overline{B}.$$

Thus,

$$\widehat{\mathfrak{M}} - \overline{A} \leq \widehat{\mathfrak{M}} - \overline{B},$$

which gives 1).

349. For \mathfrak{A} to be measurable, it is necessary and

$$\widehat{\mathfrak{A}} + \overline{A} = \widehat{\mathfrak{M}}$$

where \mathfrak{M} is any metric set $> \mathfrak{A}$, and $A = \mathfrak{M} - \mathfrak{A}$.

It is sufficient, for then 1) shows that

Let S denote the infinite series on the right of 1
let S_n denote the sum of the first n terms. Let \mathfrak{U}_n

Then $\mathfrak{U}_n \leq \mathfrak{U}$ and by 336,

$$\widehat{\mathfrak{U}}_n = S_n \leq \overline{\mathfrak{U}} \quad , \quad \text{for any } n.$$

Thus S is convergent and

$$S \leq \overline{\mathfrak{U}}.$$

On the other hand, by 339,

$$\overline{\mathfrak{U}} \leq S.$$

From 3), 4) follows that

$$S = \overline{\mathfrak{U}} = \lim S_n = \lim \widehat{\mathfrak{U}}_n.$$

We show now that \mathfrak{U} is measurable. To this end
metric set $> \mathfrak{U}$, and $\mathfrak{U}_n + A_n = \mathfrak{M}$ as usual.

Then

$$\widehat{\mathfrak{U}}_n + \widehat{A}_n = \widehat{\mathfrak{M}}.$$

But

$$A \leq A_n \quad , \quad \text{hence } \overline{A} \leq \overline{A}_n.$$

Thus 6) gives

$$\overline{A} + \widehat{\mathfrak{U}}_n \leq \widehat{\mathfrak{M}},$$

for any n . Hence

$$\overline{A} + \lim \widehat{\mathfrak{U}}_n \leq \widehat{\mathfrak{M}};$$

or using 5),

$$\overline{A} + \overline{\mathfrak{U}} \leq \widehat{\mathfrak{M}}.$$

Hence by 339,

$$\overline{A} + \overline{\mathfrak{U}} = \widehat{\mathfrak{M}}.$$

Thus by 349, \mathfrak{U} is measurable.

351. *Let*

$$\mathfrak{U} = \mathfrak{B} + \mathfrak{C};$$

then

$$\overline{\mathfrak{B}} + \overline{\mathfrak{C}} \leq \overline{\mathfrak{U}}.$$

As all the points of A are in B , and also in C , E and F , and hence in the cells of D , which thus enclose A . Let

$$\gamma_m = (d_{m1}, d_{m2} \dots) \quad , \quad \eta_n = (d_{1n}, d_{2n} \dots)$$

Let us set

$$e_m = \gamma_m + g_m \quad , \quad f_n = \eta_n + h_n.$$

Then by 350,

$$\hat{\gamma}_m = \sum_n \hat{d}_{mn} \quad , \quad \hat{\eta}_n = \sum_m \hat{d}_{mn}.$$

By 347,

$$\hat{e}_m = \hat{\gamma}_m + \hat{g}_m \quad , \quad \hat{f}_n = \hat{\eta}_n + \hat{h}_n.$$

Hence

$$\hat{\mathfrak{M}} - \sum_m \hat{e}_m = \hat{\mathfrak{M}} - \sum_{m,n} \hat{d}_{mn} - \sum_m \hat{g}_m,$$

$$\hat{\mathfrak{M}} - \sum_n \hat{f}_n = \hat{\mathfrak{M}} - \sum_{m,n} \hat{d}_{mn} - \sum_n \hat{h}_n.$$

Hence adding,

$$(\hat{\mathfrak{M}} - \sum_m \hat{e}_m) + (\hat{\mathfrak{M}} - \sum_n \hat{f}_n)$$

$$= \hat{\mathfrak{M}} - \sum \hat{d}_{mn} + [\hat{\mathfrak{M}} - (\sum \hat{g}_m + \sum \hat{h}_n)]$$

Now

$$\mathfrak{M} = U\{g_m, h_n, d_{mn}\} \quad m, n = 1, 2, \dots$$

Thus by 339, the term in [] is ≤ 0 . Thus

$$(\hat{\mathfrak{M}} - \sum_m \hat{e}_m) + (\hat{\mathfrak{M}} - \sum_n \hat{f}_n) \leq \hat{\mathfrak{M}} - \sum \hat{d}_{mn}$$

But

$$\underline{\mathfrak{B}} = \text{Max} (\hat{\mathfrak{M}} - \sum_m \hat{e}_m)$$

$$\underline{\mathfrak{C}} = \text{Max} (\hat{\mathfrak{M}} - \sum_n \hat{f}_n).$$

Thus 3) gives 1) at once.

Measurable Sets

352. 1. Let $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$. If $\mathfrak{B}, \mathfrak{C}$ are meas

2. Let $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$. If \mathfrak{A} , \mathfrak{B} are measurable, so is

$$\widehat{\mathfrak{C}} = \widehat{\mathfrak{A}} - \widehat{\mathfrak{B}}.$$

For let \mathfrak{A} lie in the metric set \mathfrak{M} . Then

$$\mathfrak{M} - \mathfrak{A} = \mathfrak{M} - (\mathfrak{B} + \mathfrak{C}) = (\mathfrak{M} - \mathfrak{C}) - \mathfrak{B}$$

Thus

$$A = C - \mathfrak{B};$$

Hence

$$C = \mathfrak{B} + A.$$

Thus C is measurable by 1. Hence \mathfrak{C} is measurable and

$$\widehat{\mathfrak{A}} = \widehat{\mathfrak{B}} + \widehat{\mathfrak{C}}.$$

From this follows 2) at once.

353. 1. Let $\mathfrak{A} = \Sigma \mathfrak{A}_n$ be the sum of an enumerable sets. Then \mathfrak{A} is measurable and

$$\widehat{\mathfrak{A}} = \Sigma \widehat{\mathfrak{A}}_n.$$

If \mathfrak{A} is the sum of a finite number of sets, the theorem is obviously true by 352, 1. In case \mathfrak{A} embraces an infinite number of sets, the reasoning of 350 may be employed.

2. Let $\mathfrak{N} = \{\mathfrak{N}_n\}$ be the union of an enumerable sets. Then \mathfrak{N} is a null set.

Follows at once from 1.

3. Let $\mathfrak{A} = \{\mathfrak{A}_n\}$ be the union of an enumerable sets whose common points two and two, form null sets. Then \mathfrak{A} is measurable and

$$\widehat{\mathfrak{A}} = \Sigma \widehat{\mathfrak{A}}_n.$$

4. Let $\mathfrak{E} = \{e_n\}$ be a non-overlapping enclosure of a measurable set. Then \mathfrak{E} is measurable and

6. If $\mathfrak{F} = \{f_n\}$ is another non-overlapping enclosure then

$$\mathfrak{D} = Dv(\mathfrak{E}, \mathfrak{F})$$

is measurable.

For the cells of \mathfrak{D} are

$$\delta_{ik} = Dv(e_i, f_k).$$

Thus δ_{ik} is metric, and

$$\widehat{\mathfrak{D}} = \Sigma \widehat{\delta}_{ik}.$$

354. 1. Harnack Sets. Let \mathfrak{A} be an interval of

$$\lambda = l_1 + l_2 + \dots$$

be a positive term series whose sum $\lambda > 0$ is $\leq l$. Cantor's set, I, 272, let us place a *black* interval of middle of \mathfrak{A} . In a similar manner let us place in remaining or *white* intervals, a black interval, whose sum is l_2 . Let us continue in this way; we get an enumerable set of black intervals \mathfrak{B} , and obviously

$$\widehat{\mathfrak{B}} = \lambda.$$

If we omit the end points from each of the black intervals, we get a set \mathfrak{B}^* , and obviously

$$\widehat{\mathfrak{B}^*} = \lambda.$$

The set

$$\mathfrak{H} = \mathfrak{A} - \mathfrak{B}^*$$

we call a *Harnack set*. This is complete by 324; and

$$\overline{\mathfrak{H}} = \widehat{\mathfrak{H}} = l - \lambda.$$

When $\lambda = l$, \mathfrak{H} is discrete, and the set reduces to Cantor's set. When $\lambda < l$, we get an apartment.

3. It is easy to extend Harnack sets to \mathfrak{R}_n . For example, in S be the unit square. On two of its adjacent sides let us put congruent Harnack sets \mathfrak{S} . We now draw lines through the points of the black intervals parallel to the sides. There results an enumerable set of black squares $\mathfrak{S} = \{S_n\}$. The sides of the squares \mathfrak{S} and their limiting points form obviously an apantactic perfect set \mathfrak{R} .

Let

$$a_1^2 + a_2^2 + \dots = m$$
be a series whose sum $0 < m \leq 1$.

We can choose \mathfrak{S} such that the square corresponding to its largest black interval has the area a_1^2 ; the four squares corresponding to the next two largest black intervals have the total area a_2^2 .

Then

$$\overline{\mathfrak{S}} = \Sigma a_n^2 = m.$$

Hence

$$\overline{\mathfrak{R}} = 1 - m = \overline{\mathfrak{R}}.$$

355. 1. If $\mathfrak{E} = \{e_m\}$ is an enclosure of \mathfrak{A} such that

$$\Sigma \hat{e}_m - \overline{\mathfrak{A}} < \epsilon,$$

is called an ϵ -enclosure. Let A be the complement of \mathfrak{A} with respect to the metric set \mathfrak{M} . Let $E = \{e_n\}$ be an ϵ -enclosure of A . We call \mathfrak{E} , E complementary ϵ -enclosures belonging to \mathfrak{A} .

2. If \mathfrak{A} is measurable, then each pair of complementary normal enclosures \mathfrak{E} , E , whose divisor $\mathfrak{D} = Dv(\mathfrak{E}, E)$, is such that

$$\hat{\mathfrak{D}} < \epsilon, \quad \epsilon \text{ small at pleasure.}$$

For let \mathfrak{E} , E be any pair of complementary $\epsilon/2$ normal enclosures. Then

$$\hat{\mathfrak{E}} - \hat{\mathfrak{A}} < \frac{\epsilon}{2}, \quad \hat{E} - \hat{A} < \frac{\epsilon}{2}.$$

Now $\mathfrak{G} = Dv(\mathfrak{E}, \mathfrak{F})$ is a normal metric enclosure over its cells g which contain points of \mathfrak{D} and \mathfrak{C} the cells of e, f . Hence

$$\widehat{g} \leq \widehat{e} + \widehat{f} < \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

Thus by 357, \mathfrak{D} is measurable.

2. Let $\mathfrak{A}, \mathfrak{B}$ be measurable.

Let $\mathfrak{D} = Dv(\mathfrak{A}, \mathfrak{B})$, $\mathfrak{U} = (\mathfrak{A}, \mathfrak{B})$.

Then $\widehat{\mathfrak{A}} + \widehat{\mathfrak{B}} = \widehat{\mathfrak{U}} + \widehat{\mathfrak{D}}$.

For $\mathfrak{U} = \mathfrak{A} + (\mathfrak{B} - \mathfrak{D})$.

Hence $\widehat{\mathfrak{U}} = \widehat{\mathfrak{A}} + \text{Meas } (\mathfrak{B} - \mathfrak{D})$
 $= \widehat{\mathfrak{A}} + \widehat{\mathfrak{B}} - \widehat{\mathfrak{D}}.$

359. Let $\mathfrak{A} = U \{ \mathfrak{A}_m \}$ be the union of an enumerable measurable cells; moreover let \mathfrak{A} be limited. Then \mathfrak{A} is measurable. If we set

$$\mathfrak{B}_1 = \mathfrak{A}_1 \quad , \quad (\mathfrak{A}_1, \mathfrak{A}_2) = \mathfrak{B}_1 + \mathfrak{B}_2,$$

$$(\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3) = \mathfrak{B}_1 + \mathfrak{B}_2 + \mathfrak{B}_3, \text{ etc.,}$$

then $\widehat{\mathfrak{A}} = \Sigma \widehat{\mathfrak{B}}_m.$

For $\mathfrak{D} = Dv(\mathfrak{A}_1, \mathfrak{A}_2)$ is measurable by 358.

Let $\mathfrak{A}_1 = \mathfrak{D} + \mathfrak{a}_1$, $\mathfrak{A}_2 = \mathfrak{D} + \mathfrak{a}_2$.

Then $\mathfrak{a}_1, \mathfrak{a}_2$ are measurable by 352, 2.

As $\mathfrak{U} = (\mathfrak{A}_1, \mathfrak{A}_2) = \mathfrak{D} + \mathfrak{a}_1 + \mathfrak{a}_2,$

\mathfrak{U} is measurable. As \mathfrak{U} and \mathfrak{B}_1 are measurable, in similar manner we show that $\mathfrak{B}_3, \mathfrak{B}_4 \dots$ are measurable.

For let

$$a_2 = \mathfrak{A}_2 - \mathfrak{A}_1, \quad a_3 = \mathfrak{A}_3 - \mathfrak{A}_2 \dots$$

For uniformity let us set $a_1 = \mathfrak{A}$. Then

$$\mathfrak{A} = \Sigma a_m.$$

As each a_n is measurable

$$\begin{aligned} \widehat{\mathfrak{A}} &= \Sigma \widehat{a}_m \\ &= \lim_{n=\infty} (\widehat{a}_1 + \dots + \widehat{a}_n) \\ &= \lim \widehat{\mathfrak{A}}_n. \end{aligned}$$

361. Let $\mathfrak{A}_1, \mathfrak{A}_2 \dots$ be measurable and their union $\mathfrak{D} = Dv \{ \mathfrak{A}_n \} > 0$, it is measurable.

For let \mathfrak{A} lie in the metric set \mathfrak{M} ;

let $\mathfrak{D} + D = \mathfrak{M}, \quad \mathfrak{A}_n + A_n = \mathfrak{M}$
as usual.

Now \mathfrak{D} denoting the points common to all the \mathfrak{A}_n can lie in all of the \mathfrak{A}_n , hence it lies in some one A_n . Thus

$$D \subseteq \{ A_n \}.$$

On the other hand, a point of $\{ A_n \}$ lies in some one \mathfrak{A}_m does not lie in \mathfrak{A}_m . Hence it does not lie in \mathfrak{D} .
 D . Hence

$$\{ A_n \} \subseteq D.$$

From 1), 2) we have $D = \{ A_n \}.$

As each A_n is measurable, so is D . Hence \mathfrak{D} is

362. If $\mathfrak{A}_1 \supseteq \mathfrak{A}_2 \supseteq \dots$ is an enumerable set of measurable sets, their divisor \mathfrak{D} is measurable, and

As
we have

$$\begin{aligned}\mathfrak{D} &= \mathfrak{M} - D, \\ \widehat{\mathfrak{D}} &= \widehat{\mathfrak{M}} - \widehat{D} \\ &= \lim (\widehat{\mathfrak{M}} - \widehat{A}_n) \\ &= \lim \widehat{\mathfrak{N}}_n.\end{aligned}$$

363. 1. The points $x = (x_1 \cdots x_m)$ such that

$$a_1 \leq x_1 \leq b_1 \quad , \quad \cdots \quad , \quad a_m \leq x_m \leq b_m$$

form a *standard rectangular cell*, whose edges have lengths

$$e_1 = b_1 - a_1 \quad , \quad \cdots \quad , \quad e_m = b_m - a_m$$

When $e_1 = e_2 = \cdots = e_m$, the cell is a *standard cell*. An enclosure of the limited set \mathfrak{N} , whose cells \mathfrak{E} are standard cells, is called a *standard enclosure*.

2. For each $\epsilon > 0$, there are standard ϵ -enclosures of the set \mathfrak{N} .

For let $\mathfrak{E} = \{e_n\}$ be any η -enclosure of \mathfrak{N} . Then

$$\Sigma \widehat{e}_n - \overline{\mathfrak{N}} < \eta.$$

Each e_n being metric, may be enclosed in a standard outer enclosure F_n , such that

$$\widehat{F}_n - \widehat{e}_n < \eta/2^n \quad , \quad n = 1, 2, \dots$$

Then $\mathfrak{F} = \{F_n\}$ is an enclosure of \mathfrak{N} , and

$$\begin{aligned}\Sigma \widehat{F}_n &< \Sigma (\widehat{e}_n + \eta/2^n) = \Sigma \widehat{e}_n + \eta \\ &< \overline{\mathfrak{N}} + 2\eta, \quad \text{by 2)}.\end{aligned}$$

But the enclosure \mathfrak{F} can be replaced by

Let

$$e_m = (e_{m,1}, e_{m,2}, e_{m,3} \dots) + e_m$$

then e_m is measurable. By this process the measurable cell e_m falls into an enumerable set of non-overlapping measurable cells, as indicated in 3). If we suppose that the same process takes place for each cell of \mathfrak{C} , we shall say we have decomposed \mathfrak{C} on \mathfrak{E} .

364. (*W. H. Young.*) Let \mathfrak{C} be any complete measurable set. Then

$$\underline{\mathfrak{A}} = \text{Max } \overline{\mathfrak{C}}.$$

For let \mathfrak{A} lie within a cube \mathfrak{M} , and let $A = \mathfrak{M} - \mathfrak{A}$ be as usual the complementary sets.

Let $\mathfrak{B} = \{b_n\}$ be a border set of \mathfrak{C} [328]. Let C be an overlapping enclosure of \mathfrak{C} ; we may suppose C to be the closure of \mathfrak{C} . Let E be a standard ϵ -enclosure of C . Let \mathfrak{A} be superimposed on E , getting a measurable enclosure A of \mathfrak{A} and A . Then

$$C = C_\Delta \geq A_\Delta.$$

Hence

$$\mathfrak{C} = \mathfrak{M} - C = \mathfrak{M} - C_\Delta \leq \mathfrak{M} - A_\Delta$$

Thus

$$\overline{\mathfrak{C}} = \overline{\mathfrak{C}}, \quad \text{by 338}$$

$$\leq \text{Meas } (\mathfrak{M} - A_\Delta)$$

$$\leq \widehat{\mathfrak{M}} - \widehat{A}_\Delta, \quad \text{by 338}$$

$$\leq \widehat{\mathfrak{M}} - \overline{A}.$$

Hence

$$\overline{\mathfrak{C}} \leq \underline{\mathfrak{A}},$$

and thus

$$\text{Let} \quad \mathfrak{K} = \mathfrak{M} - A_D + \mathfrak{F},$$

where \mathfrak{F} denotes the frontier points of A_D lying in \mathfrak{K} . Since each face of D is a null set, \mathfrak{K} is complete. Since each face of D is a null set, \mathfrak{K} is complete. Thus each set on the right of 4) is measurable, hence

$$\begin{aligned} \widehat{\mathfrak{K}} &= \widehat{\mathfrak{M}} - \widehat{A}_D + \widehat{\mathfrak{F}} \\ &= \widehat{\mathfrak{M}} - \widehat{A}_D \\ &= \widehat{\mathfrak{M}} - A - \epsilon' \quad , \quad 0 \leq \epsilon' \\ &= \mathfrak{N} - \epsilon'. \end{aligned}$$

$$\text{Thus} \quad \text{Max } \mathfrak{C} \geq \mathfrak{K} = \mathfrak{K} > \mathfrak{N} - \epsilon,$$

from which follows 3), since ϵ is small at pleasure.

365. 1. *If \mathfrak{N} is complete, it is measurable, and*

$$\widehat{\mathfrak{N}} = \mathfrak{N}.$$

For by 364,

$$\mathfrak{N} = \mathfrak{N}.$$

On the other hand,

$$\mathfrak{N} = \mathfrak{N}, \quad \text{by 338.}$$

2. *Let \mathfrak{B} be any measurable set in the limited set \mathfrak{S} .*

$$\mathfrak{N} = \text{Max } \widehat{\mathfrak{B}}.$$

$$\text{For} \quad \mathfrak{N} > \mathfrak{B} = \widehat{\mathfrak{B}}.$$

$$\text{Hence,} \quad \mathfrak{N} > \text{Max } \widehat{\mathfrak{B}}.$$

But the class of measurable components of \mathfrak{S} is a class of complete components \mathfrak{C} , since each \mathfrak{C} is

We throw the points \mathfrak{S} into two classes $\mathfrak{A} = \{a\}$, with the following properties:

1° To each a corresponds a point b symmetrical to M , and conversely.

2° If a falls in the segment δ of D_n , each of the segments δ' of D_n shall contain a point a' of \mathfrak{A} such that a' is in δ' as a is situated in δ .

3° Each δ of D_n shall contain a point a' of \mathfrak{A} situated in δ , as any given point a of \mathfrak{A} is situated in \mathfrak{E} .

4° \mathfrak{A} shall contain a point a situated in \mathfrak{E} as any point a' of \mathfrak{A} is in any δ_n .

The 1° condition states that \mathfrak{A} goes over into \mathfrak{B} about M . The 2° condition states that \mathfrak{A} falls into $2^3, \dots$ congruent subsets. The 3° condition states that \mathfrak{A}_n of \mathfrak{A} in δ_n goes over into \mathfrak{A} on stretching it in the ratio $1 : 2^n$. The condition 4° states that \mathfrak{A} goes over into \mathfrak{A}_n on stretching it in the ratio $1 : 2^n$.

We show now that \mathfrak{A} , and therefore \mathfrak{B} are not measurable. In the first place, we note that

$$\overline{\mathfrak{A}} = \overline{\mathfrak{B}},$$

by 1°. As $\mathfrak{S} = \mathfrak{A} + \mathfrak{B}$, if \mathfrak{A} or \mathfrak{B} were measurable, then \mathfrak{S} would be, and

$$\widehat{\mathfrak{A}} = \widehat{\mathfrak{B}} = \frac{1}{2}.$$

Thus if we show $\overline{\mathfrak{A}} \neq 1$ or $\overline{\mathfrak{B}} \neq 1$, neither \mathfrak{A} nor \mathfrak{B} is measurable. We show this by proving that if $\overline{\mathfrak{A}} = \alpha < 1$, then \mathfrak{B} is not measurable, and $\widehat{\mathfrak{B}} = 1$. But when \mathfrak{B} is measurable, $\widehat{\mathfrak{B}} = \frac{1}{2}$, we are led to a contradiction.

Let $\epsilon = \epsilon_1 + \epsilon_2 + \dots$ be a positive term series with terms small at pleasure. Let $\mathfrak{E}_1 = \{e_n\}$ be a non-overlapping

Each interval e_n contains one or more intervals of some D_s , such that

$$\sum_m \hat{\eta}_{nm} = \hat{e}_n - \sigma_n \quad , \quad 0 \leq \sigma_n$$

where

$$\sigma = \sum \sigma_n$$

may be taken small at pleasure.

Now each η_{nm} has a subset \mathfrak{N}_{nm} of \mathfrak{N} entirely within it. Hence there exists an enclosure \mathfrak{E}_{nm} of \mathfrak{N}_{nm} , with such that

$$\frac{\alpha_{nm}}{\alpha_1} = \frac{\hat{\eta}_{nm}}{1} \quad , \quad \text{or } \alpha_{nm} = \alpha_1 \hat{\eta}_{nm}.$$

But $\mathfrak{E}_2 = \{\mathfrak{E}_{nm}\}$ is a non-overlapping enclosure of measure

$$\begin{aligned} \alpha_2 &= \alpha_1 \sum_{n,m} \hat{\eta}_{nm} = \alpha_1 \sum_n (\hat{e}_n - \sigma_n) \\ &= \alpha_1^2 - \sigma \alpha_1 = \alpha^2 + \epsilon'_2 \quad , \quad 0 \leq \epsilon'_2 \end{aligned}$$

if σ is taken sufficiently small.

Let \mathfrak{B}_2 denote the irrational points in $\mathfrak{E}_1 - \mathfrak{E}_2$, and \mathfrak{B}_2 has no point in common with \mathfrak{B}_1 . We have

$$\begin{aligned} \widehat{\mathfrak{B}}_2 &= \widehat{\mathfrak{E}}_1 - \widehat{\mathfrak{E}}_2 = \alpha_1 - \alpha_2 \\ &= \alpha + \epsilon'_1 - \alpha^2 - \epsilon'_2 \\ &> \alpha(1 - \alpha) - \epsilon_2. \end{aligned}$$

In this way we may continue. Thus \mathfrak{B} contains a component

$$\mathfrak{B}_1 + \mathfrak{B}_2 + \dots$$

whose measure is

$$\begin{aligned} &> (1 - \alpha) \{1 + \alpha + \alpha^2 + \dots\} - \sum \epsilon_n \\ &> 1 - \epsilon. \end{aligned}$$

For by 365, 2, there exists in the sets 1), m

$$\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3 \dots$$

each of whose measures $\mathfrak{C}_n > \alpha$. Let us consider these sets, viz.:

$$\mathfrak{C}_1, \mathfrak{C}_2, \dots, \mathfrak{C}_n.$$

The points common to any two of the sets \mathfrak{C}_i and \mathfrak{C}_j form a set \mathfrak{D}_{ij} by 358, 1. Hence the union $\mathfrak{C}_{1n} = \{\mathfrak{D}_{ij}\}$ by 359. The difference of one of the sets \mathfrak{C}_i from \mathfrak{C}_{1n} is a measurable set c_1 which contains no point of any of the remaining sets of 3). Moreover

$$\widehat{c}_1 > \alpha - \mathfrak{C}_{1n}.$$

In the same way we may reason with the other sets of 3). Thus \mathfrak{A} contains n measurable sets c_i which have a common point.

Hence

$$c = c_1 + \dots + c_n$$

is a measurable set and

$$\overline{\mathfrak{A}} \geq \widehat{c} > n(\alpha - \mathfrak{C}_{1n}).$$

The first and last members give

$$\mathfrak{C}_{1n} > \alpha - \frac{1}{n} \overline{\mathfrak{A}}.$$

Thus however small $\alpha > 0$ may be, there exist

$$\mathfrak{C}_{1, \mu} \left(1 - \frac{\epsilon}{2}\right) \alpha.$$

Let us now group the sets 2) in sets of μ .

such that the points of each set in 6) lie in at least 5), and hence in at least 2^2 of the sets 1), and measures are.

$$> \left(1 - \frac{\epsilon}{2}\right) \left(1 - \frac{\epsilon}{2^2}\right) \alpha > (1 - \epsilon) \alpha$$

In this way we may continue indefinitely. The union of all the points of \mathfrak{A} , common to at least 2^k of the sets 1), etc. Let \mathfrak{B}_2 be the union of the points of \mathfrak{A} common to at least 2^2 of the sets 1), etc. In this way we get the sequence

$$\mathfrak{B}_1 \supseteq \mathfrak{B}_2 \supseteq \dots$$

each of which contains a measurable set \mathfrak{B}_k with measure $> (1 - \epsilon) \alpha$.

We have now only to apply 25 and 364.

368. As corollaries of 367 we have:

1. Let $\mathfrak{Q}_1, \mathfrak{Q}_2 \dots$ be an infinite enumerable set of cubes whose union is limited. Let each $\bar{\mathfrak{Q}}_n > \alpha$. Then there exists a set of points \mathfrak{d} whose cardinal number is $\geq \alpha$ and such that $\mathfrak{d} \supseteq \mathfrak{Q}_n$.

2. (Arzelà.) Let $y_1, y_2 \dots \doteq \eta$. On each line y_n there is an enumerable set of intervals of length δ_n . Should the number of intervals ν_n on the lines y_n be finite, let $\nu_n \doteq \infty$. In any case, let $n = 1, 2, \dots$ and the projections of these intervals on the x -axis. Then there exists at least one point $x = \xi$ in \mathfrak{A} , such that ξ is cut by an infinity of these intervals.

Associate Sets

369. 1. Let $\epsilon_1 > \epsilon_2 > \epsilon_3 \dots \doteq 0$.

2. Each outer associated set \mathfrak{A}_e is measurable, and

$$\overline{\mathfrak{A}} = \widehat{\mathfrak{A}}_e = \lim_{n \rightarrow \infty} \mathfrak{E}_n.$$

For each \mathfrak{E}_n is measurable; hence \mathfrak{A}_e is measurable by 362, and

$$\begin{aligned} \mathfrak{A}_e &= \lim \mathfrak{E}_n \\ &= \lim (\overline{\mathfrak{A}} + \epsilon'_n), \quad 0 \leq \epsilon'_n < \epsilon_n \\ &= \overline{\mathfrak{A}}, \quad \text{as } \epsilon_n \rightarrow 0. \end{aligned}$$

370. 1. Let A be the complement of \mathfrak{A} with respect to S . Let Ω containing \mathfrak{A} . Let A_e be an outer associated set of A . Then

$$\mathfrak{A}_i = \Omega - A_e$$

called an inner associated set of \mathfrak{A} . Obviously

$$\mathfrak{A}_i \leq \mathfrak{A}.$$

2. The inner associated set \mathfrak{A}_i is measurable, and

$$\widehat{\mathfrak{A}}_i = \underline{\mathfrak{A}}.$$

For A_e is measurable by 369, 2. Hence $\mathfrak{A}_i = \Omega - A_e$ is measurable. But

$$\widehat{A}_e = \overline{A}$$

369, 2. Hence

$$\widehat{\mathfrak{A}}_i = \widehat{\Omega} - \widehat{A}_e = \widehat{\Omega} - \overline{A} = \underline{\mathfrak{A}}.$$

Separated Sets

371. Let $\mathfrak{A}, \mathfrak{B}$ be two limited point sets. If there exist measurable enclosures $\mathfrak{E}, \mathfrak{F}$ of $\mathfrak{A}, \mathfrak{B}$ such that $\mathfrak{D} = Dv(\mathfrak{E}, \mathfrak{F})$

372. For \mathfrak{A} , \mathfrak{B} to be separated, it is necessary and

$\mathfrak{D} = Dv(\mathfrak{A}_e, \mathfrak{B}_e)$
is a null set.

It is sufficient. For let

$$\mathfrak{C} = (\mathfrak{A}, \mathfrak{B}) \quad , \quad \mathfrak{A}_e = \mathfrak{D} + a, \quad \mathfrak{B}_e = \mathfrak{D} +$$

Then

$$\mathfrak{C} = (a, b, \mathfrak{D})$$

is a measurable enclosure of \mathfrak{C} , consisting of three cells. Of these only \mathfrak{D} contains points of both \mathfrak{A} and \mathfrak{B} . Under the hypothesis \mathfrak{D} is a null set. Hence \mathfrak{A} , \mathfrak{B} are separated.

It is necessary. For let \mathfrak{M} be a null distinct enclosure such that those of its cells \mathfrak{N} , containing points of both \mathfrak{A} and \mathfrak{B} form a null set. Let us superimpose \mathfrak{M} on the enclosure \mathfrak{C} , forming an enclosure \mathfrak{F} of \mathfrak{A} .

The cells of \mathfrak{F} arising from a contain no point of \mathfrak{B} ; the cells arising from b contain no point of \mathfrak{A} . On the other hand, the cells arising from \mathfrak{D} , split up into three cells

$$\mathfrak{D}_a \quad , \quad \mathfrak{D}_b \quad , \quad \mathfrak{D}_{ab}.$$

The first contains no point of \mathfrak{B} , the second no point of \mathfrak{A} , the cells of the last contain both points of \mathfrak{A} , \mathfrak{B} . As \mathfrak{D}_{ab} is a null set,

$$\widehat{\mathfrak{D}}_{a,b} = 0.$$

On the other hand,

$$\mathfrak{A}_e = a + \mathfrak{D} \supseteq \mathfrak{A};$$

hence

$$a + \mathfrak{D}_a + \mathfrak{D}_{ab} \supseteq \mathfrak{A}.$$

Thus

$$\widehat{a} + \widehat{\mathfrak{D}}_a \supseteq \widehat{\mathfrak{A}},$$

by 1). Also

$$\widehat{\mathfrak{A}}_e = \widehat{a} + \widehat{\mathfrak{D}} = \widehat{\mathfrak{A}} \quad \text{by 369,}$$

This with 2) gives

373. 1. If $\mathfrak{A}, \mathfrak{B}$ are separated, then $\mathfrak{D} = Dv(\mathfrak{A}, \mathfrak{B})$ is a null set by 372. But for $\mathfrak{D}_e = Dv(\mathfrak{A}_e, \mathfrak{B}_e)$ is a null set by 372. But

2. Let $\mathfrak{A}, \mathfrak{B}$ be the Van Vleck sets in 366. We have $\bar{\mathfrak{A}} = \bar{\mathfrak{B}} = 1$. Then by 369, 2, $\widehat{\mathfrak{A}}_e = \widehat{\mathfrak{B}}_e = 1$. The set \mathfrak{D} is not a null set. Hence by 372, $\mathfrak{A}, \mathfrak{B}$ are not separated. Hence the condition that \mathfrak{D} be a null set is necessary, but not sufficient.

374. 1. Let $\{\mathfrak{A}_n\}, \{\mathfrak{B}_n\}$ be separated divisions of \mathfrak{A} . Let $\mathfrak{C}_{ik} = Dv(\mathfrak{A}_i, \mathfrak{B}_k)$. Then $\{\mathfrak{C}_{ik}\}$ is a separated division of \mathfrak{A} .

We have to show there exists a null enclosure of \mathfrak{D} . Let \mathfrak{C}_{ik} be a cell of $\{\mathfrak{C}_{ik}\}$. Now \mathfrak{C}_{ik} lies in \mathfrak{A}_i and \mathfrak{B}_k ; also \mathfrak{C}_{mn} lies in \mathfrak{A}_m and \mathfrak{B}_n . By hypothesis there exists a null enclosure \mathfrak{E} of $\mathfrak{A}_i, \mathfrak{A}_m$ and of $\mathfrak{B}_k, \mathfrak{B}_n$. Then $\mathfrak{G} = Dv(\mathfrak{E}, \mathfrak{E})$ is a null enclosure of $\mathfrak{C}_{ik}, \mathfrak{C}_{mn}$ and of $\mathfrak{A}_i, \mathfrak{A}_m$ and of $\mathfrak{B}_k, \mathfrak{B}_n$. Thus those cells of \mathfrak{G} , containing points of both $\mathfrak{A}_i, \mathfrak{A}_m$ form a null set; and those cells of \mathfrak{G} , containing points of both $\mathfrak{B}_k, \mathfrak{B}_n$ also form a null set.

Let $G = \{g\}$ denote the cells of \mathfrak{G} that contain points of $\mathfrak{C}_{ik}, \mathfrak{C}_{mn}$. Then a cell g contains points of $\mathfrak{A}_i, \mathfrak{A}_m$ and of $\mathfrak{B}_k, \mathfrak{B}_n$. Thus in either case G is a null set. Hence G forms a separated division of \mathfrak{A} .

2. Let D be a separated division of \mathfrak{A} into two parts. Let E be another separated division of \mathfrak{A} into two parts. We have seen that $F = \{f_{ik}\}$ where $f_{ik} = Dv(d_i, e_k)$ is a separated division of \mathfrak{A} . We shall say that F is obtained by composing E on D or D on E , and write $F = D + E$.

3. Let E be a separated division of the separated set \mathfrak{A} , while D is a separated division of \mathfrak{A} . If d_i is a cell of D , and $d_{ik} = Dv(d_i, e_k)$, then

$$d_i = (d_{i1}, d_{i2}, \dots) + \delta_i.$$

For let \mathfrak{D} be a null enclosure of $\mathfrak{A}_m, \mathfrak{A}_n$. Let cells of \mathfrak{D} containing points of both $\mathfrak{A}_m, \mathfrak{A}_n$. Let cells of \mathfrak{D} containing points of \mathfrak{B} ; let $\mathfrak{E}_{a,b}$ denote containing points of both $\mathfrak{B}_m, \mathfrak{B}_n$. Then

$$\mathfrak{E}_{ab} \leq \mathfrak{D}_{ab}.$$

As \mathfrak{D}_{ab} is a null set, so is \mathfrak{E}_{ab} .

376. 1. Let $\mathfrak{A} = (\mathfrak{B}, \mathfrak{C})$ be a separated division of

$$\overline{\mathfrak{A}} = \overline{\mathfrak{B}} + \overline{\mathfrak{C}}.$$

For let $\epsilon_1 > \epsilon_2 > \dots \doteq 0$. There exist ϵ_n -measures of $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$; call them respectively A_n, B_n, C_n . Then $B_n + C_n$ is an ϵ_n -enclosure of $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ simultaneously.

Since $\mathfrak{B}, \mathfrak{C}$ are separated, there exist enclosures such that those cells of $D = B + C$ containing points of \mathfrak{B} and \mathfrak{C} form a null set. Let us now superpose D with an ϵ_n -enclosure $E_n = \{e_{ns}\}$ of $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ simultaneously. Let e_{bn} denote the cells of E_n containing points of \mathfrak{B} and \mathfrak{C} ; e_{cn} those cells containing only points of \mathfrak{C} ; and e_{bc} those cells containing only points of \mathfrak{B} . Then

$$\sum_s \widehat{e}_{ns} = \sum_s \widehat{e}_{bn} + \sum_s \widehat{e}_{cn} + \sum_s \widehat{e}_{bc}.$$

As $\sum e_{bc} = 0$, we see that as $n \doteq \infty$,

$$\sum \widehat{e}_{ns} \doteq \overline{\mathfrak{A}} \quad , \quad \sum \widehat{e}_{bn} \doteq \overline{\mathfrak{B}} \quad , \quad \sum \widehat{e}_{cn} \doteq \overline{\mathfrak{C}}.$$

Hence passing to the limit $n = \infty$, in 2) we get

2. Let $\mathfrak{A} = \{\mathfrak{B}_n\}$ be a separated division of limited

On the other hand, by 1

$$\overline{\overline{\mathfrak{A}}}_n = \overline{\overline{\mathfrak{B}}}_1 + \cdots + \overline{\overline{\mathfrak{B}}}_n = B_n,$$

the sum of the first n terms of the series 2). Thus

$$B_n \leq \overline{\overline{\mathfrak{A}}},$$

and hence B is convergent by 80, 4. Thus

$$B \leq \overline{\overline{\mathfrak{A}}}.$$

On the other hand, by 339,

$$B \geq \overline{\overline{\mathfrak{A}}}.$$

The last two relations give 1).

CHAPTER XII

LEBESGUE INTEGRALS

General Theory

377. In the foregoing chapters we have developed a theory of integration which rests on the notion of content. In this chapter we propose to develop a theory of integration due to Lebesgue which rests on the notion of measure. The presentation given differs considerably from that of Lebesgue. As the reader will see, the theory of Lebesgue integrals as here presented differs from that of the theory of ordinary integrals only in employing an infinite number of cells instead of a finite number.

378. In the following we shall suppose the field of integration \mathfrak{A} to be limited, as also the integrand f lies in \mathfrak{R}_m and for brevity we set $f(x) = f(x_1 \cdots x_m)$. Let us effect a separated division of \mathfrak{A} into cells $\delta_1, \delta_2, \dots$. If each cell δ_i lies in a cube of side d we shall say D is a *separated division of norm d* .

As before, let

$$M_i = \text{Max } f, \quad m_i = \text{Min } f, \quad \omega_i = \text{Osc } f = M_i - m_i \quad \text{in } \delta_i.$$

Then

$$\bar{S}_D = \sum M_i \bar{\delta}_i, \quad \underline{S}_D = \sum m_i \bar{\delta}_i,$$

the summation extending over all the cells of \mathfrak{A} , are called the *upper and lower sums of f over \mathfrak{A} with respect to D* .

Hence

$$\Sigma m \bar{\delta}_i \leq \Sigma m_i \bar{\delta}_i \leq \Sigma M_i \bar{\delta}_i \leq \Sigma M \bar{\delta}_i.$$

Thus

$$m \Sigma \bar{\delta}_i \leq \underline{S}_D \leq \bar{S}_D \leq M \Sigma \bar{\delta}_i.$$

But

$$\Sigma \bar{\delta}_i = \bar{\mathfrak{A}},$$

by 376, 2.

380. 1. Since f is limited in \mathfrak{A} ,

$$\text{Max } \underline{S}_D, \quad \text{Min } \bar{S}_D$$

with respect to the class of all separated divisions finite. We call them respectively the *lower and integrals* of f over the field \mathfrak{A} , and write

$$\underline{\int}_{\mathfrak{A}} f = \text{Max } \underline{S}_D \quad ; \quad \bar{\int}_{\mathfrak{A}} f = \text{Min } \bar{S}_D.$$

In order to distinguish these new integrals from we have slightly modified the old symbol \int to resemble script L , or \int , in honor of the author of these integrals.

If

$$\underline{\int}_{\mathfrak{A}} f = \bar{\int}_{\mathfrak{A}} f$$

we say f is *L-integrable* over \mathfrak{A} , and denote the common value by

$$\int_{\mathfrak{A}} f,$$

which we call the *L-integral*.

The integrals treated of in Vol. I we will call *Riemann integrals* in the sense of Riemann.

2. Let f be limited over the null set \mathfrak{A} . Then f is

For let $d_1, d_2 \dots$ be an unmixed metric or complete \mathfrak{A} of norm \bar{d} . Let each cell d_i be split up into the s $d_{i1}, d_{i2} \dots$

Then since d_i is complete or metric,

$$\bar{d}_i = \bar{\bar{d}}_i = \Sigma \bar{\bar{d}}_{i\kappa}.$$

Hence using the customary notation,

$$m_i \bar{\bar{d}}_{i\kappa} \leq m_{i\kappa} \bar{\bar{d}}_{i\kappa} \leq M_{i\kappa} \bar{\bar{d}}_{i\kappa} \leq M_i \bar{\bar{d}}_{i\kappa}.$$

Thus summing over κ ,

$$m_i \bar{\bar{d}}_i \leq \Sigma_{\kappa} m_{i\kappa} \bar{\bar{d}}_{i\kappa} \leq \Sigma_{\kappa} M_{i\kappa} \bar{\bar{d}}_{i\kappa} \leq M_i \bar{\bar{d}}_i.$$

Summing over i gives

$$\Sigma m_i \bar{\bar{d}}_i \leq \Sigma_{i\kappa} m_{i\kappa} \bar{\bar{d}}_{i\kappa} \leq \Sigma_{i\kappa} M_{i\kappa} \bar{\bar{d}}_{i\kappa} \leq \Sigma M_i \bar{\bar{d}}_i.$$

Thus by definition,

$$\Sigma m_i \bar{\bar{d}}_i \leq \int_{\mathfrak{A}} f \leq \Sigma M_i \bar{\bar{d}}_i.$$

Letting now $\bar{d} \doteq 0$, we get 1).

2. Let \mathfrak{A} be metric or complete. If f is R -integrable and L -integrable and

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}} f.$$

3. In case that \mathfrak{A} is not metric or complete, the result may not hold.

Example 1. Let \mathfrak{A} denote the rational points in $(0, 1)$.

Let

$$f = 1, \text{ for } x = \frac{m}{n}, n \text{ even}$$

Example 2. Let $f = 1$ at the rational points \mathfrak{A} in

$$\int_{\mathfrak{A}} f = 1 \quad , \quad \int_{\mathfrak{A}} f = 0 \quad , \quad \text{and} \quad \int_{\mathfrak{A}} f < \int_{\mathfrak{A}} f$$

Let $g = -1$ in \mathfrak{A} . Then

$$\int_{\mathfrak{A}} g = 0 \quad , \quad \int_{\mathfrak{A}} g = -1 \quad , \quad \text{and} \quad \int_{\mathfrak{A}} g < \int_{\mathfrak{A}} g$$

Thus in 3) the L -integral is less than the R -integral
4) it is greater.

Example 3. Let $f = 1$ at the irrational points in \mathfrak{A} .
Then

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}} f,$$

although \mathfrak{A} is neither metric nor complete.

382. Let D, Δ be separated divisions of \mathfrak{A} . Let

$$E = D + \Delta = \{e_i\}.$$

Then

$$\bar{S}_E \leq \bar{S}_D, \bar{S}_\Delta$$

$$\underline{S}_E \geq \underline{S}_D, \underline{S}_\Delta.$$

For any cell d_i of D splits up into d_{i1}, d_{i2}, \dots on Δ , and

$$\bar{d}_i = \sum_{\kappa} \bar{d}_{i\kappa}.$$

But

$$M_{i\kappa} \bar{d}_{i\kappa} \leq M_i \bar{d}_i,$$

and

$$m_{i\kappa} \bar{d}_{i\kappa} \geq m_i \bar{d}_i.$$

Thus

$$\bar{S}_E \leq \bar{S}_D \quad , \quad \underline{S}_E \geq \underline{S}_D.$$

383. 1. *Extremal Sequences.* There exists a sequence

For let $\epsilon_1 > \epsilon_2 > \dots \doteq 0$. For each ϵ_n , there exists E_n such that

$$0 < \bar{S}_{E_n} - \int_{\mathfrak{A}} < \epsilon_n.$$

Let

$$E_2 + D_1 = D_2, \quad E_3 + D_2 = D_3, \dots$$

and for uniformity set $E_1 = D_1$. Then by 382,

$$\bar{S}_{D_{n+1}} \leq \bar{S}_{D_n}, \quad \bar{S}_{D_n} \leq \bar{S}_{E_n}.$$

Hence

$$0 \leq \bar{S}_{D_n} - \int_{\mathfrak{A}} < \epsilon_n.$$

Letting $n \doteq \infty$ we get 2).

Thus there exists a sequence $\{D'_n\}$ of the type 1) sequence $\{D''_n\}$ of the same type for 3). Let now D be any division of \mathfrak{A} . Obviously 2), 3) hold simultaneously for the sequence $\{D'_n\}$.

2. The sequence 1) is called an *extremal sequence*.

3. Let $\{D_n\}$ be an extremal sequence, and E any division of \mathfrak{A} . Let $E_n = D_n + E$. Then E_1, E_2, \dots is an extremal sequence also.

384. Let f be L -integrable in \mathfrak{A} . Then for any extremal sequence $\{D_n\}$,

$$\int_{\mathfrak{A}} f = \lim_{n \rightarrow \infty} \sum f(\xi_i) \bar{d}_i,$$

where d_i are the cells of D_n , and ξ_i any point of \mathfrak{A} in d_i .

For

$$m_i \leq f(\xi_i) \leq M_i.$$

Hence

$$S_n \leq \sum f(\xi_i) \bar{d}_i \leq \bar{S}_n$$

2. Let $F = \text{Max } |f|$ in \mathfrak{A} , then

$$\left| \int_{\mathfrak{A}} f \right| \leq F \overline{\mathfrak{A}}.$$

This follows from 1.

386. In order that \bar{f} be L -integrable in \mathfrak{A} , it is necessary that, for each extremal sequence $\{D_n\}$,

$$\lim_{n \rightarrow \infty} \Omega_{D_n} f = 0;$$

and it is sufficient if there exists a sequence of superimposed separated divisions $\{E_n\}$, such that

$$\lim_{n \rightarrow \infty} \Omega_{E_n} f = 0.$$

It is necessary. For

$$\int_{\mathfrak{A}} = \lim \underline{S}_{D_n} \quad , \quad \overline{\int}_{\mathfrak{A}} = \lim \overline{S}_{D_n}.$$

As f is L -integrable,

$$0 = \overline{\int}_{\mathfrak{A}} - \underline{\int}_{\mathfrak{A}} = \lim (\overline{S}_{D_n} - \underline{S}_{D_n}) = \lim \Omega_{D_n} f.$$

It is sufficient. For

$$\underline{S}_{E_n} \leq \underline{\int}_{\mathfrak{A}} \leq \overline{S}_{E_n}.$$

Both $\{\underline{S}_{E_n}\}$, $\{\overline{S}_{E_n}\}$ are limited monotone sequences. Their limits therefore exist. Hence

$$0 = \lim \Omega_{E_n} = \lim \overline{S}_{E_n} - \lim \underline{S}_{E_n}.$$

Thus

$$\underline{\int}_{\mathfrak{A}} = \overline{\int}_{\mathfrak{A}}.$$

387. In order that f be L -integrable, it is necessary and sufficient that for each $\epsilon > 0$, there exists a separated division D of \mathfrak{A} , for which

$$\Omega_D f < \epsilon. \quad (1)$$

It is necessary. For by 386, there exists an extremal sequence $\{D_n\}$, such that

$$0 \leq \Omega_{D_n} f < \epsilon \quad , \quad \text{for any } n \geq \text{some } m.$$

Thus we may take D_m for D .

It is *sufficient*. For let $\epsilon_1 > \epsilon_2 > \dots \doteq 0$. Let $\{D_n\}$ be an extremal sequence for which

$$0 \leq \Omega_{D_n} f < \epsilon_n.$$

Let $\Delta_1 = D_1$, $\Delta_2 = \Delta_1 + D_2$, $\Delta_3 = \Delta_2 + D_3 \dots$. Then $\{\Delta_n\}$ is a set of superimposed separated divisions, and obviously

$$\Omega_{\Delta_n} f < \epsilon_n \doteq 0.$$

Hence f is L -integrable by 386.

388. In order that f be L -integrable, it is necessary and sufficient that, for each pair of positive numbers ω, σ there exists a separated division D of \mathfrak{A} , such that if η_1, η_2, \dots are those cells in which $\text{Osc } f > \omega$, then

$$\Sigma \bar{\eta}_i < \sigma. \quad (1)$$

It is *necessary*. For by 387 there exists a separated division $D = \{\delta_i\}$ for which

$$\Omega_{D_n} f = \Sigma \omega_i \delta_i < \omega \sigma. \quad (2)$$

If $\theta_1, \theta_2 \dots$ denote the cells of D in which $\text{Osc } f \leq \omega$,

$$\Omega_{D_n} f = \Sigma \omega_i \eta_i + \Sigma \omega_i \bar{\theta}_i \geq \omega \Sigma \bar{\eta}_i. \quad (3)$$

This in 2) gives 1).

It is *sufficient*. For taking $\epsilon > 0$ small at pleasure, let us then take

$$\sigma = \frac{\epsilon}{2\Omega}, \quad \omega = \frac{\epsilon}{2\mathfrak{A}}, \quad (4)$$

where $\Omega = \text{Osc } f$ in \mathfrak{A} .

From 1), 3), and 4) we have, since $\omega_i \leq \Omega$,

$$\Omega_{D_n} f \leq \Sigma \Omega \bar{\eta}_i + \Sigma \omega_i \bar{\theta}_i < \sigma \Omega + \Sigma \omega \bar{\theta}_i < \sigma \Omega + \omega \mathfrak{A} = \epsilon.$$

We now apply 387.

389. 1. If f is L -integrable in \mathfrak{A} , it is in $\mathfrak{B} < \mathfrak{A}$.

For let $\{D_n\}$ be an extremal sequence of f relative to \mathfrak{A} . Then by 386,

$$\Omega_{D_n} f \doteq 0. \quad (1)$$

But the sequence $\{D_n\}$ defines a sequence of superposed separated divisions of \mathfrak{B} , which we denote by $\{E_n\}$. Obviously

$$\Omega_{E_n} f \leq \Omega_{D_n} f.$$

Hence by 1),

$$\Omega_{E_n} f \doteq 0,$$

and f is L -integrable in \mathfrak{B} by 386.

2. If f is L -integrable in \mathfrak{A} , so is $|f|$.

The proof is analogous to I, 507, using an extremal sequence for f .

390. 1. Let $\{\mathfrak{A}_n\}$ be a separated division of \mathfrak{A} into a finite or infinite number of subsets. Let f be limited in \mathfrak{A} . Then

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}_1} f + \int_{\mathfrak{A}_2} f + \dots \quad (1)$$

For let us 1° suppose that the subsets $\mathfrak{A}_1 \dots \mathfrak{A}_r$ are finite in number. Let $\{D_n\}$ be an extremal sequence of f relative to \mathfrak{A} , and $\{D_{mn}\}$ an extremal sequence relative to \mathfrak{A}_m . Let

$$E_n = D_n + D_{1n} + \dots + D_{rn}.$$

Then $\{E_n\}$ is an extremal sequence of f relative to \mathfrak{A} , and also relative to each \mathfrak{A}_m .

Now

$$\bar{S}_{\mathfrak{A}, E_n} = \bar{S}_{\mathfrak{A}_1, E_n} + \dots + \bar{S}_{\mathfrak{A}_r, E_n}.$$

Letting $n \doteq \infty$, we get 1), for this case.

Let now r be infinite. We have.

$$\bar{\mathfrak{A}} = \sum_1^{\infty} \bar{\mathfrak{A}}_m. \quad (2)$$

Let

$$\mathfrak{B}_n = (\mathfrak{A}_1 \dots \mathfrak{A}_n) \quad , \quad \mathfrak{C}_n = \mathfrak{A} - \mathfrak{B}_n.$$

Then $\mathfrak{B}_n, \mathfrak{C}_n$ form a separated division of \mathfrak{A} , and

$$\bar{\mathfrak{A}} = \bar{\mathfrak{B}}_n + \bar{\mathfrak{C}}_n.$$

If ν is taken large enough, 2) shows that

$$\bar{\mathfrak{C}}_n < \frac{\epsilon}{M} \quad , \quad n \geq \nu \quad , \quad M = \text{Max } |f| \quad \text{in } \mathfrak{A}.$$

Thus by case 1°,

$$\begin{aligned}\int_{\mathfrak{A}} f &= \int_{\mathfrak{B}_n} f + \int_{\mathfrak{C}_n} f \\ &= \int_{\mathfrak{A}_1} f + \dots + \int_{\mathfrak{A}_n} f + \epsilon',\end{aligned}\tag{3}$$

where by 385, 2

$$|\epsilon'| \leq M\bar{\mathfrak{C}}_n < \epsilon, \quad n \geq \nu.$$

Thus 1) follows from 3) in this case.

2. Let $\{\mathfrak{A}_n\}$ be a separated division of \mathfrak{A} . Then

$$\int_{\mathfrak{A}} f = \Sigma \int_{\mathfrak{A}_n} f,$$

if f is L -integrable in \mathfrak{A} , or if it is in each \mathfrak{A}_n , and limited in \mathfrak{A} .

391. 1. Let $f = g$ in \mathfrak{A} except at the points of a null set \mathfrak{N} .

Then

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}} g.\tag{1}$$

For let

$$\mathfrak{A} = \mathfrak{B} + \mathfrak{N}.$$

Then

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}} f + \int_{\mathfrak{N}} f = \int_{\mathfrak{B}} f.\tag{2}$$

Similarly

$$\int_{\mathfrak{A}} g = \int_{\mathfrak{B}} g.\tag{3}$$

But $f = g$ in \mathfrak{B} . Thus 2), 3) give 1).

$$392. \quad 1. \quad \text{If } c > 0; \quad \int_{\mathfrak{A}} cf = c \int_{\mathfrak{A}} f.$$

$$\text{If } c < 0; \quad \int_{\mathfrak{A}} cf = c \int_{\mathfrak{A}} f, \quad \int_{\mathfrak{A}} cf = c \int_{\mathfrak{A}} f.$$

The proof is similar to 3, 8, using extremal sequences.

2. If f is L -integrable in \mathfrak{A} , so is cf , and

$$\int_{\mathfrak{A}} cf = c \int_{\mathfrak{A}} f,$$

where c is a constant.

393. 1. Let $F(x) = f_1(x) + \dots + f_n(x)$, each f_m being limited in \mathfrak{A} . Then

$$\sum_{1 \leq m} \int_{\mathfrak{A}} f_m \leq \int_{\mathfrak{A}} F \leq \sum_{1 \leq m} \bar{\int}_{\mathfrak{A}} f_m. \quad (1)$$

For let $\{D_n\}$ be an extremal sequence common to F, f_1, \dots, f_n . In each cell

$$d_{n1}, \quad d_{n2}, \dots$$

of D_n we have

$$\Sigma \text{Min } f_m \leq \text{Min } F \leq \text{Max } F \leq \Sigma \text{Max } f_m.$$

Multiplying by \bar{d}_{ns} , summing over s and then letting $n \doteq \infty$, gives 1).

2. If $f_1(x), \dots, f_n(x)$ are each L -integrable in \mathfrak{A} , so is

$$F = c_1 f_1 + \dots + c_n f_n,$$

and

$$\int_{\mathfrak{A}} F = c_1 \int_{\mathfrak{A}} f_1 + \dots + c_n \int_{\mathfrak{A}} f_n.$$

394. 1.
$$\int_{\mathfrak{A}} (f + g) \leq \int_{\mathfrak{A}} f + \int_{\mathfrak{A}} g \leq \bar{\int}_{\mathfrak{A}} (f + g).$$

For using the notation of 393,

$$\text{Min } (f + g) \leq \text{Min } f + \text{Max } g \leq \text{Max } (f + g)$$

in each cell d_{ns} of D_n .

2. If g is L -integrable in \mathfrak{A} ,

$$\bar{\int}_{\mathfrak{A}} (f + g) = \bar{\int}_{\mathfrak{A}} f + \int_{\mathfrak{A}} g.$$

Reasoning similar to 3, 4, using extremal sequences.

3.
$$\int_{\mathfrak{A}} (f - g) \leq \int_{\mathfrak{A}} f - \int_{\mathfrak{A}} g.$$

$$\bar{\int}_{\mathfrak{A}} f - \bar{\int}_{\mathfrak{A}} g \leq \bar{\int}_{\mathfrak{A}} (f - g).$$

For

$$\int_{\mathfrak{A}}^{\cdot} (f - g) \leq \int_{\mathfrak{A}}^{\cdot} f + \int_{\mathfrak{A}}^{\cdot} (-g) \leq \int_{\mathfrak{A}}^{\cdot} f - \int_{\mathfrak{A}}^{\cdot} g;$$

etc.

4. If f, g are L -integrable in \mathfrak{A} , so is $f - g$, and

$$\int_{\mathfrak{A}}^{\cdot} (f - g) = \int_{\mathfrak{A}}^{\cdot} f - \int_{\mathfrak{A}}^{\cdot} g.$$

395. If f, g are L -integrable in \mathfrak{A} , so is $f \cdot g$.

Also their quotient f/g is L -integrable provided it is limited in \mathfrak{A} .

The proof of the first part of the theorem is analogous to I, 505, using extremal sequences common to both f and g . The proof of the second half is obvious and is left to the reader.

396. 1. Let f, g be limited in \mathfrak{A} , and $f \leq g$, except possibly in a null set \mathfrak{N} . Then

$$\int_{\mathfrak{A}}^{\cdot} f \leq \int_{\mathfrak{A}}^{\cdot} g. \quad (1)$$

Let us suppose first that $f \leq g$ everywhere in \mathfrak{A} .

Let $\{D_n\}$ be an extremal sequence common to both f and g . Then

$$\underline{S}_{D_n} f \leq \underline{S}_{D_n} g.$$

Letting $n \doteq \infty$, we get (1).

We consider now the *general case*. Let $\mathfrak{A} = \mathfrak{B} + \mathfrak{N}$. Then

$$\int_{\mathfrak{A}}^{\cdot} f = \int_{\mathfrak{B}}^{\cdot} f, \quad \int_{\mathfrak{A}}^{\cdot} g = \int_{\mathfrak{B}}^{\cdot} g,$$

since

$$\int_{\mathfrak{N}}^{\cdot} f = \int_{\mathfrak{N}}^{\cdot} g = 0.$$

But in \mathfrak{B} , $f \leq g$ without exception. We may therefore use the result of case 1°.

2. Let $f \geq 0$ in \mathfrak{A} . Then

$$\text{Min } g \cdot \int_{\mathfrak{A}}^{\cdot} f \leq \int_{\mathfrak{A}}^{\cdot} f \cdot g \leq \text{Max } g \cdot \int_{\mathfrak{A}}^{\cdot} f.$$

For

$$f \cdot \text{Min } g \leq fg \leq f \cdot \text{Max } g.$$

397. The relations of § also hold for L -integrals, viz.:

$$\left| \int_{\mathfrak{A}} f \right| \leq \int_{\mathfrak{A}} |f|. \quad (1)$$

$$\int_{\mathfrak{A}} f \leq \left| \int_{\mathfrak{A}} f \right|. \quad (2)$$

$$-\int_{\mathfrak{A}} |f| \leq \int_{\mathfrak{A}} f < \int_{\mathfrak{A}} |f|. \quad (3)$$

$$-\int_{\mathfrak{A}} |f| \leq \int_{\mathfrak{A}} f \leq \int_{\mathfrak{A}} |f|. \quad (4)$$

The proof is analogous to that employed for the R -integrals, using extremal sequences.

398. Let $\mathfrak{A} = (\mathfrak{B}_u, \mathfrak{C}_u)$ be a separated division for each $u \doteq 0$. Let $\overline{\mathfrak{C}}_u \doteq 0$. Then

$$\lim_{u \rightarrow 0} \int_{\mathfrak{B}_u} f = \int_{\mathfrak{A}} f.$$

For by 390, 1,

$$\int_{\mathfrak{A}} = \int_{\mathfrak{B}_u} + \int_{\mathfrak{C}_u}.$$

But by 385, 2, the last integral $\doteq 0$, since $\overline{\mathfrak{C}}_u \doteq 0$, and since f is limited.

399. Let f be limited and continuous in \mathfrak{A} , except possibly at the points of a null set \mathfrak{N} . Then f is L -integrable in \mathfrak{A} .

Let us first take $\mathfrak{N} = 0$. Then f is continuous in \mathfrak{A} . Let \mathfrak{A} lie in a standard cube Ω . If $\text{Osc } f$ is not $< \epsilon$ in \mathfrak{A} , let us divide Ω into 2^n cubes. If in one of these cubes

$$\text{Osc } f < \epsilon, \quad (1)$$

let us call it a *black cube*. A cube in which 1) does not hold we will call *white*. Each white cube we now divide in 2^n cubes. These we call black or white according as 1) holds for them or does not. In this way we continue until we reach a stage where all cubes are black, or if not we continue indefinitely. In the latter case, we get an infinite enumerable set of cubes

$$q_1, q_2, q_3 \dots \quad (2)$$

Each point a of \mathfrak{A} lies in at least one cube 2). For since f is continuous at $x = a$,

$$|f(x) - f(a)| < \epsilon/2, \quad x \text{ in } V_\delta(a).$$

Thus when the process of division has been carried so far that the diagonals of the corresponding cubes are $< \delta$, the inequality 1) holds for a cube containing a . This cube is a black cube.

Thus, in either case, each point of \mathfrak{A} lies in a black cube.

Now the cubes 2) effect a separated division D of \mathfrak{A} , and in each of its cells 1) holds. Hence f is L -integrable in \mathfrak{A} .

Let us now suppose $\mathfrak{A} > 0$. We set

$$\mathfrak{A} = \mathfrak{C} + \mathfrak{N}.$$

Then f is L -integrable in \mathfrak{C} by case 1°. It is L -integrable in \mathfrak{N} by 380, 2. Then it is L -integrable in \mathfrak{A} by 390, 1.

2. If f is L -integrable in \mathfrak{A} , we cannot say that the points of discontinuity of f form a null set.

Example. Let $f = 1$ at the irrational points \mathfrak{I} , in $\mathfrak{A} = (0, 1)$;
 $= 0$ at the other points \mathfrak{N} , in \mathfrak{A} .

Then each point of \mathfrak{A} is a point of discontinuity. But here

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{N}} + \int_{\mathfrak{I}} = \int_{\mathfrak{I}} = 1,$$

since \mathfrak{N} is a null set. Thus f is L -integrable.

400. If $f(x_1 \dots x_m)$ has limited variation in \mathfrak{A} , it is L -integrable.

For let D be a cubical division of space of norm d . Then by I, 709, there exists a fixed number V , such that

$$\Sigma \omega_i d^{m-1} \leq V$$

for any D . Let ω, σ be any pair of positive numbers. We take d such that

$$d < \frac{\sigma \omega}{V}. \quad (1)$$

Let d'_i denote those cells in which $\text{Osc } f \geq \omega$, and let the number of these cells be ν . Let η_i denote the points of \mathfrak{A} in d'_i . Then

$$\nu \omega d^{m-1} \leq \Sigma \omega_i d^{m-1} \leq V.$$

Hence

$$v \leq \frac{V}{\omega d^{m-1}}. \quad (2)$$

Thus

$$\begin{aligned} \Sigma \bar{\eta}_i &\leq v d^m \leq \frac{V d^m}{\omega d^{m-1}}, \quad \text{by 2),} \\ &\leq \frac{V d}{\omega} < \sigma, \quad \text{by 1).} \end{aligned}$$

Hence f is L -integrable by 388.

401. Let

$$\begin{aligned} \phi &= f, \text{ in } \mathfrak{A} < \mathfrak{B}; \\ &= 0, \text{ in } A = \mathfrak{B} - \mathfrak{A}. \end{aligned}$$

Then

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}} \phi, \quad (1)$$

if 1°, ϕ is L -integrable in \mathfrak{B} ; or 2°, f is L -integrable in \mathfrak{A} , and \mathfrak{A} , A are separated parts of \mathfrak{B} .

On the 1° hypothesis let $\{\mathfrak{E}_s\}$ be an extremal sequence of ϕ . Let the cells of \mathfrak{E}_s be e_1, e_2, \dots . They effect a separated division of \mathfrak{A} into cells d_1, d_2, \dots . Let m_i, M_i be the extremes of f in d_i and n_i, N_i the extremes of ϕ in e_i . Then for those cells containing at least a point of \mathfrak{A} ,

$$n_i \bar{e}_i \leq m_i \bar{d}_i \leq M_i \bar{d}_i \leq N_i \bar{e}_i, \quad (2)$$

is obviously true when $e_i = d_i$. Let $d_i < e_i$. If $m_i \leq 0$,

$$n_i \bar{e}_i \leq m_i \bar{d}_i, \quad \text{since } m_i = n_i. \quad (3)$$

If $m_i > 0$, $n_i = 0$, and 3) holds.

$$\text{If } M_i \leq 0, \quad M_i \bar{d}_i \leq N_i \bar{e}_i, \quad \text{since } N_i = 0. \quad (4)$$

If $M_i > 0$, 4) still holds, since $M_i = N_i$.

Thus 2) holds in all these cases. Summing 2) gives

$$\sum_{\mathfrak{B}} n_i \bar{e}_i \leq \int_{\mathfrak{A}} f \leq \sum_{\mathfrak{B}} N_i e_i$$

for the division \mathfrak{E}_s , since in a cell e of \mathfrak{E} , containing no point of \mathfrak{A} , $\phi = 0$. Letting $s \rightarrow \infty$, we get 1), since the end members

$$\doteq \int_{\mathfrak{B}} \phi.$$

On the 2° hypothesis,

$$\int_{\mathfrak{B}} \phi = \int_{\mathfrak{A}} \phi + \int_A \phi = \int_{\mathfrak{A}} \phi = \int_{\mathfrak{A}} f,$$

since ϕ being $= 0$ in A , is L -integrable, and we can apply 390.

402. 1. If

$$\int_{\mathfrak{A}} f = 0,$$

we call f a null function in \mathfrak{A} .

2. If $f \geq 0$ is a null function in \mathfrak{A} , the points \mathfrak{P} where $f > 0$ form a null set.

For let $\mathfrak{A} = \mathfrak{B} + \mathfrak{P}$, so that $f = 0$ in \mathfrak{B} .

By 401,

$$0 = \int_{\mathfrak{A}} f = \int_{\mathfrak{B}} f. \quad (1)$$

Let $\epsilon_1 > \epsilon_2 > \dots \doteq 0$. Let \mathfrak{P}_n denote the points of \mathfrak{P} where $f \geq \epsilon_n$. Then

$$\int_{\mathfrak{P}} \geq \int_{\mathfrak{P}_n} = 0, \quad \text{by 1).}$$

Each \mathfrak{P}_n is a null set. For

$$\int_{\mathfrak{P}_n} \geq \epsilon_n \overline{\mathfrak{P}_n} = 0.$$

Hence $\overline{\mathfrak{P}_n} = 0$.

Then

$$\mathfrak{P} = \{\mathfrak{P}_n\} = Q_1 + Q_2 + \dots$$

where $Q_1 = \mathfrak{P}_1$, $Q_2 = \mathfrak{P}_2 - \mathfrak{P}_1$, $Q_3 = \mathfrak{P}_3 - Q_2 \dots$

As each Q_n is a null set, \mathfrak{P} is a null set.

Integrand Sets

403. Let \mathfrak{A} be a limited point set lying in an m -way space \mathfrak{R}_m . Let $f(x_1 \dots x_m)$ be a limited function defined over \mathfrak{A} . Any point of \mathfrak{A} may be represented by

$$a = (a_1 \dots a_m).$$

ALL RIGHTS RESERVED
 BY THE AUTHOR
 1911

The point

$$x = (a_1 \dots a_m x_{m+1})$$

lies in an $m + 1$ way space \mathfrak{R}_{m+1} . The set of points $\{x\}$ in which x_{m+1} ranges from $-\infty$ to $+\infty$ is called an *ordinate through* a . If x_{m+1} is restricted by

$$0 \leq x_{m+1} \leq l,$$

we shall call the ordinate a *positive ordinate of length* l ; if it is restricted by

$$-l \leq x_{m+1} \leq 0,$$

it is a *negative ordinate*. The set of ordinates through all the points a of \mathfrak{A} , each having a length $= f(a)$, and taken positively or negatively, as $f(a)$ is ≥ 0 , form a point set \mathfrak{F} in \mathfrak{R}_{m+1} which we call an *integrant set*. The points of \mathfrak{F} for which x_{m+1} has a fixed value $x_{m+1} = c$ form a *section of* \mathfrak{F} , and is denoted by $\mathfrak{F}(c)$ or by \mathfrak{F}_c .

404. Let $\mathfrak{A} = \{a\}$ be a limited point set in \mathfrak{R}_m . Through each point a , let us erect a positive ordinate of constant length l , getting a set \mathfrak{D} , in \mathfrak{R}_{m+1} . Then

$$\bar{\mathfrak{D}} = l\bar{\mathfrak{A}}. \quad (1)$$

For let $\mathfrak{E}_1 > \mathfrak{E}_2 > \dots$ form a standard sequence of enclosures of \mathfrak{D} , such that

$$\bar{\mathfrak{E}}_n \doteq \bar{\mathfrak{D}}. \quad (2)$$

Let us project each section of \mathfrak{E}_n corresponding to a given value of x_{m+1} on \mathfrak{R}_m , and let \mathfrak{A}_n be their divisor. Then $\mathfrak{A}_n \geq \mathfrak{A}$. Thus

$$\bar{\mathfrak{D}} \leq \bar{\mathfrak{A}}_n l \leq \bar{\mathfrak{A}}_n l \leq \bar{\mathfrak{E}}_n.$$

Letting $n \doteq \infty$, and using 2), we get

$$\bar{\mathfrak{D}} = \bar{\mathfrak{A}} \cdot l.$$

To prove the rest of 1), let O be the complement of \mathfrak{D} with respect to some standard cube \mathfrak{Q} in \mathfrak{R}_{m+1} , of base Q in \mathfrak{R}_m .

Then, as just shown,

$$\bar{O} = l\bar{A}, \quad \text{where } A = Q - \mathfrak{A}.$$

Hence

$$\begin{aligned} \underline{\mathfrak{D}} &= \underline{\mathfrak{Q}} - \bar{O} = \underline{\mathfrak{Q}}l - l\bar{A} \\ &= l\{\underline{\mathfrak{Q}} - (\bar{Q} - \underline{\mathfrak{A}})\} \\ &= l\underline{\mathfrak{A}}. \end{aligned}$$

405. Let $f \geq 0$ be L -integrable in \mathfrak{A} . Then

$$\int_{\mathfrak{A}} f = \bar{\mathfrak{J}}, \quad (1)$$

where \mathfrak{J} is the integrand set corresponding to f .

For let $\{\delta_i\}$ be a separated division D of \mathfrak{A} . On each cell δ_i , erect a cylinder \mathfrak{C}_i of height $M_i = \text{Max } f$ in δ_i . Then by 404,

$$\bar{\mathfrak{C}}_i = \bar{\delta}_i M_i.$$

Let $\mathfrak{C} = \{\mathfrak{C}_i\}$; the \mathfrak{C}_i are separated. Hence, $\epsilon > 0$ being small at pleasure,

$$\bar{\mathfrak{J}} \leq \bar{\mathfrak{C}} = \Sigma \bar{\mathfrak{C}}_i = \Sigma \bar{\delta}_i M_i < \int_{\mathfrak{A}} f + \epsilon,$$

for a properly chosen D . Thus

$$\bar{\mathfrak{J}} \leq \int_{\mathfrak{A}} f. \quad (2)$$

Similarly we find

$$\int_{\mathfrak{A}} f \leq \bar{\mathfrak{J}}. \quad (3)$$

From 2), 3) follows 1).

406. Let $f \geq 0$ be L -integrable over the measurable field \mathfrak{A} . Then the corresponding integrand set \mathfrak{J} is measurable, and

$$\hat{\mathfrak{J}} = \int_{\mathfrak{A}} f. \quad (1)$$

For by 2) in 405,

$$\bar{\mathfrak{J}} \leq \int_{\mathfrak{A}} f.$$

Using the notation of 405, let c_n be a cylinder erected on δ_n of height $m_n = \text{Min } f$ in δ_n . Let $\mathfrak{c} = \{c_n\}$. Then $\mathfrak{c} \leq \mathfrak{J}$, and hence

$$\hat{\mathfrak{c}} \leq \hat{\mathfrak{J}}. \quad (2)$$

But \mathfrak{A} being measurable, each c_n is measurable, by 404. Hence \mathfrak{c} is by 359. Thus 2) gives

$$\hat{\mathfrak{c}} \leq \hat{\mathfrak{J}}. \quad (3)$$

Now for a properly chosen D ,

$$-\epsilon + \int_{\mathfrak{A}} f < \Sigma m_i \bar{\delta}_i = \hat{\mathfrak{c}}.$$

Hence

$$\int_{\mathfrak{A}} \leq \bar{c}, \quad (4)$$

as ϵ is arbitrarily small. From 2), 3), 4)

$$\int_{\mathfrak{A}} f \leq \underline{\mathfrak{F}} \leq \bar{\mathfrak{F}} \leq \int_{\mathfrak{A}} f,$$

from which follows 1).

Measurable Functions

407. Let $f(x_1 \dots x_m)$ be limited in the limited measurable set \mathfrak{A} . Let $\mathfrak{A}_{\lambda\mu}$ denote the points of \mathfrak{A} at which

$$\lambda \leq f < \mu.$$

If each $\mathfrak{A}_{\lambda\mu}$ is measurable, we say f is measurable in \mathfrak{A} .

We should bear in mind that when f is measurable in \mathfrak{A} , necessarily \mathfrak{A} itself is measurable, by hypothesis.

408. 1. If f is measurable in \mathfrak{A} , the points \mathfrak{C} of \mathfrak{A} , at which $f = C$, form a measurable set.

For let \mathfrak{A}_n denote the points where

$$- \epsilon_n + C \leq f < C + \epsilon_n,$$

where

$$\epsilon_1 > \epsilon_2 > \dots \doteq 0.$$

Then by hypothesis, \mathfrak{A}_n is measurable. But $\mathfrak{C} = Dv\{\mathfrak{A}_n\}$. Hence \mathfrak{C} is measurable by 361.

2. If f is measurable in \mathfrak{A} , the set of points where

$$\lambda \leq f \leq \mu$$

is measurable, and conversely.

Follows from 1, and 407.

3. If the points \mathfrak{A}_λ in \mathfrak{A} where $f \geq \lambda$ form a measurable set for each λ , f is measurable in \mathfrak{A} .

For $\mathfrak{A}_{\lambda\mu}$ having the same meaning as in 407,

$$\mathfrak{A}_{\lambda\mu} = \mathfrak{A}_\lambda - \mathfrak{A}_\mu.$$

Each set on the right being measurable, so is $\mathfrak{A}_{\lambda\mu}$.

409. 1. If f is measurable in \mathfrak{A} , it is L -integrable.

For setting $m = \text{Min } f$, $M = \text{Max } f$ in \mathfrak{A} , let us effect a division D of the interval $\mathfrak{F} = (m, M)$ of norm d , by interpolating a finite number of points

$$m_1 < m_2 < m_3 < \dots$$

Let us call the resulting segments, as well as their lengths,

$$d_1, d_2, d_3 \dots$$

Let \mathfrak{A}_i denote the points of \mathfrak{A} in which

$$m_{i-1} \leq f < m_i, \quad i = 1, 2, \dots; \quad m_0 = m.$$

We now form the sums

$$s_D = \sum m_{i-1} \mathfrak{A}_i, \quad \bar{s}_D = \sum m_i \mathfrak{A}_i.$$

Obviously

$$s_D \leq \int_{\mathfrak{A}} f \leq \bar{s}_D. \quad (1)$$

$$\begin{aligned} \text{But } \bar{s}_D - s_D &= m_1 \mathfrak{A}_1 + m_2 \mathfrak{A}_2 + \dots - \{m \mathfrak{A}_1 + m_1 \mathfrak{A}_2 + \dots\} \\ &= \mathfrak{A}_1(m_1 - m) + \mathfrak{A}_2(m_2 - m_1) + \dots \\ &\leq d \{ \mathfrak{A}_1 + \mathfrak{A}_2 + \dots \} \\ &\leq d \mathfrak{A} \\ &\doteq 0, \quad \text{as } d \doteq 0. \end{aligned} \quad (2)$$

We may now apply 387.

2. If f is measurable in \mathfrak{A}

$$\int_{\mathfrak{A}} f = \lim \sum m_{i-1} \mathfrak{A}_i = \lim \sum m_i \mathfrak{A}_i, \quad (3)$$

using the notation in 1.

This follows from 1), 2) in 1.

3. The relation 3) is taken by Lebesgue as definition of his integrals. His theory is restricted to measurable fields and to measurable functions. For Lebesgue's own development of his theory the reader is referred to his paper, *Intégrale, Longueur, Aire, Annali di Mat.*, Ser. 3, vol. 7 (1902); and to his book, *Leçons sur l'Intégration*. Paris, 1904. He may also consult the excellent account of it in *Hobson's book, The Theory of Functions of a Real Variable*. Cambridge, England, 1907.

Semi-Divisors and Quasi-Divisors

410. 1. The convergence of infinite series leads to the two following classes of point sets.

Let
$$F = \sum f_i(x_1 \dots x_m) = \sum_1^n f_i + \sum_{n+1}^{\infty} f_i = F_n + \bar{F}_n, \quad (1)$$
 each f_i being defined in \mathfrak{A} .

Let us take $\epsilon > 0$ small at pleasure, and then fix it.

Let us denote by \mathfrak{A}_n the points of \mathfrak{A} at which

$$-\epsilon \leq \bar{F}_n(x) \leq \epsilon. \quad (2)$$

Of course \mathfrak{A}_n may not exist. We are thus led in general to the sets

$$\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3 \dots \quad (3)$$

The complementary set $A_n = \mathfrak{A} - \mathfrak{A}_n$ will denote the points where

$$|\bar{F}_n(x)| > \epsilon. \quad (4)$$

If now F is convergent at x , there exists a ν such that this point lies in

$$\mathfrak{A}_\nu, \mathfrak{A}_{\nu+1}, \mathfrak{A}_{\nu+2} \dots \quad (5)$$

The totality of the points of convergence forms a set which has this property: corresponding to each of its points x , there exists a ν such that x lies in the set 5). A set having this property is called the *semi-divisor* of the sets 3), and is denoted by

$$\text{Sdv } \{\mathfrak{A}_n\}.$$

Suppose now, on the other hand, that 1) does not converge at the point x in \mathfrak{A} . Then there exists an infinite set of indices

$$n_1 < n_2 < \dots \doteq \infty,$$

such that

$$|\bar{F}_{n_s}(x)| > \epsilon.$$

Thus, the point x lies in an infinity of the sets

$$A_1, A_2, A_3 \dots \quad (6)$$

The totality of points such that each lies in an infinity of the sets 6) is called the *quasi-divisor* of 6) and is denoted by

$$\text{Qdv } \{A_n\}.$$

Obviously,

$$\text{Sdv } \{\mathfrak{A}_n\} + \text{Qdv } \{A_n\} = \mathfrak{A}. \quad (7)$$

We may generalize these remarks at once. Since $F(x)$ is nothing but

$$\lim F_n(x),$$

we can apply these notions to the case that the functions $f_i(x_1 \cdots x_m)$ are defined in \mathfrak{A} , and that

$$\lim f_i = \phi.$$

2. We may go still farther and proceed in the following abstract manner.

The divisor \mathfrak{D} of the point sets

$$\mathfrak{A}_1, \mathfrak{A}_2, \dots \quad (1)$$

is the set of points lying in all the sets 1).

The totality of points each of which lies in an infinity of the sets 1) is called the quasi-divisor and is denoted by

$$Qdv \{ \mathfrak{A}_n \}. \quad (2)$$

The totality of points α , to each of which correspond an index m_α , such that α lies in

$$\mathfrak{A}_{m_\alpha}, \mathfrak{A}_{m_\alpha+1}, \dots$$

forms a set called the semi-divisor of 1), and is denoted by

$$Sdv \{ \mathfrak{A}_n \}. \quad (3)$$

If we denote 2), 3) by \mathfrak{Q} and \mathfrak{S} respectively, we have, obviously,

$$\mathfrak{D} \leq \mathfrak{S} \leq \mathfrak{Q}. \quad (4)$$

3. In the special case that $\mathfrak{A}_1 > \mathfrak{A}_2 > \dots$ we have

$$\mathfrak{Q} = \mathfrak{S} = \mathfrak{D}. \quad (5)$$

For denoting the complementary sets by the corresponding Roman letters, we have

$$D = A_1 + Dv(\mathfrak{A}_1, A_2) + Dv(\mathfrak{A}_2, A_3) + \dots$$

But Q has precisely the same expression.

Thus $\mathfrak{Q} = \mathfrak{D}$, and hence by 4), $\mathfrak{S} = \mathfrak{D}$.

4. Let $\mathfrak{A}_n + A_n = \mathfrak{B}$, $n = 1, 2, \dots$ Then

$$\text{Qdv } \{\mathfrak{A}_n\} + \text{Sdv } \{A_n\} = \mathfrak{B}.$$

For each point b of \mathfrak{B} lies

either 1° only in a finite number of \mathfrak{A}_n , or in none at all,
or 2° in an infinite number of \mathfrak{A}_n .

In the 1° case, b does not lie in $\mathfrak{A}_s, \mathfrak{A}_{s+1} \dots$; hence it lies in $A_s, A_{s+1} \dots$ In the 2° case b lies obviously in $\text{Qdv } \{\mathfrak{A}_n\}$.

5. If $\mathfrak{A}_1, \mathfrak{A}_2 \dots$ are measurable, and their union is limited,

$$\Omega = \text{Qdv } \{\mathfrak{A}_n\} \quad , \quad \mathfrak{S} = \text{Sdv } \{\mathfrak{A}_n\}$$

are measurable.

For let $\mathfrak{D}_n = Dv(\mathfrak{A}_n, \mathfrak{A}_{n+1} \dots)$. Then $\mathfrak{S} = \{\mathfrak{D}_n\}$.

But \mathfrak{S} is measurable, as each \mathfrak{D}_n is. Thus $\text{Sdv } \{A_n\}$ is measurable, and hence Ω is by 4.

6. Let $\Omega = \text{Qdv } \{\mathfrak{A}_n\}$, each \mathfrak{A}_n being measurable, and their union limited. If there are an infinity of the \mathfrak{A}_n , say

$$\mathfrak{A}_{i_1}, \mathfrak{A}_{i_2} \dots \quad ; \quad i_1 < i_2 < \dots$$

whose measure is $\geq \alpha$, then

$$\widehat{\Omega} \geq \alpha. \tag{6}$$

For let $\mathfrak{B}_n = (\mathfrak{A}_{i_n}, \mathfrak{A}_{i_{n+1}} \dots)$, then $\widehat{\mathfrak{B}}_n \geq \alpha$.

Let $\mathfrak{B} = Dv\{\mathfrak{B}_n\}$. As $\mathfrak{B}_n \geq \mathfrak{B}_{n+1}$,

$$\widehat{\mathfrak{B}} = \lim \widehat{\mathfrak{B}}_n \geq \alpha \tag{7}$$

by 362. As $\Omega \geq \mathfrak{B}$ we have 6) at once, from 7).

Limit Functions

411. Let $\lim_{t \rightarrow \tau} f(x_1 \dots x_m, t_1 \dots t_n) = \phi(x_1 \dots x_m),$

as x ranges over \mathfrak{A} , τ finite or infinite. Let f be measurable in \mathfrak{A} and numerically $\leq M$, for each t near τ . Then ϕ is measurable in \mathfrak{A} also.

To prove this we show that the points \mathfrak{B} of \mathfrak{A} where

$$\lambda \leq \phi \leq \mu \tag{1}$$

form a measurable set for each λ, μ . For simplicity let τ be finite. Let $t_1, t_2 \dots \doteq \tau$; also let $\epsilon_1 > \epsilon_2 > \dots \doteq 0$. Let $\mathfrak{G}_{n,s}$ denote the points of \mathfrak{A} where

$$\lambda - \epsilon_n \leq f(x, t_s) \leq \mu + \epsilon_n. \quad (2)$$

Then for each point x of \mathfrak{B} , there is an s_0 such that 2) holds for any t_s , if $s \geq s_0$. Let $\mathfrak{G}_n = \text{Sdv}_s \{ \mathfrak{G}_{ns} \}$. Then $\mathfrak{B} \subset \mathfrak{G}_n$. But the \mathfrak{G}_{ns} being measurable, \mathfrak{G}_n is by 410, 5. Finally $\mathfrak{B} = \text{Dv} \{ \mathfrak{G}_n \}$, and hence \mathfrak{B} is measurable.

412. Let
$$\lim_{t \rightarrow \tau} f(x_1 \dots x_m, t_1 \dots t_n) = \phi(x_1 \dots x_m),$$

for x in \mathfrak{A} , and τ finite or infinite. Let $t', t'' \dots \doteq \tau$. Let each $f_s = f(x, t^{(s)})$ be measurable, and numerically $\leq M$. Let $\phi = f_s + g_s$. Let \mathfrak{G}_s denote the points where

$$|g_s| > \epsilon.$$

Then for each $\epsilon > 0$,
$$\lim_{s \rightarrow \infty} \widehat{\mathfrak{G}}_s = 0. \quad (1)$$

For by 411, ϕ is measurable, hence g_s is measurable in \mathfrak{A} , hence \mathfrak{G}_s is measurable.

Suppose now that 1) does not hold. Then

$$\overline{\lim}_{s \rightarrow \infty} \widehat{\mathfrak{G}}_s = l > 0.$$

Then there are an infinity of the \mathfrak{G}_s , as $\mathfrak{G}_{s_1}, \mathfrak{G}_{s_2} \dots$ whose measures are $\geq \lambda > 0$. Then by 410, 6, the measure of

$$\mathfrak{G} = \text{Qdv} \{ \mathfrak{G}_s \}$$

is $\geq \lambda$. But this is not so, since $f_s \doteq \phi$, at each point of \mathfrak{A} .

413. 1. Let
$$\lim_{t \rightarrow \tau} f(x_1 \dots x_m, t_1 \dots t_n) = \phi(x_1 \dots x_m),$$

for x in \mathfrak{A} , and τ finite or infinite.

Let
$$t', t'' \dots \doteq \tau. \quad (1)$$

If each $f_s = f(x, t^{(s)})$ is measurable, and numerically $\leq M$ in \mathfrak{A} for each sequence 1), then

$$\int_{\mathfrak{A}} \phi = \lim_{t \rightarrow \tau} \int_{\mathfrak{A}} f(x, t). \quad (2)$$

For set
and let

$$\phi = f_s + g_s,$$

$$|g_s| \leq N, \quad s = 1, 2, \dots$$

Then as in 412, ϕ and g_s are measurable in \mathfrak{A} . Then by 409, they are L -integrable, and

$$\int_{\mathfrak{A}} \phi = \int_{\mathfrak{A}} f_s + \int_{\mathfrak{A}} g_s. \quad (3)$$

Let \mathfrak{B}_s denote the points of \mathfrak{A} , at which

$$|g_s| \geq \epsilon;$$

and let $\mathfrak{B}_s + B_s = \mathfrak{A}$. Then \mathfrak{B}_s, B_s are measurable, since g_s is. Thus by 390,

$$\int_{\mathfrak{A}} g_s = \int_{\mathfrak{B}_s} g_s + \int_{B_s} g_s.$$

Hence

$$\left| \int_{\mathfrak{A}} g_s \right| \leq N\widehat{\mathfrak{B}}_s + \epsilon\widehat{B}_s \leq N\widehat{\mathfrak{B}}_s + \epsilon\widehat{\mathfrak{A}}.$$

By 412, $\widehat{\mathfrak{B}}_s \doteq 0$. Thus

$$\lim_{s \rightarrow \infty} \int_{\mathfrak{A}} g_s = 0.$$

Hence passing to the limit in 3), we get 2), for the sequence 1). Since we can do this for every sequence of points t which $\doteq \tau$, the relation 2) holds.

2. Let

$$F = \Sigma f_{i_1, \dots, i_n}(x_1 \dots x_m)$$

converge in \mathfrak{A} . If each term f_i is measurable, and each $|F_\lambda| \leq M$, then F is L -integrable, and

$$\int_{\mathfrak{A}} F = \Sigma \int_{\mathfrak{A}} f_i.$$

Iterated Integrals

414. In Vol. I, 732, seq. we have seen that the relation,

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}} \int_{\mathfrak{C}} f,$$

holds when f is R -integrable in the metric field \mathfrak{A} . This result was extended to *iterable* fields in 14 of the present volume. We

wish now to generalize still further to the case that f is L -integrable in the *measurable* field \mathfrak{A} . The method employed is due to Dr. W. A. Wilson,* and is essentially simpler than that employed by Lebesgue.

1. Let $x = (z_1 \cdots z_s)$ denote a point in s -way space \mathfrak{N}_s , $s = m + n$. If we denote the first m coördinates by $x_1 \cdots x_m$, and the remaining coördinates by $y_1 \cdots y_n$, we have

$$z = (x_1 \cdots x_m y_1 \cdots y_n).$$

The points

$$x = (x_1 \cdots x_m 0 0 \cdots 0)$$

range over an m -way space \mathfrak{N}_m , when z ranges over \mathfrak{N}_s . We call x the *projection of z on \mathfrak{N}_m* .

Let z range over a point set \mathfrak{A} lying in \mathfrak{N}_s , then x will range over a set \mathfrak{B} in \mathfrak{N}_m , called the *projection of \mathfrak{A} on \mathfrak{N}_m* . The points of \mathfrak{A} whose projection is x is called the *section of \mathfrak{A} corresponding to x* . We may denote it by

$$\mathfrak{A}(x), \text{ or more shortly by } \mathfrak{C}.$$

We write

$$\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$$

to denote that \mathfrak{A} is conceived of as formed of the sections \mathfrak{C} , corresponding to the different points of its projection \mathfrak{B} .

2. Let Ω denote a standard cube containing \mathfrak{A} , let q denote its projection on \mathfrak{N}_m . Then $\mathfrak{B} \leq q$. Suppose each section $\mathfrak{A}(x)$ is measurable. It will be convenient to let $\hat{\mathfrak{A}}(x)$ denote a function of x defined over q such that

$$\begin{aligned} \hat{\mathfrak{A}}(x) &= \text{Meas } \mathfrak{A}(x) = \mathfrak{C} && \text{when } x \text{ lies in } \mathfrak{B}, \\ &= 0 && \text{when } x \text{ lies in } q - \mathfrak{B}. \end{aligned}$$

This function therefore is equal to the measure of the section of \mathfrak{A} corresponding to the point x , when such a section exists; and when not, the function = 0.

When each section $\mathfrak{A}(x)$ is not measurable, we can introduce the functions

$$\overline{\mathfrak{A}}(x) \quad , \quad \underline{\mathfrak{A}}(x).$$

* Dr. Wilson's results were obtained in August, 1900, and were presented by me in the course of an address which I had the honor to give at the Second Decennial Celebration of Clark University, September, 1900.

Here the first $= \bar{\mathfrak{E}}$ when a section exists, otherwise it $= 0$, in q . A similar definition holds for the other function.

3. Let us note that the sections

$$\mathfrak{A}_o(x) \quad , \quad \mathfrak{A}_i(x),$$

where $\mathfrak{A}_o, \mathfrak{A}_i$ are the outer and inner associated sets belonging to \mathfrak{A} , are always measurable.

For $\mathfrak{A}_o = Dv\{\mathfrak{E}_n\}$, where each \mathfrak{E}_n is a standard enclosure, each of whose cells e_{nm} is rectangular. But the sections $e_{nm}(x)$ are also rectangular. Hence

$$\mathfrak{A}_o(x) = Dv\{e_{nm}(x)\},$$

being the divisor of measurable sets, is measurable.

415. Let \mathfrak{A}_o be an outer associated set of \mathfrak{A} , both lying in the standard cube \mathfrak{Q} . Then $\widehat{\mathfrak{A}}_o(x)$ is L -integrable in q , and

$$\bar{\mathfrak{A}} = \int_q \widehat{\mathfrak{A}}_o(x). \quad (1)$$

For let $\{\mathfrak{E}_n\}$ be a sequence of standard enclosures of \mathfrak{A} , and $\mathfrak{E}_n = \{e_{nm}\}$. Then

$$\widehat{\mathfrak{E}}_n = \sum_m \widehat{e}_{nm} \quad (2)$$

and

$$\widehat{\mathfrak{E}}_n(x) = \sum_m \widehat{e}_{nm}(x). \quad (3)$$

Now e_{nm} being a standard cell, $\widehat{e}_{nm}(x)$ has a constant value > 0 for all x contained in the projection of e_{nm} on q . It is thus continuous in q except for a discrete set. It thus has an R -integral, and

$$\widehat{e}_{nm} = \int_q \widehat{e}_{nm}(x).$$

This in 2) gives

$$\begin{aligned} \widehat{\mathfrak{E}}_n &= \sum \int_q \widehat{e}_{nm}(x) \\ &= \int_q \sum \widehat{e}_{nm}(x), \quad \text{by 413, 2,} \\ &= \int_q \widehat{\mathfrak{E}}_n(x), \end{aligned} \quad (4)$$

by 3).

On the other hand, $\mathfrak{E}(x)$ is a measurable function by 411. Also

$$\begin{aligned}\overline{\mathfrak{N}} &= \mathfrak{N}_e = \lim \mathfrak{E}_n \\ &= \lim \int_{\mathfrak{q}} \mathfrak{E}_n(x) \\ &= \int_{\mathfrak{q}} \lim \mathfrak{E}_n(x), \quad \text{by 413, 1.}\end{aligned}\quad (5)$$

Now

$$\mathfrak{N}_o(x) = \lim_{n \rightarrow \infty} \mathfrak{E}_n(x).$$

Thus this in 5) gives 1).

416. Let \mathfrak{N} lie in the standard cube \mathfrak{Q} . Let \mathfrak{N}_i be an inner associated set. Then $\mathfrak{N}_i(x)$ is L -integrable in \mathfrak{q} , and

$$\mathfrak{N} = \int_{\mathfrak{q}} \mathfrak{N}_i(x).$$

For

$$\mathfrak{Q} = \mathfrak{N}_i + A_o.$$

Thus

$$\mathfrak{N}_i(x) = \widehat{\mathfrak{Q}}(x) - \widehat{A}_o(x).$$

Hence $\mathfrak{N}_i(x)$ is L -integrable in \mathfrak{q} , and

$$\begin{aligned}\int_{\mathfrak{q}} \mathfrak{N}_i(x) &= \int_{\mathfrak{q}} \widehat{\mathfrak{Q}}(x) - \int_{\mathfrak{q}} \widehat{A}_o(x) \\ &= \widehat{\mathfrak{Q}} - A_o, \quad \text{by 415,} \\ &= \mathfrak{N}_i = \mathfrak{N} \quad \text{by 370, 2.}\end{aligned}$$

417. Let measurable \mathfrak{N} lie in the standard cube \mathfrak{Q} .

Then

$$\mathfrak{N} = \int_{\mathfrak{q}} \mathfrak{N}(x). \quad (1)$$

For

$$\mathfrak{N}_i(x) \leq \mathfrak{N}(x) \leq \mathfrak{N}_o(x).$$

Hence

$$\mathfrak{N} = \int_{\mathfrak{q}} \mathfrak{N}_i(x) \leq \int_{\mathfrak{q}} \mathfrak{N}(x) \leq \int_{\mathfrak{q}} \mathfrak{N}_o(x) = \overline{\mathfrak{N}}, \quad (2)$$

using 396, 1, and 415, 416. From 2) we conclude 1) at once.

418. Let $\mathfrak{N} = \mathfrak{B} \cdot \mathfrak{E}$ be measurable. Then $\overline{\mathfrak{E}}$ are L -integrable in \mathfrak{B} , and

$$\mathfrak{N} = \int_{\mathfrak{B}} \overline{\mathfrak{E}}.$$

Here the first $= \bar{\mathfrak{E}}$ when a section exists, otherwise it $= 0$, in q . A similar definition holds for the other function.

3. Let us note that the sections

$$\mathfrak{A}_o(x) \quad , \quad \mathfrak{A}_i(x),$$

where $\mathfrak{A}_o, \mathfrak{A}_i$ are the outer and inner associated sets belonging to \mathfrak{A} , are always measurable.

For $\mathfrak{A}_o = Dv\{\mathfrak{E}_n\}$, where each \mathfrak{E}_n is a standard enclosure, each of whose cells e_{nm} is rectangular. But the sections $e_{nm}(x)$ are also rectangular. Hence

$$\mathfrak{A}_o(x) = Dv\{e_{nm}(x)\},$$

being the divisor of measurable sets, is measurable.

415. Let \mathfrak{A}_o be an outer associated set of \mathfrak{A} , both lying in the standard cube Ω . Then $\bar{\mathfrak{A}}_o(x)$ is L -integrable in q , and

$$\bar{\mathfrak{A}} = \int_q \bar{\mathfrak{A}}_o(x). \quad (1)$$

For let $\{\mathfrak{E}_n\}$ be a sequence of standard enclosures of \mathfrak{A} , and $\mathfrak{E}_n = \{e_{nm}\}$. Then

$$\bar{\mathfrak{E}}_n = \sum_m \bar{e}_{nm} \quad (2)$$

and

$$\bar{\mathfrak{E}}_n(x) = \sum_m \bar{e}_{nm}(x). \quad (3)$$

Now e_{nm} being a standard cell, $\bar{e}_{nm}(x)$ has a constant value > 0 for all x contained in the projection of e_{nm} on q . It is thus continuous in q except for a discrete set. It thus has an R -integral, and

$$\bar{e}_{nm} = \int_q \bar{e}_{nm}(x).$$

This in 2) gives

$$\begin{aligned} \bar{\mathfrak{E}}_n &= \sum \int_q \bar{e}_{nm}(x) \\ &= \int_q \sum \bar{e}_{nm}(x), \quad \text{by 413, 2,} \\ &= \int_q \bar{\mathfrak{E}}_n(x), \end{aligned} \quad (4)$$

by 3).

On the other hand, $\mathfrak{E}(x)$ is a measurable function by 411. Also

$$\begin{aligned}\overline{\mathfrak{A}} &= \overline{\mathfrak{A}}_o = \lim \mathfrak{E}_n \\ &= \lim \int_{\mathfrak{q}} \mathfrak{E}_n(x) \\ &= \int_{\mathfrak{q}} \lim \mathfrak{E}_n(x), \quad \text{by 413, 1.}\end{aligned}\quad (5)$$

Now $\widehat{\mathfrak{A}}_o(x) = \lim_{n \rightarrow \infty} \mathfrak{E}_n(x).$

Thus this in 5) gives 1).

416. Let \mathfrak{A} lie in the standard cube \mathfrak{Q} . Let \mathfrak{A}_i be an inner associated set. Then $\widehat{\mathfrak{A}}_i(x)$ is L -integrable in \mathfrak{q} , and

$$\mathfrak{A} = \int_{\mathfrak{q}} \widehat{\mathfrak{A}}_i(x).$$

For $\mathfrak{Q} = \mathfrak{A}_i + A_o.$

Thus $\widehat{\mathfrak{A}}_i(x) = \widehat{\mathfrak{Q}}(x) - \widehat{A}_o(x).$

Hence $\widehat{\mathfrak{A}}_i(x)$ is L -integrable in \mathfrak{q} , and

$$\begin{aligned}\int_{\mathfrak{q}} \widehat{\mathfrak{A}}_i(x) &= \int_{\mathfrak{q}} \widehat{\mathfrak{Q}}(x) - \int_{\mathfrak{q}} \widehat{A}_o(x) \\ &= \widehat{\mathfrak{Q}} - A_o, \quad \text{by 415,} \\ &= \mathfrak{A}_i = \mathfrak{A} \quad \text{by 370, 2.}\end{aligned}$$

417. Let measurable \mathfrak{A} lie in the standard cube \mathfrak{Q} . Then

$$\widehat{\mathfrak{A}} = \int_{\mathfrak{q}} \overline{\mathfrak{A}}(x). \quad (1)$$

For $\mathfrak{A}_i(x) \leq \mathfrak{A}(x) \leq \mathfrak{A}_o(x).$

Hence $\mathfrak{A} = \int_{\mathfrak{q}} \widehat{\mathfrak{A}}_i(x) \leq \int_{\mathfrak{q}} \overline{\mathfrak{A}}(x) \leq \int_{\mathfrak{q}} \widehat{\mathfrak{A}}_o(x) = \overline{\mathfrak{A}}, \quad (2)$

using 396, 1, and 415, 416. From 2) we conclude 1) at once.

418. Let $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ be measurable. Then $\overline{\mathfrak{C}}$ are L -integrable in \mathfrak{B} , and

$$\widehat{\mathfrak{A}} = \int_{\mathfrak{B}} \overline{\mathfrak{C}}.$$

For by 417,

$$\begin{aligned}\widehat{\mathfrak{A}} &= \int_{\mathfrak{A}} \overline{\mathfrak{A}}(x) \\ &= \int_{\mathfrak{B}} \overline{\mathfrak{C}},\end{aligned}$$

by 401.

419. If $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ is measurable, the points of \mathfrak{B} at which \mathfrak{C} is not measurable form a null set \mathfrak{N} .

For by 418,

$$\widehat{\mathfrak{A}} = \int_{\mathfrak{B}} \overline{\mathfrak{C}} = \int_{\mathfrak{B}} \mathfrak{C}.$$

Hence

$$0 = \int_{\mathfrak{B}} (\overline{\mathfrak{C}} - \mathfrak{C}).$$

Thus

$$\phi = \overline{\mathfrak{C}} - \mathfrak{C}$$

is a null function in \mathfrak{B} , and by 402, 2, points where $\phi > 0$ form a null set.

420. Let $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ be measurable. Let \mathfrak{b} denote the points of \mathfrak{B} for which the corresponding sections \mathfrak{C} are measurable. Then

$$\widehat{\mathfrak{A}} = \int_{\mathfrak{b}} \overline{\mathfrak{C}}.$$

For by 419,

$$\mathfrak{B} = \mathfrak{b} + \mathfrak{N},$$

and \mathfrak{N} is a null set. Hence by 418,

$$\begin{aligned}\widehat{\mathfrak{A}} &= \int_{\mathfrak{B}} \overline{\mathfrak{C}} = \int_{\mathfrak{b}} \overline{\mathfrak{C}} + \int_{\mathfrak{N}} \overline{\mathfrak{C}} \\ &= \int_{\mathfrak{b}} \overline{\mathfrak{C}}.\end{aligned}$$

421. Let $f \geq 0$ in \mathfrak{A} . If the integrand set \mathfrak{J} , corresponding to f be measurable, then f is L -integrable in \mathfrak{A} , and

$$\widehat{\mathfrak{J}} = \int_{\mathfrak{A}} f.$$

For the points of \mathfrak{J} lying in an $m+1$ way space \mathfrak{R}_{m+1} may be denoted by

$$x = (y_1 \cdots y_m, z),$$

where $y = (y_1 \cdots y_m)$ ranges over \mathfrak{R}_m , in which \mathfrak{A} lies. Thus \mathfrak{A} may be regarded as the projection of \mathfrak{J} on \mathfrak{R}_m . To each point y

of \mathfrak{A} corresponds a section $\mathfrak{F}(y)$, which for brevity may be denoted by \mathfrak{R} . Thus we may write

$$\mathfrak{F} = \mathfrak{A} \cdot \mathfrak{R}.$$

As \mathfrak{R} is nothing but an ordinate through y of length $f(y)$, we have by 419,

$$\widehat{\mathfrak{F}} = \int_{\mathfrak{A}} \overline{\mathfrak{R}} = \int_{\mathfrak{A}} f.$$

422. Let f be L -integrable over the measurable field $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$. Let \mathfrak{b} denote those points of \mathfrak{B} , for which f is L -integrable over the corresponding sections \mathfrak{C} . Then

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{b}} \int_{\mathfrak{C}} f. \quad (1)$$

Moreover $\mathfrak{N} = \mathfrak{B} - \mathfrak{b}$ is a null set.

Let us 1° suppose $f \geq 0$. Then by 406, \mathfrak{F} is measurable and

$$\widehat{\mathfrak{F}} = \int_{\mathfrak{A}} f. \quad (2)$$

Let β denote the points of \mathfrak{B} for which $\mathfrak{F}(x)$ is measurable. Then by 420,

$$\widehat{\mathfrak{F}} = \int_{\beta} \widehat{\mathfrak{F}}(x). \quad (3)$$

By 419, the points

$$\mathfrak{P} = \mathfrak{B} - \beta \quad (4)$$

form a null set.

On the other hand, $\mathfrak{F}(x)$ is the integrand set of f , for $\mathfrak{A}(x) = \mathfrak{C}$. Hence by 421, for any x in β ,

$$\widehat{\mathfrak{F}}(x) = \int_{\mathfrak{C}} f, \quad (5)$$

and

$$\beta \leq \mathfrak{b}. \quad (6)$$

From 2), 3), 5) we have

$$\int_{\mathfrak{A}} f = \int_{\beta} \int_{\mathfrak{C}} f. \quad (7)$$

From 6) we have

$$\mathfrak{N} = \mathfrak{B} - \mathfrak{b} \leq \mathfrak{B} - \beta = \mathfrak{P},$$

a null set by 4). Let us set

$$\mathfrak{b} = \beta + \mathfrak{n}.$$

Then π lying in the null set \mathfrak{B} , is a null set. Hence

$$\underline{\mathcal{L}}_{\mathfrak{B}} \underline{\mathcal{L}}_{\mathfrak{C}} + \underline{\mathcal{L}}_{\pi} \underline{\mathcal{L}}_{\mathfrak{C}} = \underline{\mathcal{L}}_{\mathfrak{b}} \underline{\mathcal{L}}_{\mathfrak{C}}.$$

This with 7) gives 1).

Let f be now unrestricted as to sign. We take $C > 0$, such that the auxiliary function

$$g = f + C \geq 0, \quad \text{in } \mathfrak{A}.$$

Then f, g are simultaneously L -integrable over any section \mathfrak{C} . Thus by case 1°

$$\int_{\mathfrak{A}} (f + C) = \int_{\mathfrak{b}} \int_{\mathfrak{C}} (f + C). \quad (8)$$

Now

$$\int_{\mathfrak{A}} (f + C) = \int_{\mathfrak{A}} f + \int_{\mathfrak{A}} C = \int_{\mathfrak{A}} f + C \widehat{\mathfrak{A}}, \quad (9)$$

$$\int_{\mathfrak{C}} (f + C) = \int_{\mathfrak{C}} f + C \overline{\mathfrak{C}}. \quad (10)$$

By 418, $\overline{\mathfrak{C}}$ is L -integrable in \mathfrak{B} , and hence in \mathfrak{b} . Thus

$$\int_{\mathfrak{b}} \int_{\mathfrak{C}} (f + C) = \int_{\mathfrak{b}} \int_{\mathfrak{C}} f + C \int_{\mathfrak{b}} \overline{\mathfrak{C}}. \quad (11)$$

As \mathfrak{b} differs from \mathfrak{B} by a null set,

$$\int_{\mathfrak{b}} \overline{\mathfrak{C}} = \int_{\mathfrak{B}} \overline{\mathfrak{C}} = \widehat{\mathfrak{A}}, \quad (12)$$

by 418. From 8), 9), 10), 11), 12) we have 1).

423. If f is L -integrable over the measurable set $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$, then

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f. \quad (1)$$

For by 422,

$$\int_{\mathfrak{A}} = \int_{\mathfrak{b}} \underline{\int}_{\mathfrak{C}}. \quad (2)$$

As $\mathfrak{B} - \mathfrak{b} = \mathfrak{N}$ is a null set,

$$\int_{\mathfrak{N}} \underline{\int}_{\mathfrak{C}} f = 0$$

may be added to the right side of 2) without altering its value. Thus

$$\int_{\mathfrak{A}} = \int_{\mathfrak{B}} \int_{\mathfrak{C}} + \int_{\mathfrak{A}} \int_{\mathfrak{C}} = \int_{\mathfrak{B}} \int_{\mathfrak{C}}.$$

424. 1. (*W. A. Wilson.*) If $f(x_1 \cdots x_m)$ is L -integrable in measurable \mathfrak{A} , f is measurable in \mathfrak{A} .

Let us first suppose that $f \geq 0$. We begin by showing that the set of points \mathfrak{A}_λ of \mathfrak{A} at which $f \geq \lambda$, is measurable. Then by 408, 3, f is measurable in \mathfrak{A} .

Now f being L -integrable in \mathfrak{A} , its integrand set \mathfrak{F} is measurable by 406. Let \mathfrak{F}_λ be the section of \mathfrak{F} corresponding to $x_{m+1} = \lambda$. Then the projection of \mathfrak{F}_λ on \mathfrak{R}_m is \mathfrak{A}_λ . Since \mathfrak{F} is measurable, the sections \mathfrak{F}_λ are measurable, except at most over a null set L of values of λ , by 419. Thus there exists a sequence

$$\lambda_1 < \lambda_2 < \cdots \doteq \lambda$$

none of whose terms lies in L . Hence each \mathfrak{F}_{λ_n} is measurable, and hence \mathfrak{A}_{λ_n} is also.

As $\mathfrak{A}_{\lambda_{n+1}} \leq \mathfrak{A}_{\lambda_n}$, each point of \mathfrak{A}_λ lies in

$$\mathfrak{D} = Dv\{\mathfrak{A}_{\lambda_n}\}, \quad (1)$$

so that

$$\mathfrak{A}_\lambda \leq \mathfrak{D}. \quad (2)$$

On the other hand, each point d of \mathfrak{D} lies in \mathfrak{A}_λ . For if not, $f(d) < \lambda$.

There thus exists an s such that

$$f(d) < \lambda_s < \lambda. \quad (3)$$

But then d does not lie in \mathfrak{A}_{λ_s} , for otherwise $f(d) \geq \lambda_s$, which contradicts 3). But not lying in \mathfrak{A}_{λ_s} , d cannot lie in \mathfrak{D} , and this contradicts our hypothesis. Thus

$$\mathfrak{D} \leq \mathfrak{A}_\lambda. \quad (4)$$

From 2), 4) we have

$$\mathfrak{D} = \mathfrak{A}_\lambda.$$

But then from 1), \mathfrak{A}_λ is measurable.

Let the sign of f be now unrestricted.

Since f is limited, we may choose the constant C , such that

$$g = f(x) + C \geq 0, \text{ in } \mathfrak{A}.$$

Then g is L -integrable, and hence, by case 1°, g is measurable. Hence f , differing only by a constant from g , is also measurable.

2. Let \mathfrak{A} be measurable. If f is L -integrable in \mathfrak{A} , it is measurable in \mathfrak{A} , and conversely.

This follows from 1 and 409, 1.

3. From 2 and 409, 3, we have at once the theorem :

When the field of integration is measurable, an L -integrable function is integrable in Lebesgue's sense; and conversely; moreover, both have the same value.

Remark. In the theory which has been developed in the foregoing pages, the reader will note that neither the field of integration nor the integrand needs to be measurable. This is not so in Lebesgue's theory. In removing this restriction, we have been able to develop a theory entirely analogous to Riemann's theory of integration, and to extend this to a theory of upper and lower integration. We have thus a perfect counterpart of the theory developed in Chapter XIII of vol. I.

4. Let \mathfrak{A} be metric or complete. If $f(x_1 \dots x_m)$ is limited and R -integrable, it is a measurable function in \mathfrak{A} .

For by 381, 2, it is L -integrable. Also since \mathfrak{A} is metric or complete, \mathfrak{A} is measurable. We now apply 1.

IMPROPER L -INTEGRALS

Upper and Lower Integrals

425. 1. We propose now to consider the case that the integrand $f(x_1 \dots x_m)$ is not limited in the limited field of integration \mathfrak{A} . In chapter II we have treated this case for R -integrals. To extend the definitions and theorems there given to L -integrals, we have in general only to replace metric or complete sets by measurable sets; discrete sets by null sets; unmixed sets by separated sets;

divisions by separated divisions; sequences of superposed divisions by extremal sequences; etc.

28 we may define an improper L -integral in any of the ways there given, making such changes as just indicated. Following we shall employ only the 3^o Type of definition. Explicit we define as follows:

$f(x_1 \dots x_m)$ be defined for each point of the limited set \mathfrak{A} . Denote the points of \mathfrak{A} at which

$$- \alpha \leq f(x_1 \dots x_m) \leq \beta \quad \alpha, \beta > 0. \quad (1)$$

ts

$$\lim_{\alpha, \beta \rightarrow \infty} \int_{\mathfrak{A}_{\alpha\beta}} f, \quad \lim_{\alpha, \beta \rightarrow \infty} \int_{\mathfrak{A}_{\alpha\beta}} \bar{f} \quad (2)$$

they exist, we call the *lower* and *upper* (improper) L -integrals and denote them by

$$\underline{\int}_{\mathfrak{A}} f, \quad \overline{\int}_{\mathfrak{A}} f.$$

If the two limits 2) exist and are equal, we denote their value by

$$\int_{\mathfrak{A}} f$$

f is (improperly) L -integrable in \mathfrak{A} , etc.

In order to use the demonstrations of Chapter II without trouble, we introduce the term *separated function*. A function is called a *separated function* when the fields $\mathfrak{A}_{\alpha\beta}$ defined by 1) are separated parts of \mathfrak{A} .

We have defined measurable functions in 407 in the case that \mathfrak{A} is a limited field. We may extend it to unlimited functions by requiring that the fields $\mathfrak{A}_{\alpha\beta}$ are measurable however large α, β are.

Being so, we see that measurable functions are special cases of separated functions.

If the field \mathfrak{A} of integration is measurable, $\mathfrak{A}_{\alpha\beta}$ is a measurable part of \mathfrak{A} , if it is a separated part. From this follows the following result:

If f is a separated function in the measurable field \mathfrak{A} , it is L -integrable in each $\mathfrak{A}_{\alpha\beta}$.

From this follows also the theorem:

Let f be a separated function in the measurable field \mathfrak{A} . If either the lower or upper integral of f over \mathfrak{A} is convergent, f is L -integrable in \mathfrak{A} , and

$$\int_{\mathfrak{A}} f = \lim_{\alpha, \beta \rightarrow \infty} \int_{\mathfrak{A}_{\alpha\beta}} f.$$

426. To illustrate how the theorems on improper R -integrals give rise to analogous theorems on improper L -integrals, which may be demonstrated along the same lines as used in Chapter II, let us consider the analogue of 38, 2, viz.:

If f is a separated function such that $\int_{\mathfrak{A}} f$ converges, so do $\int_{\mathfrak{P}} f$.

Let $\{E_n\}$ be an extremal sequence common to both

$$\int_{\mathfrak{A}_{\alpha\beta'}}, \quad \int_{\mathfrak{A}_{\alpha\beta}} \quad \beta' > \beta.$$

Let e denote the cells of E_n containing a point of \mathfrak{P}_β ; e' those cells containing a point of $\mathfrak{P}_{\beta'}$; δ those cells containing a point of $\mathfrak{A}_{\alpha\beta}$ but none of $\mathfrak{P}_{\beta'}$. Then

$$\int_{\mathfrak{A}_{\alpha\beta'}} = \lim_{n \rightarrow \infty} \{ \sum M'_e \cdot e + \sum M'_{e'} \cdot e' + \sum M'_\delta \cdot \delta \}.$$

In this manner we may continue using the proof of 38, and so establish our theorem.

427. As another illustration let us prove the theorem analogous to 46, viz.:

Let $\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_n$ form a separated division of \mathfrak{A} . If f is a separated function in \mathfrak{A} , then

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}_1} f + \dots + \int_{\mathfrak{A}_n} f,$$

provided the integral on the left exists, or all the integrals on the right exist.

For let $\mathfrak{A}_{i, \alpha\beta}$ denote the points of $\mathfrak{A}_{i\alpha\beta}$ in \mathfrak{A}_i . Then by 390, 1,

$$\int_{\mathfrak{A}_{i\alpha\beta}} = \int_{\mathfrak{A}_{i, \alpha\beta}} + \dots + \int_{\mathfrak{A}_{i, n, \alpha\beta}}.$$

In this way we continue with the reasoning of 46.

428. In this way we can proceed with the other theorems; in each case the requisite modification is quite obvious, by a consideration of the demonstration of the corresponding theorem in R -integrals given in Chapter II.

This is also true when we come to treat of *iterated* integrals along the lines of 70-78. We have seen, in 425, 2, that if \mathfrak{A} is measurable, *upper* and *lower* integrals of separated functions do not exist as such; they reduce to L -integrals. We may still have a theory analogous to iterated R -integrals, by extending the notion of iterable fields, using the notion of upper measure. To this end we define:

A limited point set at $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ is *submeasurable* with respect to \mathfrak{B} , when

$$\overline{\mathfrak{A}} = \int_{\mathfrak{B}} \overline{\mathfrak{C}}.$$

We do not care to urge this point at present, but prefer to pass on at once to the much more interesting case of L -integrals over measurable fields.

L-Integrals

429. These we may define for our purpose as follows:

Let $f(x_1 \dots x_m)$ be defined over the limited measurable set \mathfrak{A} . As usual let $\mathfrak{A}_{\alpha\beta}$ denote the points of \mathfrak{A} at which

$$-\alpha \leq f \leq \beta, \quad \alpha, \beta \geq 0.$$

Let each $\mathfrak{A}_{\alpha\beta}$ be measurable, and let f have a proper L -integral in each $\mathfrak{A}_{\alpha\beta}$. Then the improper integral of f over \mathfrak{A} is

$$\int_{\mathfrak{A}} f = \lim_{\alpha, \beta \rightarrow \infty} \int_{\mathfrak{A}_{\alpha\beta}} f, \quad (1)$$

when this limit exists. We shall also say that the integral on the left of 1) is *convergent*.

On this hypothesis, the reader will note at once that the demonstrations of Chapter II admit ready adaptation; in fact some of the theorems require no demonstration, as they follow easily from results already obtained.

430. Let us group together for reference the following theorems, analogous to those on improper R -integrals.

1. If f is (improperly) L -integrable in \mathfrak{A} , it is in any measurable part of \mathfrak{A} .

2. If g, h denote as usual the non-negative functions associated with f , then

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}} g - \int_{\mathfrak{A}} h. \quad (1)$$

3. If $\int_{\mathfrak{A}} f$ is convergent, so is $\int_{\mathfrak{A}} |f|$, and conversely.

4. When convergent,

$$\left| \int_{\mathfrak{A}} f \right| \leq \int_{\mathfrak{A}} |f|. \quad (2)$$

5. If $\int_{\mathfrak{A}} f$ is convergent, then

$$\epsilon > 0, \quad \sigma > 0, \quad \left| \int_{\mathfrak{B}} f \right| \leq \epsilon,$$

for any measurable $\mathfrak{B} < \mathfrak{A}$, such that $\widehat{\mathfrak{B}} < \sigma$.

6. Let $\mathfrak{A} = (\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_n)$ be a separated division of \mathfrak{A} , each \mathfrak{A}_i being measurable. Then

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}_1} f + \dots + \int_{\mathfrak{A}_n} f, \quad (3)$$

provided the integral on the left exists, or all the integrals on the right exist.

7. Let $\mathfrak{A} = \{\mathfrak{A}_n\}$ be a separated division of \mathfrak{A} , into an enumerable infinite set of measurable sets \mathfrak{A}_n . Then

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}_1} f + \int_{\mathfrak{A}_2} f + \dots \quad (4)$$

provided the integral on the left exists.

8. If $f \leq g$ in \mathfrak{A} , except possibly at a null set, then

$$\int_{\mathfrak{A}} f \leq \int_{\mathfrak{A}} g, \quad (5)$$

when convergent.

431. 1. To show how simple the proofs run in the present case, let us consider, in the first place, the theorem analogous to 38, 2, viz. :

If $\int_{\mathfrak{A}} f$ converges, so do $\int_{\mathfrak{B}} f$ and $\int_{\mathfrak{A}} f$.

The rather difficult proof of 38, 2 can be replaced by the following simpler one. Since

$$\mathfrak{A}_{\alpha\beta} = \mathfrak{B}_{\beta} + \mathfrak{A}_{\alpha} \quad (1)$$

is a *separated* division of $\mathfrak{A}_{\alpha\beta}$, we have

$$\begin{aligned} \int_{\mathfrak{A}_{\alpha\beta}} &= \int_{\mathfrak{B}_{\beta}} + \int_{\mathfrak{A}_{\alpha}}, \\ \int_{\mathfrak{A}_{\alpha\beta}} &= \int_{\mathfrak{B}_{\beta}} + \int_{\mathfrak{A}_{\alpha}}. \end{aligned}$$

Hence

$$\left| \int_{\mathfrak{A}_{\alpha\beta}} - \int_{\mathfrak{A}_{\alpha\beta'}} \right| = \left| \int_{\mathfrak{B}_{\beta}} - \int_{\mathfrak{B}_{\beta'}} \right|.$$

But the left side is $< \epsilon$, for a sufficiently large α , and $\beta, \beta' >$ some β_0 . This shows that $\int_{\mathfrak{B}}$ is convergent. Similarly we show the other integral converges.

2. This form of proof could not be used in 38, 2, since 1) in general is not an unmixed division of $\mathfrak{A}_{\alpha\beta}$.

3. In a similar manner we may establish the theorem analogous to 39, viz. :

If $\int_{\mathfrak{B}} f$ and $\int_{\mathfrak{A}} f$ converge, so does $\int_{\mathfrak{A}} f$.

4. Let us look at the demonstration of the theorem analogous to 43, 1, viz. :

$$\int_{\mathfrak{A}} g = \int_{\mathfrak{B}} f \quad ; \quad \int_{\mathfrak{A}} h = - \int_{\mathfrak{A}} f,$$

provided the integral on either side of these equations converges.

Let us prove the first relation. Let \mathfrak{B}_β denote the points of \mathfrak{A} at which $f \leq \beta$. Then

$$\mathfrak{B}_\beta = \mathfrak{A} + \mathfrak{P}_\beta$$

is a separated division of \mathfrak{B}_β , and hence

$$\int_{\mathfrak{B}_\beta} g = \int_{\mathfrak{A}} g + \int_{\mathfrak{P}_\beta} g = \int_{\mathfrak{P}_\beta} g = \int_{\mathfrak{P}_\beta} f, \text{ etc.}$$

5. It is now obvious that the analogue of 44, 1 is the relation 1) in 430.

6. The analogue of 46 is the relation 3) in 430. Its demonstration is precisely similar to that in 46.

7. We now establish 430, 7. Let

$$\mathfrak{B}_m = (\mathfrak{A}_1, \mathfrak{A}_2 \dots \mathfrak{A}_m).$$

Then

$$\mathfrak{A} = \mathfrak{B}_m + B_m$$

is a separated division of \mathfrak{A} , and we may take m so large that $\widehat{B}_m < \sigma$, an arbitrarily small positive number. Hence by 430, 5, we may take m so large that

$$\left| \int_{B_m} f \right| < \epsilon.$$

Thus

$$\begin{aligned} \int_{\mathfrak{A}} f &= \int_{\mathfrak{B}_m} f + \int_{B_m} f \\ &= \int_{\mathfrak{A}_1} f + \dots + \int_{\mathfrak{A}_m} f + \epsilon' \quad , \quad |\epsilon'| < \epsilon. \end{aligned}$$

From this our theorem follows at once.

Iterated Integrals

432. 1. Let us see how the reasoning of Chapter II may be extended to this case. We will of course suppose that the field of integration $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ is measurable. Then by 419, the points of \mathfrak{B} for which the sections are not measurable form a null set. Since the integral of any function over a null set is zero, we may therefore in our reasoning suppose that every \mathfrak{C} is measurable.

Since \mathfrak{A} is measurable, there exists a sequence of complete components $A_m = B_m C_m$ in \mathfrak{A} , such that the measure of $A = \{A_m\}$ is $\widehat{\mathfrak{A}}$.

Since A_m is complete, its projection B_m is complete, by I, 717, 4. The points of B_m for which the corresponding sections C_m are not measurable form a null set ν_m . Hence the union $\{\nu_m\}$ is a null set. Thus we may suppose, without loss of generality in our demonstrations, that \mathfrak{A} is such that every section in each A_m is measurable.

Now from

$$0 = \mathfrak{A} - \hat{A} = \int_{\mathfrak{B}} \mathfrak{E} - \int_{\mathfrak{B}} \hat{C} = \int_{\mathfrak{B}} (\mathfrak{E} - \hat{C}),$$

we see that those points of \mathfrak{B} where $\mathfrak{E} > \hat{C}$ form a null set. We may therefore suppose that $\mathfrak{E} = \hat{C}$ everywhere. Then $\mathfrak{E} - \hat{C}$ is a null set at each point; we may thus adjoin them to \hat{C} . Thus we may suppose that $\mathfrak{E} = \hat{C}$ at each point of \mathfrak{B} , and that $\mathfrak{B} = B$ is the union of an enumerable set of complete sets B_m .

As we shall suppose that

$$\int_{\mathfrak{A}} f$$

is convergent, let

$$\alpha_1 < \alpha_2 < \dots \doteq \infty,$$

$$\beta_1 < \beta_2 < \dots \doteq \infty.$$

Let us look at the sets $\mathfrak{A}_{\alpha_n}, \mathfrak{B}_{\beta_n}$, which we shall denote by \mathfrak{A}_n . These are measurable by 429. Moreover, the reasoning of 72, 2 shows that without loss of generality we may suppose that \mathfrak{A} is such that $\mathfrak{B}_n = \mathfrak{B}$. We may also suppose that each \mathfrak{C}_n is measurable, as above.

2. Let us finally consider the integrals

$$\int_{\mathfrak{C}} f. \quad (1)$$

These may not exist at every point of \mathfrak{B} , because f does not admit a proper or an improper integral at this point. It will suffice for our purpose to suppose that 1) does not exist at a null set in \mathfrak{B} . Then without loss of generality we may suppose in our demonstrations that 1) converges at each point of \mathfrak{B} .

On these assumptions let us see how the theorems 73, 74, 75, and 76 are to be modified, in order that the proofs there given may be adapted to the present case.

433. 1. The first of these may be replaced by this :

Let $B_{\sigma, n}$ denote the points of \mathfrak{B} at which $\widehat{c}_n > \sigma$. Then

$$\lim_{n \rightarrow \infty} \overline{B}_{\sigma, n} = 0.$$

For by 419,

$$\widehat{\mathfrak{A}} = \int_{\mathfrak{B}} \widehat{\mathfrak{C}},$$

as by hypothesis the sections \mathfrak{C} are measurable. Moreover, by hypothesis

$$\mathfrak{C} = \mathfrak{C}_n + c_n$$

is a separated division of \mathfrak{C} , each set on the right being measurable. Thus the proof in 73 applies at once.

2. The theorem of 74 becomes :

Let the integrals

$$\int_{\mathfrak{C}} f, \quad f \geq 0$$

be limited in the complete set \mathfrak{B} . Let \mathfrak{C}_n denote the points of \mathfrak{B} at which

$$\int_{c_n} f \leq \epsilon.$$

Then

$$\lim_{n \rightarrow \infty} \overline{\mathfrak{C}}_n = \widehat{\mathfrak{B}}.$$

The proof is analogous to that in 74. Instead of a cubical division of the space \mathfrak{R}_p , we use a standard enclosure. The sets \mathfrak{B}_n are now measurable, and thus

$$b = Dv\{\mathfrak{B}_n\}$$

is measurable. Thus $\overline{b}_n \doteq \widehat{b}$. The rest of the proof is as in 74.

3. The theorem of 75 becomes :

Let the integral

$$\int_{\mathfrak{C}} f, \quad f \geq 0$$

be limited in complete \mathfrak{B} . Then

$$\lim_{n \rightarrow \infty} \int_{\mathfrak{B}} \int_{c_n} f = 0.$$

The proof is entirely similar to that in 75, except that we use extremal sequences, instead of cubical divisions.

4. As a corollary of 3 we have

Let the integral

$$\int_{\mathfrak{C}} f, \quad f \geq 0$$

be limited and L -integrable in \mathfrak{B} . Let $\mathfrak{B} = \{B_m\}$ the union of an enumerable set of complete sets. Then

$$\lim_{n \rightarrow \infty} \int_{\mathfrak{B}} \int_{\mathfrak{C}_n} f = 0.$$

For if $\mathfrak{B}_m = (B_1, B_2 \dots B_m)$, and $\mathfrak{B} = \mathfrak{B}_m + \mathfrak{D}_m$, we have

$$\int_{\mathfrak{B}} \int_{\mathfrak{C}_n} = \int_{\mathfrak{B}_m} \int_{\mathfrak{C}_n} + \int_{\mathfrak{D}_m} \int_{\mathfrak{C}_n}.$$

But for m sufficiently large, \mathfrak{D}_m is small at pleasure. Hence

$$\int_{\mathfrak{D}_m} \int_{\mathfrak{C}_n} < \int_{\mathfrak{D}_m} \int_{\mathfrak{C}} < \epsilon.$$

We have now only to apply 3.

434. 1. We are now in position to prove the analogue of 76, viz. :

Let $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ be measurable. Let $\int_{\mathfrak{A}} f$ be convergent. Let the integrals $\int_{\mathfrak{C}} f$ converge in \mathfrak{B} , except possibly at a null set. Then

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}} \int_{\mathfrak{C}} f, \quad (1)$$

provided the integral on the right is convergent.

We follow along the line of proof in 76, and begin by taking $f \geq 0$ in \mathfrak{A} . By 423, we have

$$\int_{\mathfrak{A}_n} f = \int_{\mathfrak{B}} \int_{\mathfrak{C}_n} f;$$

hence

$$\int_{\mathfrak{A}} f = \lim_{n \rightarrow \infty} \int_{\mathfrak{B}} \int_{\mathfrak{C}_n} f. \quad (2)$$

Now $\epsilon > 0$ being small at pleasure,

$$\begin{aligned} -\epsilon + \int_{\mathfrak{B}} \int_{\mathfrak{C}} f &< \int_{\mathfrak{B}'} \int_{\mathfrak{C}} f, \quad \text{for } G > \text{some } G_0, \\ &\leq \int_{\mathfrak{B}'} \left\{ \int_{\mathfrak{C}_n} + \int_{\mathfrak{C}_n} \right\} \\ &\leq \int_{\mathfrak{B}} \int_{\mathfrak{C}_n} + \int_{\mathfrak{B}} \int_{\mathfrak{C}_n}. \end{aligned}$$

Since we have seen that we may regard \mathfrak{B} as the union of an enumerable set of complete sets, we see that the last term on the right $\doteq 0$, as $n \doteq \infty$, by 433, 4. Thus

$$\int_{\mathfrak{B}} \int_{\mathfrak{C}} \leq \lim \int_{\mathfrak{B}} \int_{\mathfrak{C}_n} = \int_{\mathfrak{A}}, \quad (3)$$

by 2). On the other hand,

$$\int_{\mathfrak{B}} \int_{\mathfrak{C}_n} \leq \int_{\mathfrak{B}} \int_{\mathfrak{C}}.$$

Hence

$$\int_{\mathfrak{A}} = \lim \int_{\mathfrak{B}} \int_{\mathfrak{C}_n} \leq \int_{\mathfrak{B}} \int_{\mathfrak{C}}. \quad (4)$$

From 3) and 4) we have 1), when $f \geq 0$.

The general case is now obviously true. For

$$\mathfrak{A} = \mathfrak{P} + \mathfrak{N},$$

where $f \geq 0$ in \mathfrak{P} , and < 0 in \mathfrak{N} . Here \mathfrak{P} and \mathfrak{N} are measurable. We have therefore only to use 1) for each of these fields and add the results.

2. The theorem 1 states that if

$$\int_{\mathfrak{A}} f, \quad \int_{\mathfrak{B}} \int_{\mathfrak{C}} f,$$

both converge, they are equal. *Hobson** in a remarkable paper on Lebesgue Integrals has shown that it is only necessary to assume the convergence of the first integral; the convergence of the second follows then as a necessary consequence.

* *Proceedings of the London Mathematical Society*, Ser. 2, vol. 8 (1900), p. 81.

435. We close this chapter by proving a theorem due to Lebesgue, which is of fundamental importance in the theory of Fourier's Series.

Let $f(x)$ be properly or improperly L -integrable in the interval $\Gamma = (a < b)$. Then

$$\begin{aligned}\lim_{\delta \rightarrow 0} J_f &= \lim_{\delta \rightarrow 0} \int_a^{\beta} |f(x + \delta) - f(x)| dx \\ &= \lim_{\delta \rightarrow 0} \int_a^{\beta} |\Delta f| dx = 0, \quad a < \beta < \beta + \delta < b.\end{aligned}\quad (1)$$

For in the first place,

$$J_f \leq \int_a^{\beta} |f(x + \delta)| dx + \int_a^{\beta} |f| dx \leq 2 \int_a^{\beta} |f| dx. \quad (2)$$

Next we note that

$$\begin{aligned}|f(x + \delta) - f(x)| - |g(x + \delta) - g(x)| \\ \leq |(f(x + \delta) - g(x + \delta)) - (f(x) - g(x))|.\end{aligned}$$

Hence

$$\begin{aligned}\int_a^{\beta} |\Delta f| dx - \int_a^{\beta} |\Delta g| dx &\leq \int_a^{\beta} |\Delta(f - g)| dx, \\ J_f - J_g &\leq J_{f-g}.\end{aligned}\quad (3)$$

From 2), 3) we have

$$J_f < J_g + 2 \int_a^{\beta} |f - g| dx. \quad (4)$$

Let now

$$\begin{aligned}g &= f && \text{for } |f| \leq G, \\ &= 0 && \text{for } |f| > G.\end{aligned}$$

Then by 4),

$$\begin{aligned}J_f &\leq J_g + 2 \int_a^{\beta} |f - g| dx \\ &\leq J_g + \epsilon',\end{aligned}$$

where ϵ' is small at pleasure, for G sufficiently large. Thus the theorem is established, if we prove it for a limited function, $|g(x)| < G$.

Let us therefore effect a division of the interval $\Gamma = (-G, G)$, of norm d , by interpolating the points

$$-G < c_1 < c_2 < \dots < G,$$

causing Γ to fall into the intervals

$$\gamma_1, \gamma_2, \gamma_3 \dots$$

Let $h_m = c_m$ for those values of x for which $g(x)$ falls in the interval γ_m , and $= 0$ elsewhere in \mathfrak{A} . Then

$$\begin{aligned} J_g &\leq \Sigma J_{h_i} + 2 \int_a^\beta (g - \Sigma h_i) dx \\ &\leq \Sigma J_{h_i} + 2 d\mathfrak{A} \\ &< \Sigma J_{h_i} + \epsilon', \quad \epsilon' \text{ small at pleasure,} \end{aligned}$$

for d sufficiently small.

Thus we have reduced the demonstration of our theorem to a function $h(x)$ which takes on but two values in \mathfrak{A} , say 0 and γ .

Let \mathfrak{E} be a $\sigma/4$ enclosure of the points where $h = \gamma$, while \mathfrak{F} may denote a finite number of intervals of \mathfrak{E} such that $\mathfrak{F} - \mathfrak{E} < \sigma/4$.

Let $\phi = \gamma$ in \mathfrak{E} , and elsewhere $= 0$; let $\psi = \gamma$ in \mathfrak{F} , and elsewhere $= 0$. Thus using 4),

$$\begin{aligned} J_h &\leq J_\phi + \int_a^\beta |h - \phi| \\ &\leq J_\phi + \frac{\sigma}{2} \gamma, \end{aligned}$$

since $h = \phi$ in (α, β) , except at points of measure $< \sigma/4$. Similarly

$$J_\psi < J_\psi + \frac{\sigma}{2} \gamma.$$

$$\text{Thus} \quad J_h \leq J_\psi + \sigma\gamma < J_\psi + \epsilon,$$

for σ sufficiently small.

Thus the demonstration is reduced to proving it for a ψ which is continuous, except at a finite number of points. But for such a function, it is obviously true.

CHAPTER XIII

FOURIER'S SERIES

Preliminary Remarks

436. 1. Let us suppose that the limited function $f(x)$ can be developed into a series of the type

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots \\ + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots \quad (1)$$

which is valid in the interval $\mathfrak{H} = (-\pi, \pi)$. If it is also known that this series can be integrated termwise, the coefficients a_n, b_n can be found at once as follows. By hypothesis

$$\int_{-\pi}^{\pi} f(x) dx = a_0 \int_{-\pi}^{\pi} dx + a_1 \int_{-\pi}^{\pi} \cos x dx + \dots \\ + b_1 \int_{-\pi}^{\pi} \sin x dx + \dots$$

As the terms on the right all vanish except the first, we have

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx. \quad (2')$$

Let us now multiply 1) by $\cos nx$ and integrate.

$$\int_{-\pi}^{\pi} f(x) \cos nx dx = a_0 \int_{-\pi}^{\pi} \cos nx dx + a_1 \int_{-\pi}^{\pi} \cos x \cos nx dx + \dots \\ + b_1 \int_{-\pi}^{\pi} \sin x \cos nx dx + \dots$$

Now
$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = 0, \quad m \neq n,$$

$$\int_{-\pi}^{\pi} \cos^2 nx dx = \pi,$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx dx = 0.$$

Thus all the terms on the right of the last series vanish except the one containing a_n . Hence

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx. \quad (2'')$$

Finally multiplying 1) by $\sin nx$, integrating, and using the relations

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = 0, \quad m \neq n,$$

$$\int_{-\pi}^{\pi} \sin^2 nx dx = \pi,$$

we get

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx. \quad (2''')$$

Thus under our present hypothesis,

$$\begin{aligned} f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u) du + \frac{1}{\pi} \sum_1^{\infty} \cos nx \int_{-\pi}^{\pi} f(u) \cos n u du \\ + \frac{1}{\pi} \sum_1^{\infty} \sin nx \int_{-\pi}^{\pi} f(u) \sin n u du. \end{aligned} \quad (3)$$

The series on the right is known as *Fourier's series*; the coefficients 2) are called *Fourier's coefficients* or *constants*. When the relation 3) holds for a set of points \mathfrak{B} , we say $f(x)$ can be developed in a Fourier's series in \mathfrak{B} , or Fourier's development is valid in \mathfrak{B} .

2. Fourier thought that every continuous function in \mathfrak{A} could be developed into a trigonometric series of the type 3). The demonstration he gave is not rigorous. Later *Dirichlet* showed that such a development is possible, provided the continuous function has only a finite number of oscillations in \mathfrak{A} . The function still regarded as limited may also have a finite number of *discontinuities of the first kind*, i.e. where

$$f(a+0) \quad , \quad f(a-0) \quad (4)$$

exist, but one at least is $\neq f(a)$.

At such a point a , Fourier's series converges to

$$\frac{1}{2} \{f(a+0) + f(a-0)\}.$$

Jordan has extended Dirichlet's results to functions having limited variation in \mathfrak{A} . Thus Fourier's development is valid in certain cases when f has an infinite number of oscillations or points of discontinuity. Fourier's development is also valid in certain cases when f is not limited in \mathfrak{A} , as we shall see in the following sections.

We have supposed that $f(x)$ is given in the interval $\mathfrak{A} = (-\pi, \pi)$. This restriction was made only for convenience. For if $f(x)$ is given in the interval $\mathfrak{B} = (a < b)$, we have only to change the variable by means of the relation

$$u = \frac{\pi(2x - a - b)}{b - a}.$$

Then when x ranges over \mathfrak{B} , u will range over \mathfrak{A} .

Suppose f is an *even function* in \mathfrak{A} ; its development in Fourier's series will contain only cosine terms. For

$$f(x) = \sum_0^{\infty} (a_n \cos nx + b_n \sin nx),$$

$$f(-x) = \sum_0^{\infty} (a_n \cos nx - b_n \sin nx).$$

Adding and remembering that $f(x) = f(-x)$ in \mathfrak{A} , we get

$$f(x) = \frac{1}{2} \sum_0^{\infty} a_n \cos nx, \quad f \text{ even}.$$

Similarly if f is *odd*, its development in Fourier's series will contain only sine terms;

$$f(x) = \frac{1}{2} \sum_1^{\infty} b_n \sin nx, \quad f \text{ odd}.$$

Let us note that if $f(x)$ is given only in $\mathfrak{B} = (0, \pi)$, and has limited variation in \mathfrak{B} , we may develop f either as a sine or a cosine series in \mathfrak{B} . For let

$$\begin{aligned} g(x) &= f(x) \quad , \quad x \text{ in } \mathfrak{B} \\ &= f(-x) \quad , \quad x \text{ in } (-\pi, 0). \end{aligned}$$

Then g is an even function in \mathfrak{A} and has limited variation. Using Jordan's result, we see g can be developed in a cosine series valid in \mathfrak{A} . Hence f can be developed in a cosine series valid in \mathfrak{B} .

In a similar manner, let

$$\begin{aligned} h(x) &= f(x) \quad , \quad x \text{ in } \mathfrak{B} \\ &= -f(-x) \quad , \quad -\pi \leq x < 0. \end{aligned}$$

Then h is an odd function in \mathfrak{A} , and Fourier's development contains only sine terms.

Unless $f(0) = 0$, the Fourier series will not converge to $f(0)$ but to 0, on account of the discontinuity at $x = 0$. The same is true for $x = \pi$.

If f can be developed in Fourier's series valid in $\mathfrak{A} = (-\pi, \pi)$, the series \mathfrak{B}) will converge for all x , since its terms admit the period 2π . Thus \mathfrak{B}) will represent $f(x)$ in \mathfrak{A} , but will not represent it unless f also admits the period 2π . The series \mathfrak{B}) defines a periodic function admitting 2π as a period.

EXAMPLES

437. We give now some examples. They may be verified by the reader under the assumption made in 436. Their justification will be given later

Example 1. $f(x) = x \quad , \quad \text{for } -\pi \leq x \leq \pi.$

Then

$$x = 2 \left\{ \frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right\}.$$

If we set $x = \frac{\pi}{2}$, we get *Leibnitz's formula*,

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Example 2. $f(x) = x \quad , \quad 0 \leq x \leq \pi$

$$= -x \quad , \quad -\pi \leq x \leq 0.$$

Then

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \left\{ \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right\}.$$

If we set $x = 0$, we get

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Example 3.

$$\begin{aligned} f(x) &= 1 \quad , \quad 0 < x < \pi \\ &= 0 \quad , \quad x = 0, \pm \pi \\ &= -1 \quad , \quad -\pi < x < 0. \end{aligned}$$

Then

$$f(x) = \frac{4}{\pi} \left\{ \frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right\}.$$

Example 4.

$$\begin{aligned} f(x) &= x \quad , \quad 0 \leq x \leq \frac{\pi}{2} \\ &= \pi - x \quad , \quad \frac{\pi}{2} \leq x < \pi. \end{aligned}$$

By defining f as an odd function, it can be developed in a sine series, valid in $(0, \pi)$. We find

$$f(x) = \frac{4}{\pi} \left\{ \frac{\sin x}{1} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \dots \right\}.$$

Example 5.

$$\begin{aligned} f(x) &= 1 \quad , \quad 0 \leq x \leq \frac{\pi}{2} \\ &= -1 \quad , \quad \frac{\pi}{2} < x < \pi. \end{aligned}$$

By defining f as an even function, we get a development in cosines,

$$f(x) = \frac{4}{\pi} \left\{ \frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \dots \right\},$$

valid in $(0, \pi)$.

Example 6.

$$f(x) = \frac{1}{2}(\pi - x) \quad , \quad 0 < x \leq \pi.$$

By defining f as an odd function we get a development in sines,

$$f(x) = \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots$$

valid in $(-\pi, \pi)$.

Example 7.

$$\begin{aligned} \text{Let } f(x) &= \frac{\pi}{3} \quad , \quad 0 < x < \frac{\pi}{3} \\ &= 0 \quad , \quad \frac{\pi}{3} < x < \frac{2\pi}{3} \\ &= -\frac{\pi}{3} \quad , \quad \frac{2\pi}{3} < x < \pi. \end{aligned}$$

Developing f as a sine series, we get

$$f(x) = \sin 2x + \frac{\sin 4x}{2} + \frac{\sin 8x}{4} + \dots$$

valid in $(0, \pi)$.

Example 8. $f(x) = e^x$, in $(-\pi, \pi)$.

• We find

$$f(x) = \frac{2 \sinh \pi}{\pi} \left\{ \frac{1}{2} - \frac{1}{1+1^2} \cos x + \frac{1}{1+2^2} \cos 2x - \frac{1}{1+3^2} \cos 3x + \dots \right. \\ \left. + \frac{1}{1+1^2} \sin x - \frac{2}{1+2^2} \sin 2x + \frac{3}{1+3^2} \sin 3x - \dots \right\}$$

valid for $-\pi < x < \pi$.

Example 9. We find

$$\cos \mu x = \frac{2\mu}{\pi} \sin \pi \mu \left\{ \frac{1}{2\mu^2} - \frac{\cos x}{\mu^2 - 1} + \frac{\cos 2x}{\mu^2 - 2^2} - \frac{\cos 3x}{\mu^2 - 3^2} + \dots \right\}$$

valid for $-\pi \leq x \leq \pi$, $\mu^2 \neq 1, 2^2, 3^2, \dots$

Let us set $x = \pi$, and replace μ by x ; we get

$$\frac{\pi}{2x} \cot \pi x = \frac{1}{2x^2} + \frac{1}{x^2 - 1^2} + \frac{1}{x^2 - 2^2} + \frac{1}{x^2 - 3^2} + \dots$$

a decomposition of $\cot \pi x$ into partial fractions, a result already found in 216.

Example 10. We find

$$\sin x = \frac{2}{\pi} \left\{ 1 - \frac{2 \cos 2x}{1 \cdot 3} - \frac{2 \cos 4x}{3 \cdot 5} - \frac{2 \cos 6x}{5 \cdot 7} \dots \right\},$$

valid for $0 \leq x \leq \pi$.

Summation of Fourier's Series

438. In order to justify the development of $f(x)$ in Fourier's series F , we will actually sum the F series and show that it converges to $f(x)$ in certain cases. To this end let us suppose that $f(x)$ is given in the interval $\mathfrak{A} = (-\pi, \pi)$, and let us extend f by giving it the period 2π . Moreover, at the points of discontinuity of the first kind, let us suppose

$$f(x) = \frac{1}{2} \{ f(x+0) + f(x-0) \}.$$

then the function

$$\phi(u) = f(x + 2u) + f(x - 2v) - 2f(x)$$

continuous at $u = 0$, and has the value 0, at points of continuity, and at points of discontinuity of 1° kind of f . Finally let us suppose that f is (properly or improperly) L -integrable in \mathfrak{A} ; this condition being necessary, in order to make the Fourier coefficients a_n, b_n have a sense.

Let

$$\begin{aligned} F = F(x) &= \frac{1}{2} a_0 + a_1 \cos x + a_2 \cos 2x + \dots \\ &\quad + b_1 \sin x + b_2 \sin 2x + \dots \quad (1) \\ &= \frac{1}{2} a_0 + \sum_1^{\infty} (a_n \cos nx + b_n \sin nx), \end{aligned}$$

where we will now write

$$a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx dx, \quad (2')$$

$$b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx dx. \quad (2'')$$

Since $f(x)$ is periodic, the coefficients a_n, b_n have the same value whatever c is chosen. If we make $c = -\pi$, these integrals reduce to those given in 436.

We may write

$$\begin{aligned} F &= \frac{1}{\pi} \int_c^{c+2\pi} f(t) dt \left\{ \frac{1}{2} + \sum_1^{\infty} (\cos nx \cos nt + \sin nx \sin nt) \right\} \\ &= \frac{1}{\pi} \int_c^{c+2\pi} \left\{ \frac{1}{2} + \sum_1^{\infty} \cos n(t-x) \right\} f(t) dt. \quad (3) \end{aligned}$$

as

$$F_n = \frac{1}{\pi} \int_c^{c+2\pi} P_n \cdot f(t) dt,$$

are

$$P_n = \frac{1}{2} + \sum_1^n \cos m(t-x). \quad (4)$$

provided

$$\sin \frac{1}{2}(t-x) \neq 0, \quad (5)$$

we may write

$$\begin{aligned} &= \frac{1}{2 \sin \frac{1}{2}(t-x)} \left\{ \sin \frac{1}{2}(t-x) + \sum_1^m 2 \sin \frac{1}{2}(t-x) \cos m(t-x) \right\} \\ &= \frac{1}{2 \sin \frac{1}{2}(t-x)} \left[\sin \frac{1}{2}(t-x) \right. \\ &\quad \left. + \sum_1^n \left\{ \sin \frac{2m+1}{2}(t-x) - \sin \frac{2m-1}{2}(t-x) \right\} \right]. \end{aligned}$$

Thus

$$P_n = \frac{\sin \frac{1}{2}(2n+1)(t-x)}{2 \sin \frac{1}{2}(t-x)}, \quad (6)$$

if 5) holds. Let us see what happens when 5) does not hold. In this case $\frac{1}{2}(t-x)$ is a multiple of π . As both t and x lie in $(c, c+2\pi)$, this is only possible for three singular values:

$$t = x \quad ; \quad t = c, \quad x = c + 2\pi \quad ; \quad t = c + 2\pi, \quad x = c.$$

For these singular values 4) gives

$$P_n = \frac{2n+1}{2}. \quad (7)$$

As P_n is a continuous function of t, x , the expression on the right of 6) must converge to the value 7) as x, t converge to these singular values. We will therefore assign to the expression on the right of 6) the value 7), for the above singular values. Then in all cases

$$F_n = \frac{1}{\pi} \int_c^{c+2\pi} \frac{\sin \frac{1}{2}(2n+1)(t-x)}{2 \sin \frac{1}{2}(t-x)} f(t) dt.$$

Let us set

$$2n+1 = \nu, \quad t-x = u.$$

Then

$$F_n = \frac{1}{\pi} \int_{\frac{1}{2}(c-x)}^{\frac{1}{2}(c-x)+\pi} f(x+2u) \frac{\sin \nu u}{\sin u} du.$$

Let us choose c so that

$$c-x = -\pi,$$

then

$$\pi F_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \int_{-\frac{\pi}{2}}^0 + \int_0^{\frac{\pi}{2}}.$$

Replacing u by $-u$ in the first integral on the right, it becomes

$$\int_0^{\frac{\pi}{2}} f(x-2u) \frac{\sin \nu u}{\sin u} du.$$

Thus we get

$$F_n = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \{f(x+2u) + f(x-2u)\} \frac{\sin \nu u}{\sin u} du. \quad (8)$$

Let us now introduce the term $-2f(x)$ under the sign of integration in order to replace the brace by $\phi(u)$. To this end let us

Let x be an arbitrary but fixed value and consider the Fourier series for the function

$$g(t) = f(x), \text{ a constant.}$$

If we denote the Fourier series corresponding to the g function

$$G = \frac{1}{2}g_0 + g_1 \cos t + g_2 \cos 2t + \dots \\ + h_1 \sin t + h_2 \sin 2t + \dots$$

have

$$g_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x) dt = 2f(x),$$

$$g_n = \frac{f(x)}{\pi} \int_c^{c+2\pi} \cos nt dt = 0,$$

$$h_n = \frac{f(x)}{\pi} \int_c^{c+2\pi} \sin nt dt = 0.$$

Thus the sum of the first $n+1$ terms of the Fourier series belonging to $g(t)$ reduces to

$$G_n = f(x). \quad (9)$$

But this sum is also given by 8), if we replace

$$f(x+2u) + f(x-2u) \\ g(x+2u) + g(x-2u) = 2f(x),$$

since g is a constant. We get thus

$$G_n = \frac{1}{\pi} \int_0^{\pi} 2f(t) \frac{\sin nu}{\sin u} du. \quad (10)$$

Let us therefore subtract $f(x)$ from both sides of 8), using 9),

$$D_n(x) - f(x) = \frac{1}{\pi} \int_0^{\pi} \{f(x+2u) + f(x-2u) - 2f(x)\} \frac{\sin nu}{\sin u} du.$$

Setting

$$D_n(x) = \pi \{F_n(x) - f(x)\}, \quad (11)$$

we have

$$D_n(x) = \int_0^{\pi} \phi(u) \frac{\sin nu}{\sin u} du. \quad (12)$$

We have thus the theorem:

For the Fourier Series to converge to $f(x)$ at the point x , it is necessary and sufficient that $D_n(x) \rightarrow 0$, as $n \rightarrow \infty$.

*Validity of Fourier's Development**

439. The integral on the right side of 438, 12), on which the validity of Fourier's development at the point x depends, is a special case of the integral

$$J_n = \int_{\mathfrak{B}} g(u) \sin n u d u \quad , \quad \mathfrak{B} = (a < b). \quad (1)$$

In fact J_n goes over into D_n , if we set

$$g = \frac{\phi}{\sin u} \quad , \quad a = 0, \quad b = \frac{\pi}{2}.$$

To evaluate J_n let us break \mathfrak{B} up into the intervals

$$\mathfrak{B}_0 = \left(a, a + \frac{\pi}{n}\right) \quad , \quad \mathfrak{B}_1 = \left(a + \frac{\pi}{n}, a + \frac{2\pi}{n}\right) \cdots \mathfrak{B}_r = \left(a + r \frac{\pi}{n}, b\right).$$

These intervals are equal except the last, which is shorter than the others unless $b - a$ is a multiple of π/n . We have thus

$$J_n = \int_{\mathfrak{B}_0} + \int_{\mathfrak{B}_1} + \cdots + \int_{\mathfrak{B}_r}.$$

If we set

$$v = u + \frac{\pi}{n},$$

we see that while v ranges over \mathfrak{B}_{2s} , u ranges over \mathfrak{B}_{2s-1} . This substitution enables us to replace the integrals over \mathfrak{B}_{2s} by those over \mathfrak{B}_{2s-1} , since

$$\int_{\mathfrak{B}_{2s}} g(v) \sin n v d v = - \int_{\mathfrak{B}_{2s-1}} g\left(u + \frac{\pi}{n}\right) \sin n u d u.$$

Hence grouping the integrals in pairs, we get

$$\begin{aligned} J_n = \int_{\mathfrak{B}_0} g(u) \sin n u d u + \sum_s \int_{\mathfrak{B}_{2s-1}} \left\{ g(u) - g\left(u + \frac{\pi}{n}\right) \right\} \sin n u d u \\ + \int_{\mathfrak{B}_r} g(u) \sin n u d u, \end{aligned}$$

* The presentation given in 430-448 is due in the main to *Lebesgue*. Cf. his classic paper, *Mathematische Annalen*, vol. 61 (1905), p. 261. Also his *Leçons sur les Séries Trigonométriques*, Paris, 1906.

where \mathfrak{B}' is \mathfrak{B}_r or $\mathfrak{B}_{r-1} + \mathfrak{B}_r$, depending on the parity of r . Now

$$\left| \int_{\mathfrak{B}_0}^{\cdot} |g| \right| \leq \int_{\mathfrak{B}_0}^{\cdot} |g|, \quad (2)$$

$$\begin{aligned} \sum \int_{\mathfrak{B}_{2s-1}}^{\cdot} \left\{ g(u) - g\left(u + \frac{\pi}{n}\right) \right\} \sin nu du & \left| \right. \\ & < \sum \int_{\mathfrak{B}_{2s-1}}^{\cdot} \left| g\left(u + \frac{\pi}{n}\right) - g(u) \right| du \\ & \leq \int_{u=1, \frac{\pi}{n}}^{u=1, \frac{\pi}{n}} \left| g\left(u + \frac{\pi}{n}\right) - g(u) \right| du. \end{aligned} \quad (3)$$

$$\left| \int_{\mathfrak{B}'}^{\cdot} |g| \right| \leq \int_{\mathfrak{B}'}^{\cdot} |g|. \quad (4)$$

Thus $J_n \doteq 0$, if the three integrals 2), 3), 4) $\doteq 0$. Moreover, these three integrals are *uniformly evanescent* with respect to some point set $\mathfrak{E} \leq \mathfrak{B}$, J_n is also uniformly evanescent in \mathfrak{E} . In particular we note the theorem

$J_n \doteq 0$, if g is L -integrable in \mathfrak{B} .

We are now in a position to draw some important conclusions with respect to Fourier's series.

440. 1. Let $f(x)$ be L -integrable in $(c, c + 2\pi)$. Then the Fourier constants $a_n, b_n \doteq 0$, as $n \doteq \infty$.

For

$$a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nax dx$$

a special case of the J_n integral. As f is L -integrable, we need only apply the theorem at the close of the last article. Similar reasoning applies to b_n .

2. For a given value of x in $\mathfrak{A} = (-\pi, \pi)$ let

$$\psi(u) = \frac{\phi(u)}{\sin u}, \quad (1)$$

ψ L -integrable in $\mathfrak{B} = \left(0, \frac{\pi}{2}\right)$. Then Fourier's development is valid at the point x .

For by 438, Fourier's series $\doteq f(x)$ at the point x , if $D_n(x) \doteq 0$. But D_n is a special case of J_n for which the g function is integrable. We thus need only apply 439.

3. For a given x in $\mathfrak{A} = (-\pi, \pi)$, let

$$\chi(u) = \frac{\phi(u)}{u} \quad (2)$$

be L -integrable in $\mathfrak{B} = (0, \frac{\pi}{2})$. Then Fourier's development is valid at the point x .

For let $\delta > 0$, then

$$\begin{aligned} \int_0^\delta |\psi| du &= \int_0^\delta \left| \frac{\phi(u)}{\sin u} \right| \leq \int_0^\delta \frac{|\phi(u)|(1+\eta)}{u} du \\ &\leq (1+\eta) \int_0^\delta |\chi(u)| du \\ &\doteq 0, \quad \text{as } \delta \doteq 0, \quad \text{by hypothesis.} \end{aligned}$$

4. For a given x in $\mathfrak{A} = (-\pi, \pi)$, let

$$\omega(u) = \frac{f(x+u) - f(x)}{u} \quad (3)$$

be L -integrable in \mathfrak{A} . Then Fourier's development is valid at the point x .

For

$$\begin{aligned} \chi(u) &= \frac{f(x+2u) - f(x)}{u} + \frac{f(x-2u) - f(x)}{u} \\ &= 2[\omega(2u) + \omega(-2u)]. \end{aligned}$$

Thus χ is L -integrable in $(0, \frac{\pi}{2})$, as it is the difference of two integrable functions.

441. (Lebesgue). For a given x in $\mathfrak{A} = (-\pi, \pi)$ let

$$\begin{aligned} 1^\circ \quad \lim_{n=\infty} n \int_0^{\frac{\pi}{n}} |\phi(u)| du &= 0; \\ 2^\circ \quad \lim_{\delta=0} \int_\delta^\eta |\psi(u+\delta) - \psi(u)| du &= 0 \end{aligned}$$

for some η such that

$$0 < \delta < \eta \leq \frac{\pi}{2}.$$

Then Fourier's development is valid at the point x .

For as we have seen,

$$|D_n| \leq \int_0^{\frac{\pi}{\nu}} \left| \frac{\phi(u)}{\sin u} \sin \nu u \right| du + \int_{\frac{\pi}{\nu}}^{\frac{\pi}{2}} \left| \psi\left(u + \frac{\pi}{n}\right) - \psi(u) \right| du \\ + \int_{\beta_n}^{\frac{\pi}{2}} \left| \frac{\phi(u)}{\sin u} \sin \nu u \right| du \leq D' + D'' + D''',$$

where β_n is a certain number which $\doteq \frac{\pi}{2}$, as $n \doteq \infty$.

Let us first consider D' . Since $0 \leq u \leq \frac{\pi}{\nu}$, we have $0 \leq \nu u \leq \pi$. Hence

$$\frac{\sin \nu u}{\sin u} = \frac{\nu u - \frac{\nu^3 u^3}{6} + \sigma \frac{\nu^5 u^5}{24}}{u - \frac{u^3}{6} + \tau \frac{u^5}{24}}, \quad 0 < \sigma, \tau \leq 1 \\ = \nu \frac{1 - \frac{\nu^2 u^2}{6} \left(1 - \frac{\sigma \nu u}{4}\right)}{1 - \frac{u^2}{6} \left(1 - \frac{\tau u}{4}\right)} = \nu \frac{1 - s \frac{u^2}{6}}{1 - t \frac{u^2}{6}} \\ \leq \nu, \text{ provided } s \geq t.$$

But this is indeed so. For

$$1 - \frac{\nu \sigma u}{4} \geq 1 - \frac{\pi}{4}.$$

Hence

$$s \geq \nu^2 \left(1 - \frac{\pi}{4}\right) > 1 > t, \quad \text{if } \nu \geq 5.$$

Thus

$$D' < \nu \int_0^{\frac{\pi}{\nu}} |\phi| du \doteq 0, \text{ by hypothesis.}$$

We now turn to D'' . We have

$$D'' = \int_{\frac{\pi}{\nu}}^{\frac{\pi}{2}} = \int_{\frac{\pi}{\nu}}^{\frac{\pi}{\eta}} + \int_{\frac{\pi}{\eta}}^{\frac{\pi}{2}}, \quad \frac{\pi}{\nu} < \eta < \frac{\pi}{2}.$$

Now f being L -integrable,

$$\left| \psi\left(u + \frac{\pi}{n}\right) - \psi(u) \right|$$

is L -integrable in $\left(\eta, \frac{\pi}{2}\right)$. Thus

$$\lim_{n=\infty} \int_{\eta}^{\frac{\pi}{2}} = 0.$$

But by condition 2°,

$$\lim_{\nu=\infty} \int_{\frac{\pi}{\nu}}^{\eta} = 0.$$

Thus

$$\lim_{\delta=0} D'' = 0.$$

Finally we consider D''' . But the integrand is an integrable function in $\left(\beta, \frac{\pi}{2}\right)$. Thus it $\doteq 0$ as $n \doteq \infty$.

442. 1. *The validity of Fourier's development at the point x depends only on the nature of f in a vicinity of x , of norm δ as small as we please.*

For the conditions of the theorem in 441 depend only on the value of f in such a vicinity.

2. Let us call a point x at which the function

$$\phi(u) = f(x + 2u) + f(x - 2u) - 2f(x)$$

is continuous at $u = 0$, and has the value 0, a *regular point*.

In 438, we saw that if x is a point of discontinuity of the first kind for $f(x)$, then x is a regular point.

3. *Fourier's development is valid at a regular point x , provided for some η*

$$\lim_{\delta=0} \int_{\delta}^{\eta} |\psi(u + \delta) - \psi(u)| du = 0, \quad 0 < \delta < \eta \leq \frac{\pi}{2}.$$

For at a regular point x , $\phi(u)$ is continuous at $u = 0$, and $= 0$ for $u = 0$. Now

$$\lim_{h=0} \frac{1}{h} \int_0^h |\phi(u)| du = |\phi(0)| = 0.$$

Thus

$$n \int_0^{\frac{\pi}{n}} |\phi(u)| du = \pi \cdot \frac{1}{\pi} \cdot \int_0^{\frac{\pi}{n}} |\phi| du$$

$$\doteq \pi |\phi(0)| = 0.$$

Hence condition 1° of 441 is satisfied.

Limited Variation

443. 1. Before going farther we must introduce a few notions relative to the variation of a function $f(x)$ defined over an interval $\mathfrak{A} = (a < b)$. Let us effect a division D of \mathfrak{A} into subintervals, r interpolating a finite number of points $a_1 < a_2 < \dots$. The sum

$$V_D = \sum |f(a_i) - f(a_{i+1})| \quad (1)$$

called *the variation of f in \mathfrak{A} for the division D* . If

$$\text{Max } V_D \quad (2)$$

finite with respect to the class of all finite divisions of \mathfrak{A} , we say *has finite variation in \mathfrak{A}* . When 2) is finite, we denote its value by

$$\text{Var } f, \text{ or } V_f, \text{ or } V$$

and call it *the variation of f in \mathfrak{A}* .

We shall show in § 5 that finite variation means the same thing as limited variation introduced in I, 509. We use the term finite variation in sections 1 to 4 only for clearness.

2. A most important property of functions having finite variation is brought out by the following geometric consideration.

Let us take two monotone increasing curves A, B such that one of them crosses the other a finite or infinite number of times. If $f(x), g(x)$ are the continuous functions having these curves as graphs, it is obvious that

$$d(x) = f(x) - g(x)$$

is a continuous function which changes its sign, when the curves A, B cross each other. Thus we can construct functions in infinite variety, which oscillate infinitely often in a given interval, and which are the difference of two monotone increasing functions.

For simplicity we have taken the curves A, B continuous. A moment's reflection will show that this is not necessary.

Since $d(x)$ is the difference of two monotone increasing functions, its variation is obviously finite. Jordan has proved the following *fundamental theorem*.

3. If $f(x)$ has finite variation in the interval $\mathfrak{A} = (a < b)$, there exists an infinity of limited monotone increasing functions $g(x), h(x)$ such that

$$f = g - h. \quad (1)$$

For let D be a finite division of \mathfrak{A} . Let

$$\begin{aligned} P_D &= \text{sum of terms } \{f(a_{m+1}) - f(a_m)\} \text{ which are } \geq 0, \\ -N_D &= \dots \dots \dots < 0. \end{aligned}$$

Then
$$V_D = \Sigma |f(a_{m+1}) - f(a_m)| = P_D + N_D. \quad (2)$$

Also

$$\{f(a_1) - f(a)\} + \{f(a_2) - f(a_1)\} + \dots + \{f(b) - f(a_n)\} = P_D - N_D.$$

On the left the sum is telescopic, hence

$$f(b) - f(a) = P_D - N_D. \quad (3)$$

From 2), 3) we have

$$V_D = 2 P_D + f(a) - f(b) = 2 N_D + f(b) - f(a). \quad (4)$$

Let now
$$\text{Max } P_D = P, \quad \text{Max } N_D = N$$

with respect to the class of finite divisions D .

We call them the *positive* and *negative variation* of $f(x)$ in \mathfrak{A} . Then 4) shows that

$$V = 2 P + f(a) - f(b), \quad V = 2 N + f(b) - f(a). \quad (5)$$

Adding these, we get
$$V = P + N. \quad (6)$$

From 5) we have

$$f(b) - f(a) = P - N. \quad (7)$$

Instead of the interval $\mathfrak{A} = (a < b)$, let us take the interval $(a < x)$, where x lies in \mathfrak{A} . Replacing b by x in 7), we have

$$f(x) - f(a) = P(x) - N(x). \quad (8)$$

Obviously $P(x)$, $N(x)$ are monotone increasing functions.
 Let $\mu(x)$ be a monotone increasing function in \mathfrak{A} . If we set

$$\begin{aligned} g(x) &= f(a) + P(x) + \mu(x) \\ h(x) &= N(x) + \mu(x), \end{aligned} \quad (9)$$

we get 1) from 8) at once.

4. From 8) we have

$$\begin{aligned} |f(x)| &\leq |f(a)| + P(x) + N(x) \\ &\leq |f(a)| + V(x). \end{aligned} \quad (10)$$

5. We can now show that *when $f(x)$ has finite variation in the interval $\mathfrak{A} = (a < b)$ it has limited variation and conversely.*

For if f has finite variation in \mathfrak{A} we can set

$$f(x) = \phi(x) - \psi(x),$$

where ϕ, ψ are monotone increasing in \mathfrak{A} . Then if \mathfrak{A} is divided into the intervals $\delta_1, \delta_2 \dots$ we have

$$\text{Osc } f \leq \text{Osc } \phi + \text{Osc } \psi, \quad \text{in } \delta_i.$$

But

$$\text{Osc } \phi = \Delta \phi, \quad \text{Osc } \psi = \Delta \psi, \quad \text{in } \delta_i$$

since these functions are monotone. Hence summing over all the intervals δ_i ,

$$\begin{aligned} \Sigma \text{Osc } f &\leq \Sigma \Delta \phi + \Sigma \Delta \psi \\ &\leq \{\phi(b) - \phi(a)\} + \{\psi(b) - \psi(a)\} \\ &\leq \text{some } M, \text{ for any division.} \end{aligned}$$

Hence f has limited variation.

If f has limited variation in \mathfrak{A} ,

$$|\Delta f| \leq \text{Osc } f, \quad \text{in } \delta_i.$$

Hence

$$\Sigma |\Delta f| \leq \Sigma \text{Osc } f \leq \text{some } M.$$

Hence f has finite variation.

6. *If $f(x)$ has limited variation in the interval \mathfrak{A} , its points of discontinuity form a pantactic set in \mathfrak{A} .*

This follows from 5, and I, 508.

7. Let $a < b < c$; then if f has finite variation in (a, c) ,

$$V_{a,b}f + V_{b,c}f = V_{a,c}f, \quad (11)$$

where $V_{a,b}$ means the variation of f in the interval (a, b) , etc.

For

$$V_{ac}f = \text{Max } V_Df$$

with respect to the class of all finite divisions D of (a, c) . The divisions D fall into two classes:

- 1° those divisions E containing the point b ,
- 2° the divisions F which do not.

Let Δ be a division obtained by interpolating one or more points in the interval. Obviously

$$V_{\Delta}f \geq V_Df.$$

Let now G be obtained from a division F by adding the point b . Then

$$V_Gf \geq V_Ff.$$

Hence

$$\text{Max}_E V_E \geq \text{Max}_F V_F.$$

Hence to find $V_{a,c}f$, we may consider only the class E . Let now E_1 be a division of (a, b) , and E_2 a division of (b, c) . Then $E_1 + E_2$ is a division of class E . Conversely each division of class E gives a division of (a, b) , (b, c) . Now

$$V_Ef = V_{E_1}f + V_{E_2}f.$$

From this 11) follows at once.

444. We establish now a few simple relations concerning the variation of two functions in an interval $\mathfrak{A} = (a < b)$.

$$1. \quad V(f+g) = Vf + Vg. \quad (1)$$

$$\text{For} \quad \Sigma |(f_{i+1} + g) - (f_i + g)| = \Sigma |f_{i+1} - f_i|,$$

where for brevity we set

$$f_i = f(a_i).$$

$$2. \quad V(cf) = |c| Vf. \quad (2)$$

For

$$\Sigma |cf_{i+1} - cf_i| = |c| \Sigma |f_{i+1} - f_i|.$$

Let f, g be monotone increasing functions in \mathfrak{A} . Then

$$V(f+g) = Vf + Vg. \quad (3)$$

$$\begin{aligned} \text{or } \Sigma |(f_{i+1} + g_{i+1}) - (f_i + g_i)| &= \Sigma |(f_{i+1} - f_i) + (g_{i+1} - g_i)| \\ &= \Sigma |f_{i+1} - f_i| + \Sigma |g_{i+1} - g_i|. \end{aligned}$$

For any two functions f, g having limited variation,

$$V(f+g) \leq Vf + Vg. \quad (4)$$

Let f, f_1 have limited variation in $\mathfrak{A} = (a, b)$.

$$\begin{aligned} \text{Let } \alpha &= |f(a)|, \quad \alpha_1 = |f_1(a)|. \\ \text{Then } V(ff_1) &\leq (\alpha + Vf)(\alpha_1 + Vf_1). \end{aligned} \quad (5)$$

or by 443, 8) we have

$$f = P - N + A, \quad f_1 = P_1 - N_1 + A_1,$$

$$\text{or } A = f(a), \quad A_1 = f_1(a).$$

Thus

$$= PP_1 - PN_1 + PA_1 - NP_1 + NN_1 - NA_1 + AP_1 - AN_1 + AA_1.$$

Hence by 2, 4,

$$\begin{aligned} Vff_1 &\leq VP P_1 + VP N_1 + VP A_1 + \dots \\ &\leq V(P P_1 + P N_1 + \dots), \quad \text{by 3} \\ &\leq P P_1 + P N_1 + P \alpha_1 + \dots \\ &\leq (P + N + \alpha)(P_1 + N_1 + \alpha_1). \end{aligned}$$

$$\text{but } Vf = P + N, \quad \text{hence, etc.}$$

15. Fourier's development is valid at the regular point x , if there

is a $0 < \xi \leq \frac{\pi}{2}$, such that in $(0, \xi)$ the variation $V(u)$ of $\psi(u)$

is limited, and such that $u V(u) \doteq 0, u \doteq 0$.

By 442, we have only to show that

$$\Psi = \int_{\delta}^{\eta} |\psi(u + \delta) - \psi(u)| du \quad 0 < \delta < \eta \leq \frac{\pi}{2}$$

vanishes with δ .

Let us *first* suppose that $\psi(u)$ is monotone in some $(0, \xi)$, say monotone increasing. Similar reasoning will apply, if it is monotone decreasing. Then, taking $0 < \eta + \delta < \xi$,

$$\Psi = \int_{\delta}^{\eta} \{\psi(u + \delta) - \psi(u)\} du = \int_{\delta}^{\eta} \psi(u + \delta) du - \int_{\delta}^{\eta} \psi(u) du.$$

In the second integral from the end, set $v = u + \delta$.

$$\text{Then} \quad \int_{\delta}^{\eta} \psi(u + \delta) du = \int_{2\delta}^{\eta+\delta} \psi(v) dv.$$

$$\begin{aligned} \text{Hence,} \quad \Psi &= \int_{2\delta}^{\eta+\delta} \psi(u) du - \int_{\delta}^{\eta} \psi(u) du \\ &= \int_{2\delta}^{\eta+\delta} - \int_{\delta}^{2\delta} - \int_{2\delta}^{\eta}. \end{aligned}$$

$$\text{Thus} \quad |\Psi(u)| \leq \int_{\delta}^{2\delta} |\psi| du + \int_{\eta}^{\eta+\delta} |\psi| du = \Psi_1 + \Psi_2.$$

We will consider the integrals on the right separately. Let

$$\phi_m = \text{Max} |\phi|, \quad \text{in } (\delta, 2\delta).$$

Then

$$\Psi_1 \leq \int_{\delta}^{2\delta} \frac{|\phi|}{\sin u} du \leq \phi_m \int_{\delta}^{2\delta} \frac{du}{\sin u}.$$

Now

$$\sin u = u - \sigma' u^2, \quad 0 < \sigma' < \frac{1}{2}.$$

Hence,

$$\frac{1}{\sin u} = \frac{1}{u} + \sigma u, \quad |\sigma| < \text{some } M.$$

Thus,

$$\begin{aligned} \Psi_1 &\leq \phi_m \left\{ \int_{\delta}^{2\delta} \frac{du}{u} + M \int_{\delta}^{2\delta} u du \right\} \\ &< \phi_m \{ \log 2 + M' \delta^2 \} \\ &\doteq 0, \quad \text{as } \delta \doteq 0, \quad \text{since } \phi(u) \doteq 0, \end{aligned}$$

as x is a regular point.

We turn now to Ψ_2 . In $(\eta, \eta + \delta)$, δ, η sufficiently small,

$$\sin u > u - \frac{1}{8} u^2 > \eta(1 - \eta^2).$$

Thus, if $\phi_\mu = \text{Max } |\phi| \text{ in } (\eta, \eta + \delta)$,

$$\Psi_2 \leq \frac{\phi_\mu}{\eta(1-\eta)} \int_{\eta}^{\eta+\delta} du = \frac{\phi_\mu \delta}{\eta(1-\eta)} \doteq 0$$

with δ .

Thus, when ψ is monotone in some $(0, \xi)$, Fourier's development is valid. But obviously when ψ is monotone, the condition that $uV(u) \doteq 0$ is satisfied. Our theorem is thus established in this case.

Let us now consider the case that the variation $V(u)$ of ψ is limited in (u, ξ) .

From 443, 10), we have

$$|\psi(u)| \leq |\psi(\xi)| + V(u).$$

As before we have

$$|\Psi| < \int_{\delta}^{2\delta} |\psi| du + \int_{\eta}^{\eta+\delta} |\psi| du = \Psi_1 + \Psi_2.$$

By hypothesis there exists for each $\epsilon > 0$, a $\delta_0 > 0$, such that

$$uV(u) \leq \epsilon, \quad \text{for any } 0 < u \leq \delta_0.$$

Hence,

$$V(u) \leq \frac{\epsilon}{u}, \quad 0 < u \leq \delta_0.$$

Thus,

$$\begin{aligned} \Psi_1 &\leq \int_{\delta}^{2\delta} |\psi(\xi)| du + \int_{\delta}^{2\delta} V(u) du \\ &\leq |\psi(\xi)| \delta + \epsilon \int_{\delta}^{2\delta} \frac{du}{u} \\ &= |\psi(\xi)| \delta + \epsilon \log 2. \end{aligned}$$

Let us turn now to Ψ_2 . Since $V(u)$ is the sum of two limited monotone decreasing functions P, N in (u, ξ) , it is integrable.

Thus,

$$\Psi_2 \leq |\psi(\xi)| \int_{\eta}^{\eta+\delta} du + \int_{\eta}^{\eta+\delta} V(u) du \leq \delta \{ |\psi(\xi)| + V(\eta) \}$$

is evanescent with δ .

Fourier's development is valid at the regular point x , if limited variation in some interval $(0 < \xi)$, $\xi < \frac{\pi}{2}$.

$0 < u < \gamma < \xi$, then

$$V_{u\xi}\psi = V_{u\gamma}\psi + V_{\gamma\xi}\psi.$$

$$\psi = \phi(u) \cdot \frac{1}{\sin u}.$$

$$V_{u\gamma}\psi \leq \{ V_{u\gamma}\phi + |\phi(\gamma)| \} \left\{ V_{u\gamma} \frac{1}{\sin u} + \frac{1}{\sin \gamma} \right\}.$$

in u being monotone,

$$V_{u\gamma} \frac{1}{\sin u} = \frac{1}{\sin u} - \frac{1}{\sin \gamma}.$$

$$V_{u\gamma}\psi \leq \frac{V_{u\gamma}\phi + |\phi(\gamma)|}{\sin u} = V_1.$$

early,

$$V_{\gamma\xi}\psi \leq \frac{V_{\gamma\xi}\phi + |\phi(\xi)|}{\sin \gamma} = V_2.$$

$$0 < \frac{u}{\sin u} < M, \text{ in } (0^*, \xi).$$

theorem now follows by 445. For we may take γ so small

$$V_{0\gamma}\phi < \frac{\epsilon}{4M}, \quad |\phi(\gamma)| < \frac{\epsilon}{4M}. \quad (1)$$

s for any $u < \gamma$,

$$uV_1 < \frac{\epsilon}{2}.$$

the other hand, M being sufficiently large, and γ chosen as and then fixed,

$$V_2 < M.$$

us

$$uV_2 < \frac{\epsilon}{2},$$

< some δ' . Hence

$$uV_{u\xi}\psi < \epsilon,$$

< $u < \text{some } \delta$.

(Jordan.) Fourier's development is valid at the regular point $f(x)$ has limited variation in some domain of x .

For $\phi(u) = \{f(x+2u) - f(u)\} + \{f(x-2u) - f(u)\}$

as limited variation also.

3. *Fourier's development is valid at every point of $\mathfrak{A} = (0, 2\pi)$, if f is limited and has only a finite number of oscillations in \mathfrak{A} .*

Other Criteria

447. Let $X = \int_{\delta}^{\eta} |\chi(u+\delta) - \chi(u)| du$, $\chi(u) = \frac{\phi(u)}{u}$,

$$\Psi = \int_{\delta}^{\eta} |\psi(u+\delta) - \psi(u)| du \quad , \quad 0 < \delta < \eta \leq \frac{\pi}{2}.$$

If $X \doteq 0$ as $\delta \doteq 0$, so does Ψ , and conversely.

For

$$\begin{aligned} \chi(u+\delta) - \chi(u) &= \psi(u+\delta) \frac{\sin(u+\delta)}{u+\delta} - \psi(u) \frac{\sin u}{u} \\ &= \{\psi(u+\delta) - \psi(u)\} \frac{\sin(u+\delta)}{u+\delta} + \rho, \end{aligned}$$

here

$$\rho = \psi(u) \left\{ \frac{\sin(u+\delta)}{u+\delta} - \frac{\sin u}{u} \right\}.$$

Obviously X and Ψ are simultaneously evanescent with δ , provided

$$R = \int_{\delta}^{\eta} |\rho| \doteq 0 \quad , \quad \text{as } \delta \doteq 0.$$

Let

$$Z(u) = \frac{\sin u}{u}.$$

Then

$$\rho = \psi(u) \{Z(u+\delta) - Z(u)\}$$

$$= \delta \psi(u) Z'(v) \quad , \quad u < v < u + \delta.$$

Now

$$\begin{aligned} Z'(v) &= \frac{v \cos v - \sin v}{v^2} = \frac{v \left(1 - \frac{v^2}{2} + \dots\right) - v \left(1 - \frac{v^2}{6} + \dots\right)}{v^2} \\ &= -\frac{1}{3} v + \frac{1}{30} v^3 + \dots \end{aligned}$$

Thus

$$|Z'(v)| < Mv < M \cdot 2u.$$

Hence

$$|\rho| < \frac{2u\delta|\phi|M}{\sin u} < 2\delta|\phi|M.$$

As

$$|\phi| \leq |f(x+2u)| + |f(x-2u)| + 2|f(x)|,$$

$$R < 2\delta M \int_{\delta}^{\eta} |\phi| \doteq 0, \quad \text{with } \delta.$$

448. (*Lipschitz-Dini.*) At the regular point x , Fourier's development is valid, if for each $\epsilon > 0$, there exists a $\delta_0 > 0$, such that for each $0 < \delta < \delta_0$,

$$|\phi(u+\delta) - \phi(u)| < \frac{\epsilon}{|\log \delta|}, \quad \text{for any } u \text{ in } (\delta, \delta_0).$$

For

$$\begin{aligned} |\chi(u+\delta) - \chi(u)| &= \left| \frac{\phi(u+\delta) - \phi(u)}{u+\delta} + \left\{ \frac{1}{u+\delta} - \frac{1}{u} \right\} \phi(u) \right| \\ &< \frac{|\phi(u+\delta) - \phi(u)|}{u} + \delta \frac{|\phi(u)|}{u^2}. \end{aligned}$$

Now x being a regular point, there exists an η' such that

$$|\phi(u)| < \epsilon, \quad \text{for } u \text{ in any } (\delta, \eta').$$

Thus taking

$$\eta > \delta_0, \eta',$$

$$\begin{aligned} X &= \int_{\delta}^{\eta} |\chi(u+\delta) - \chi(u)| du < \frac{\epsilon}{|\log \delta|} \int_{\delta}^{\eta} \frac{du}{u} + \epsilon \delta \int_{\delta}^{\eta} \frac{du}{u^2} \\ &< \epsilon \frac{\log \eta - \log \delta}{|\log \delta|} + \epsilon \delta \left(\frac{1}{\delta} - \frac{1}{\eta} \right) \\ &< 2\epsilon, \quad \text{for any } \delta < \eta. \end{aligned}$$

Thus

$$X \doteq 0, \text{ as } \delta \doteq 0.$$

Uniqueness of Fourier's Development

449. Suppose $f(x)$ can be developed in Fourier's series

$$f(x) = \frac{1}{2}a_0 + \sum_1^{\infty} (a_n \cos nx + b_n \sin nx), \quad (1)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, \quad (2)$$

id in $\mathfrak{N} = (-\pi, \pi)$. We ask can $f(x)$ be developed in a similar series

$$f(x) = \frac{1}{2} a'_0 + \sum_1^{\infty} (a'_n \cos nx + b'_n \sin nx), \quad (3)$$

valid in \mathfrak{N} , where the coefficients are not Fourier's coefficients, least not all of them.

Suppose this were true. Subtracting 1), 3) we get

$$0 = \frac{1}{2} (a_0 - a'_0) + \sum \{ (a_n - a'_n) \cos nx + (b_n - b'_n) \sin nx \} = 0, \\ c_0 + \sum \{ c_n \cos nx + d_n \sin nx \} = 0, \quad \text{in } \mathfrak{N}. \quad (4)$$

us it would be possible for a trigonometric series of the type to vanish without all the coefficients c_m, d_m vanishing.

For a power series

$$p_0 + p_1 x + p_2 x^2 + \dots \quad (5)$$

vanish in an interval about the origin, however small, we know that all the coefficients p_m in 5) must $= 0$.

We propose to show now that a similar theorem holds for a trigonometric series. In fact we shall prove the *fundamental*

Theorem 1. Suppose it is known that the series 4) converges to 0 at all the points of $\mathfrak{N} = (-\pi, \pi)$, except at a reducible set \mathfrak{N} . Then the coefficients c_m, d_m are all 0, and the series 4) $\equiv 0$ at all the points of \mathfrak{N} .

From this we deduce at once as corollaries:

Theorem 2. Let \mathfrak{N} be a reducible set in \mathfrak{N} . Let the series

$$\alpha_0 + \sum_1^{\infty} \{ \alpha_n \cos nx + \beta_n \sin nx \} \quad (6)$$

converge in \mathfrak{N} , except possibly at the points \mathfrak{N} . Then 6) defines a function $F(x)$ in $\mathfrak{N} - \mathfrak{N}$.

If the series

$$\alpha'_0 + \sum \{ \alpha'_n \cos nx + \beta'_n \sin nx \}$$

converges to $F(x)$ in $\mathfrak{N} - \mathfrak{N}$, its coefficients are respectively equal to those in 6).

Theorem 3. If $f(x)$ admits a development in Fourier's series for set $\mathfrak{N} - \mathfrak{N}$, any other development of $f(x)$ of the type 6), valid in $\mathfrak{N} - \mathfrak{N}$ is necessarily Fourier's series, i.e. the coefficients α_m, β_m have values given in 2).

In order to establish the fundamental theorem, we shall make use of some results due to *Riemann*, *G. Cantor*, *Harnack* and *Schwarz* as extended by later writers. Before doing this let us prove the easy

Theorem 4. If $f(x)$ admits a development in Fourier's series which is uniformly convergent in $\mathfrak{A} = (-\pi, \pi)$, it admits no other development of the type 3), which is also uniformly convergent in \mathfrak{A} .

For then the corresponding series 4) is uniformly convergent in \mathfrak{A} , and may be integrated termwise. Thus making use of the method employed in 436, we see that all the coefficients in 4) vanish.

450. 1. Before attempting to prove the fundamental theorem which states that the coefficients a_n, b_n are 0, we will first show that the coefficients of any trigonometric series which converges in \mathfrak{A} , except possibly at a point set of a certain type, must be such that they $\doteq 0$, as $n \doteq \infty$. We have already seen, in 440, 1, that this is indeed so in the case of Fourier's series, whether it converges or not. It is not the case with every trigonometric series as the following example shows, viz. :

$$\sum_1^{\infty} \sin n! x. \quad (1)$$

When $x = \frac{m\pi}{r!}$ all the terms, beginning with the $r!$ th, vanish, and hence 1) is convergent at such points. Thus 1) is convergent at a pantactic set of points. In this series the coefficients a_n of the *cosine terms* are all 0, while the coefficients of the sine terms b_n , are 0 or 1. Thus b_n does not $\doteq 0$, as $n \doteq \infty$.

2. Before enunciating the theorem on the convergence of the coefficients of a trigonometric series to 0, we need the notion of *divergence of a series* due to *Harnack*.

Let $A = a_1 + a_2 + \dots$ (2)
be a series of real terms. Let g_n, G_n be the minimum and maximum of all the terms

$$A_{n+1}, A_{n+2}, \dots$$

where as usual A_n is the sum of the first n terms of 2). Obviously

$$g_n \leq g_{n+1}, \quad G_n \geq G_{n+1}.$$

Thus the two sequences $\{g_n\}$, $\{G_n\}$ are monotone, and if limited, their terms converge to fixed values. Let us say

$$g_n \doteq g, \quad G_n \doteq G.$$

The difference

$$d = G - g$$

is called the *divergence* of the series 2).

3. For the series 2) to converge it is necessary and sufficient that its divergence $d = 0$.

For if A is convergent,

$$- \epsilon + A \leq A_{n+p} \leq A + \epsilon, \quad p = 1, 2, \dots$$

thus

$$- \epsilon + A \leq g_n \leq G_n \leq A + \epsilon.$$

Thus the limits G , g exist, and

$$G - g \leq 2\epsilon; \quad \text{or } G = g,$$

since $\epsilon > 0$ is small at pleasure.

Suppose now $d = 0$. Then by hypothesis, G , g exist and are equal. There exists, therefore, an n , such that

$$g - \epsilon \leq g_n < G_n < G + \epsilon,$$

$$G_n - g_n \leq 2\epsilon.$$

Thus

$$|A_{n+p} - A_n| \leq 2\epsilon, \quad p = 1, 2, \dots$$

and A is convergent.

451. Let the series

$$\sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

converge such that for each $\delta > 0$, there exists a subinterval of

$$\mathfrak{H} = (-\pi, \pi)$$

in which each point of which its divergence $d < \delta$. Then a_n , $b_n \doteq 0$, as $n \rightarrow \infty$.

For, as in 450, there exists for each x an m_x , such that

$$|a_n \cos nx + b_n \sin nx| < \frac{\delta}{8}, \quad n > m_x \quad (1)$$

for any point x in some interval \mathfrak{B} of \mathfrak{A} . Thus if b is an inner point of \mathfrak{B} , $x = b + \beta$ will lie in \mathfrak{B} , if β lies in some interval $B = (p, q)$. Now

$$\begin{aligned} a_n \cos n(b + \beta) + b_n \sin n(b + \beta) \\ = (a_n \cos nb + b_n \sin nb) \cos n\beta - (a_n \sin nb - b_n \cos nb) \sin n\beta. \end{aligned}$$

$$\begin{aligned} a_n \cos n(b - \beta) + b_n \sin n(b - \beta) \\ = (a_n \cos nb + b_n \sin nb) \cos n\beta + (a_n \sin nb - b_n \cos nb) \sin n\beta. \end{aligned}$$

Adding and subtracting these equations, and using 1) we have

$$|(a_n \cos nb + b_n \sin nb) \cos n\beta| < \frac{\delta}{4},$$

$$|(a_n \sin nb - b_n \cos nb) \sin n\beta| < \frac{\delta}{4},$$

for all $n > m_x$. Let us multiply the first of these inequalities by $\cos nb \sin n\beta$, and the second by $\sin nb \cos n\beta$, and add. We get

$$|a_n \sin n\beta_1| < \delta, \quad \beta_1 = 2\beta, \quad n > m_x. \quad (2)$$

Again if we multiply the first inequality by $\sin nb \sin n\beta$, and the second by $\cos nb \cos n\beta$, and subtract, we get

$$|b_n \sin n\beta_1| < \delta, \quad n > m_x. \quad (3)$$

From 2), 3), we can infer that for any $\epsilon > 0$

$$|a_n| < \epsilon, \quad |b_n| < \epsilon, \quad n > \text{some } m, \quad (4)$$

or what is the same, that $a_n, b_n \doteq 0$.

For suppose that the first inequality of 4) did not hold. Then there exists a sequence

$$n_1 < n_2 < \dots \doteq \infty \quad (5)$$

such that on setting

$$|a_{n_r}| = \delta + \delta'_{n_r}, \quad \epsilon - \delta = \delta'$$

we will have

$$\delta_{n_r} \geq \delta'. \quad (6)$$

If this be so, we can show that there exists a sequence

$$\nu_1 < \nu_2 < \dots \doteq \infty$$

in 5), such that for some β' in B ,

$$|a_{\nu_r} \sin \nu_r \beta'| \geq \delta, \quad (7)$$

which contradicts 2). To this end we note that $\gamma_0 > 0$ may be chosen so small that for any r and any $|\gamma| \leq \gamma_0$,

$$|a_r| \cos \gamma \geq (\delta + \delta') \cos \gamma_0 > \delta. \quad (8)$$

Let us take the integer ν_1 so that

$$\nu_1 > \frac{\pi + 2\gamma_0}{q - p}. \quad (9)$$

Then

$$\frac{2}{\pi}(\nu_1(q - p) - 2\gamma_0) > 2.$$

Thus at least one odd integer lies in the interval determined by two numbers

$$\frac{2}{\pi}(p\nu_1 + \gamma_0) \quad , \quad \frac{2}{\pi}(q\nu_1 - \gamma_0).$$

Let m_1 be such an integer. Then

$$\frac{2}{\pi}(p\nu_1 + \gamma_0) \leq m_1 \leq \frac{2}{\pi}(q\nu_1 - \gamma_0). \quad (10)$$

We set

$$p_1 = \frac{1}{\nu_1} \left(m_1 \frac{\pi}{2} - \gamma_0 \right) \quad , \quad q_1 = \frac{1}{\nu_1} \left(m_1 \frac{\pi}{2} + \gamma_0 \right) \quad (11)$$

so that the interval $B_1 = (p_1, q_1)$ lies in B . The length of B_1 is $2\gamma_0/\nu_1$. Then for any β in B_1 ,

$$\nu_1 \beta = m_1 \frac{\pi}{2} + \gamma_1 \quad , \quad |\gamma_1| \leq \gamma_0.$$

Thus by 8),

$$|a_{\nu_1} \sin \nu_1 \beta| = |a_{\nu_1}| \cos \gamma_1 > \delta. \quad (12)$$

Thus we may reason on B_1 as we have on B . We determine ν_2 , replacing p, q by p_1, q_1 . We determine the odd integer m_2 , replacing p, q, ν_1 by p_1, q_1, ν_2 . The relation 11) determines the new interval $B_2 = (p_2, q_2)$, on replacing m_1, ν_1 by m_2, ν_2 . The length of B_2 is $2\gamma_0/\nu_2$, and B_2 lies in B_1 . For this relation, and for any β in B_2 we have, similar to 12),

$$|a_{\nu_2} \sin \nu_2 \beta| > \delta.$$

In this way we may continue indefinitely. The intervals $B_2 > \dots \ni$ to a point β' , and obviously for this β' , the rela-

tion 7) holds for any x . In a similar manner we see that if b_n does not $\doteq 0$, the relation 3) cannot hold.

452. As corollaries of the last theorem we have :

1. Let the series

$$\sum_0^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1)$$

be such that for each $\delta > 0$, the points in $\mathfrak{A} = (-\pi, \pi)$ at which the divergence of 1) is $\geq \delta$, form an apantactic set in \mathfrak{A} . Then $a_n, b_n \doteq 0$, as $n \doteq \infty$.

2. Let the series 1) converge in \mathfrak{A} , except possibly at the points of a reducible set \mathfrak{R} . Then $a_n, b_n \doteq 0$.

For \mathfrak{R} being reducible [318, 6], there exists in \mathfrak{A} an interval \mathfrak{B} in which 1) converges at every point. We now apply 451.

453. Let
$$F(x) = \sum_0^{\infty} (a_n \cos nx + b_n \sin nx)$$

at the points of $\mathfrak{A} = (-\pi, \pi)$, where the series is convergent. At the other points of \mathfrak{A} , let $F(x)$ have an arbitrarily assigned value, lying between the two limits of indetermination g, G of the series. If F is R -integrable in \mathfrak{A} , the coefficients $a_n, b_n \doteq 0$.

For there exists a division of \mathfrak{A} , such that the sum of those intervals in which $\text{Osc } F \geq \omega$ is $< \sigma$. There is therefore an interval \mathfrak{B} in which $\text{Osc } F < \omega$. If \mathfrak{R} is an inner interval of \mathfrak{B} , the divergence of the above series is $< \omega$ at each point of \mathfrak{R} . We now apply 451.

454. Riemann's Theorem.

Let $F(x) = \frac{1}{2} a_0 + \sum_1^{\infty} (a_n \cos nx + b_n \sin nx) = \Sigma A_n$ converge at each point of $\mathfrak{A} = (-\pi, \pi)$, except possibly at the points of a reducible set \mathfrak{R} . The series obtained by integrating this series termwise, we denote by

$$G(x) = \frac{1}{4} a_0 x^2 - \sum_1^{\infty} \frac{1}{n^2} (a_n \cos nx + b_n \sin nx) = \frac{1}{2} A_0 x^2 - \sum_1^{\infty} \frac{A_n}{n^2}.$$

Then G is continuous in \mathfrak{A} .

$$\Phi(u) = G(x+2u) + G(x-2u) - 2G(x). \quad (1)$$

at each point of $\mathfrak{B} = \mathfrak{A} - \mathfrak{N}$,

$$\lim_{u \rightarrow 0} \frac{\Phi(u)}{4u^2} = F(x); \quad (2)$$

at each point of \mathfrak{N} ,

$$\lim_{u \rightarrow 0} \frac{\Phi(u)}{u} = 0. \quad (3)$$

, in the first place, since \mathfrak{N} is a reducible set, $a_n, b_n \doteq 0$. The G is therefore uniformly convergent in \mathfrak{N} , and is thus a continuous function.

us now compute Φ . We have

$$\begin{aligned} a_n \cos n(x+2u) + a_n \cos n(x-2u) - 2a_n \cos nx \\ = 2a_n \cos nx (\cos 2nu - 1) \\ = -4a_n \cos nx \sin^2 nu. \end{aligned}$$

$$\begin{aligned} b_n \sin n(x+2u) + b_n \sin n(x-2u) - 2b_n \sin nx \\ = 2b_n \sin nx (\cos 2nu - 1) \\ = -4b_n \sin nx \sin^2 nu. \end{aligned}$$

$$\frac{\Phi(u)}{4u^2} = \sum_0^\infty A_n \left(\frac{\sin nu}{nu} \right)^2,$$

agree to give the coefficient of A_0 the value 1. Let us choose an arbitrary but fixed value in \mathfrak{B} . Then for each $\epsilon > 0$, there exists an m such that

$$A_0 + A_1 + \dots + A_{n-1} = F(x) + \epsilon_n, \quad |\epsilon_n| < \epsilon, \quad n \geq m.$$

$$A_n = \epsilon_{n+1} - \epsilon_n.$$

$$\begin{aligned} \frac{\Phi}{4u^2} &= F(x) + \epsilon_1 + \sum_1^\infty (\epsilon_{n+1} - \epsilon_n) \left(\frac{\sin nu}{nu} \right)^2 \\ &= F(x) + \sum_1^\infty \epsilon_n \left\{ \left[\frac{\sin (n-1)u}{(n-1)u} \right]^2 - \left[\frac{\sin nu}{nu} \right]^2 \right\} \\ &= F(x) + S \end{aligned} \quad (4)$$

$$u < \frac{\pi}{m}, \text{ so that } m < \frac{\pi}{u};$$

and break S into three parts

$$S_1 = \sum_1^m, \quad S_2 = \sum_{m+1}^{\kappa}, \quad S_3 = \sum_{\kappa+1}^{\infty},$$

where κ is the greatest integer $< \pi/u$, and then consider each sum separately, as $u \neq 0$.

Obviously $\lim_{u=0} S_1 = 0$.

As to the second sum, the number of its terms increases indefinitely as $u \rightarrow 0$.

For any u ,

$$\begin{aligned} |S_2| &< \epsilon \sum_{m+1}^{\kappa} \{ \dots \} \\ &< \epsilon \left\{ \left[\frac{\sin mu}{mu} \right]^2 - \left[\frac{\sin \kappa u}{\kappa u} \right]^2 \right\} \\ &< \epsilon \left[\frac{\sin mu}{mu} \right]^2 < \epsilon, \end{aligned}$$

since each term in the brace is positive. In fact

$$\frac{\sin v}{v}$$

is a decreasing function of v as v ranges from 0 to π , and

$$nu \leq \kappa u \leq \pi, \quad n = m, m+1, \dots, \kappa.$$

Finally we consider S_3 . We may write the general term as follows:

$$\begin{aligned} &\epsilon_n \left\{ \left[\frac{\sin (n-1)u}{(n-1)u} \right]^2 - \left[\frac{\sin (n-1)u}{nu} \right]^2 \right\} \\ &+ \epsilon_n \left\{ \left[\frac{\sin (n-1)u}{nu} \right]^2 - \left[\frac{\sin nu}{nu} \right]^2 \right\}. \end{aligned}$$

Now

$$\frac{\sin^2 (n-1)u - \sin^2 nu}{n^2 u^2} = \frac{-\sin (2n-1)u \sin u}{n^2 u^2} \leq \left| \frac{1}{n^2 u} \right|.$$

$$|S_3| \leq \epsilon \sum_{k=1}^{\infty} \left\{ \frac{1}{(n-1)^2} - \frac{1}{n^2} \right\} + \epsilon \sum_{k=1}^{\infty} \frac{1}{n^2} \\ \leq \frac{\epsilon}{\kappa^2 n^2} + \frac{\epsilon}{\kappa n},$$

$$\sum_{k=1}^{\infty} \frac{1}{n^2} < \int_{\kappa}^{\infty} \frac{dx}{x^2} = \frac{1}{\kappa}.$$

$$\kappa \geq \frac{\pi}{n} - 1, \quad \text{or } \kappa n \geq \pi - n.$$

$$|S_3| \leq \epsilon \left\{ \frac{1}{(\pi - n)^2} + \frac{1}{\pi - n} \right\}.$$

$$S = S_1 + S_2 + S_3 \doteq 0, \text{ as } n \doteq 0,$$

h proves the limit 2), on using 4).

o prove the limit 3), we have

$$\frac{\Phi(n)}{nu} = \sum_0^{\infty} n A_n \left(\frac{\sin nu}{nu} \right)^2 = T.$$

et us give n a definite value and break T into three sums.

$$T_1 = \sum_1^m,$$

o m is chosen so that

$$|A_n| < \epsilon, \quad n > m;$$

$$T_2 = \sum_{m+1}^{\lambda},$$

o λ is the greatest integer such that

$$\lambda n \leq 1;$$

$$T_3 = \sum_{\lambda+1}^{\infty}.$$

viously for some M ,

$$|T_1| \leq nM.$$

$$|T_2| \leq \epsilon n \lambda \leq \epsilon,$$

$$\left(\frac{\sin nu}{nu} \right)^2 < 1.$$

As to the last sum,

$$|T_3| \leq \frac{\epsilon}{u} \sum_{n=\lambda+1}^{\infty} \frac{1}{n^2} \leq \epsilon \lambda \cdot \frac{1}{\lambda}, \quad \text{since } \frac{1}{u} \leq \lambda,$$

$$< \epsilon.$$

Thus

$$T \doteq 0, \quad \text{as } u \doteq 0.$$

455. Schwarz-Liuroth Theorem.

In $\mathfrak{A} = (a < b)$ let the continuous function $f(x)$ be such that

$$S(x, u) = \frac{f(x+u) + f(x-u) - 2f(x)}{u^2} \doteq 0, \quad \text{as } u \doteq 0, \quad (1)$$

except possibly at an enumerable set \mathfrak{E} in \mathfrak{A} . At the points \mathfrak{E} , let

$$uS(x, u) \doteq 0 \quad \text{as } u \doteq 0. \quad (2)$$

Then f is a linear function in \mathfrak{A} .

Let us first suppose with Schwarz that $\mathfrak{E} = 0$. We introduce the auxiliary function,

$$g(x) = \eta L(x) - \frac{1}{2} c(x-a)(x-b),$$

where

$$L(x) = f(x) - f(a) - \frac{x-a}{b-a} \{f(b) - f(a)\},$$

$\eta = \pm 1$, and c is an arbitrary constant.

The function $g(x)$ is continuous in \mathfrak{A} , and $g(a) = g(b) = 0$. Moreover

$$\frac{g(x+u) + g(x-u) - 2g(x)}{u^2} \doteq c, \quad \text{as } u \doteq 0.$$

Thus for all $0 < u < \text{some } \delta$,

$$G = g(x+u) + g(x-u) - 2g(x) > 0. \quad (3)$$

From this follows that $g(x) \leq 0$ in \mathfrak{A} . For if $g(x) > 0$, at any point in \mathfrak{A} , it takes on its maximum value at some point ξ within \mathfrak{A} .

Thus

$$g(\xi+u) - g(\xi) \leq 0, \quad g(\xi-u) - g(\xi) < 0,$$

for $0 < u < \delta$, δ being sufficiently small. Adding these two inequalities gives $G \leq 0$, which contradicts 3). Thus $g \leq 0$ in \mathfrak{A} .

Let us now suppose $L \neq 0$ for some x in \mathfrak{A} . We take c so small that

$$\operatorname{sgn} g = \operatorname{sgn} \eta L = \eta \operatorname{sgn} L.$$

at η is at pleasure ± 1 , hence the supposition that $L \neq 0$ is inadmissible. Hence $L = 0$ in \mathfrak{A} , or

$$f(x) = f(a) - \frac{x-a}{b-a} \{f(b) - f(a)\} \quad (4)$$

is a linear function.

Let us now suppose with Lüröth that $\mathfrak{E} > 0$. We introduce the binary continuous function.

$$h(x) = L(x) + c(x-a)^2, \quad c > 0.$$

$$h(a) = 0, \quad h(b) = c(b-a)^2.$$

Suppose at some inner point ξ of \mathfrak{A}

$$L(\xi) > 0. \quad (5)$$

This leads to a contradiction, as we proceed to show. For then

$$h(\xi) - h(b) = L(\xi) + c\{(\xi-a)^2 - (b-a)^2\} > 0,$$

hence

$$c = \frac{L(\xi)}{(b-a)^2 - (\xi-a)^2} > 0.$$

We shall take c so that this inequality is satisfied, i.e. c lies in the interval $\mathfrak{C} = (0^*, c^*)$. Thus

$$h(\xi) > h(b) > h(a).$$

Since $h(x)$ takes on its maximum value at some inner point e

Hence for $\delta > 0$ sufficiently small,

$$h(e+u) - h(e) < 0, \quad h(e-u) - h(e) < 0, \quad 0 < u \leq \delta. \quad (6)$$

$$H(e, u) = \frac{h(e+u) + h(e-u) - 2h(e)}{u^2} \leq 0. \quad (7)$$

But if e is a point of $\mathfrak{A} - \mathfrak{E}$,

$$\lim_{u \rightarrow 0} H(e, u) = 2c > 0.$$

This contradicts 7), which requires that

$$\lim_{u \rightarrow 0} H(e, u) \leq 0.$$

Hence e is a point of \mathfrak{E} . Hence by 2),

$$\frac{h(e+u)-h(e)}{u} + \frac{h(e-u)-h(e)}{u} \doteq 0, \quad \text{as } u \doteq 0.$$

By 6), both terms have the same sign. Hence each term $\doteq 0$.
Thus for $u > 0$

$$0 = \lim_{u \rightarrow 0} \frac{h(e \pm u) - h(e)}{\pm u} = \lim_{\pm u} \frac{f(e \pm u) - f(e)}{\pm u} - \frac{f(b) - f(a)}{b - a} + 2c(e - a).$$

Hence

$$f'(e) = \frac{f(b) - f(a)}{b - a} + 2c(e - a). \quad (8)$$

Thus to each c in the interval \mathfrak{E} , corresponds an e in \mathfrak{E} , at which point the derivative of $f(x)$ exists and has the value given on the right of 8). On the other hand, two different c 's, say c and c' , in \mathfrak{E} cannot correspond to the same e in \mathfrak{E} .

For then 8) shows that

$$c(e - a) = c'(e - a),$$

or as

$$c > a, \quad c' = c.$$

Thus there is a uniform correspondence between \mathfrak{E} whose cardinal number is c , and \mathfrak{E} whose cardinal number is c , which is absurd. Thus the supposition 5) is impossible. In a similar manner, the assumption that $L < 0$ at some point in \mathfrak{A} , leads to a contradiction. Hence $L = 0$ in \mathfrak{A} , and 4) again holds, which proves the theorem.

456. Cantor's Theorem. Let

$$\frac{1}{2} a_0 + \sum_1^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1)$$

converge to 0 in $\mathfrak{A} = (-\pi, \pi)$, except possibly at a reducible set \mathfrak{A} , where nothing is asserted regarding its convergence. Then it converges to 0 at every point in \mathfrak{A} , and all its coefficients

$$a_0, a_1, a_2 \dots b_1, b_2, b_3 \dots = 0.$$

For by 452, 2, $a_n, b_n \doteq 0$. Then Riemann's function

$$f(x) = \frac{1}{4} a_0 x^2 - \sum \frac{1}{n^2} (a_n \cos nx + b_n \sin nx)$$

satisfies the conditions of the Schwarz-Lüroth theorem, 455, since \mathfrak{R} is enumerable. Thus $f(x)$ is a linear function of x in \mathfrak{A} , and has the form $\alpha + \beta x$. Hence

$$\alpha + \beta x - \frac{1}{4} a_0 x^2 = - \sum_1^{\infty} \frac{1}{n^2} (a_n \cos nx + b_n \sin nx). \quad (2)$$

The right side admits the period 2π , and is therefore periodic.

Its period ω must be 0. For if $\omega > 0$, the left side has this period, which is absurd. Hence $\omega = 0$, and the left side reduces to a constant, which gives $\beta = 0$, $a_0 = 0$. But in $\mathfrak{A} - \mathfrak{R}$, the right side of 1) has the sum 0. Hence $\alpha = 0$. Thus the right side of 2) vanishes in \mathfrak{A} . As it converges uniformly in \mathfrak{A} , we may determine its coefficients as in 436. This gives

$$a_n = 0 \quad , \quad b_n = 0 \quad , \quad n = 1, 2 \dots$$

CHAPTER XIV

DISCONTINUOUS FUNCTIONS

Properties of Continuous Functions

457. 1. In Chapter VII of Volume I we have discussed some of the elementary properties of continuous and discontinuous functions. In the present chapter further developments will be given, paying particular attention to discontinuous functions. Here the results of Baire* are of foremost importance. Lebesgue† has shown how some of these may be obtained by simpler considerations, and we have accordingly adopted them.

2. Let us begin by observing that the definition of a continuous function given in I, 339, may be extended to sets having isolated points, if we use I, 339, 2 as definition.

Let therefore $f(x_1 \dots x_m)$ be defined over \mathfrak{A} , being either limited or unlimited. Let a be any point of \mathfrak{A} . If for each $\epsilon > 0$, there exists a $\delta > 0$, such that

$$|f(x) - f(a)| < \epsilon, \quad \text{for any } x \text{ in } I_\delta(a),$$

we say f is continuous at a .

By the definition it follows at once that f is continuous at each isolated point of \mathfrak{A} . Moreover, when a is a proper limiting point of \mathfrak{A} , the definition here given coincides with that given in I, 339. If f is continuous at each point of \mathfrak{A} , we say it is continuous in \mathfrak{A} . The definition of discontinuity given in I, 347, shall still hold, except that we must regard isolated points as points of continuity.

* "Sur les Fonctions de Variables réelles," *Annali di Mat.*, Ser. 3, vol. 3 (1890).

Also his *Leçons sur les Fonctions Discontinues*. Paris, 1906.

† *Bulletin de la Société Mathématique de France*, vol. 32 (1904), p. 220.

3. The reader will observe that the theorems I, 350 to 354 inclusive, are valid not only for limited perfect domains, but also for limited complete sets.

458. 1. If $f(x_1 \dots x_m)$ is continuous in the limited set \mathfrak{A} , and its values are known at the points of $\mathfrak{B} < \mathfrak{A}$, then f is known at all points of \mathfrak{B}' lying in \mathfrak{A} .

For let $b_1, b_2, b_3 \dots$ be points of \mathfrak{B} , whose limiting point b lies in \mathfrak{A} . Then

$$\lim_{n \rightarrow \infty} f(b_n) = f(b).$$

2. If f is known for a dense set \mathfrak{B} in \mathfrak{A} , and is continuous in \mathfrak{A} , f is known throughout \mathfrak{A} .

For

$$\mathfrak{B}' \geq \mathfrak{A}.$$

3. If $f(x_1 \dots x_m)$ is continuous in the complete set \mathfrak{A} , the points \mathfrak{B} in \mathfrak{A} where $f = c$, a constant, form a complete set. If \mathfrak{A} is an interval, there is a first and a last point of \mathfrak{B} .

For $f = c$ at $x = \alpha_1, \alpha_2 \dots$ which $\doteq \alpha$; we have therefore

$$f(\alpha) = \lim_{n \rightarrow \infty} f(\alpha_n) = c.$$

459. The points of continuity \mathfrak{C} of $f(x_1 \dots x_m)$ in \mathfrak{A} lie in a deleted enclosure \mathfrak{R} . If \mathfrak{A} is complete, $\mathfrak{R} = \mathfrak{C}$.

For let $\epsilon_1 > \epsilon_2 > \dots \doteq 0$. For each ϵ_n , and for each point of continuity c in \mathfrak{A} , there exists a cube \mathfrak{Q} whose center is c , such that

$$\text{Osc } f < \epsilon_n, \quad \text{in } \mathfrak{Q}.$$

Thus the points of continuity of f lie in an enumerable non-overlapping set of complete metric cells, in each of which $\text{Osc } f < \epsilon_n$. Let \mathfrak{Q}_n be the inner points of this enclosure. Then each point of the deleted enclosure

$$\mathfrak{R} = Dv \{ \mathfrak{Q}_n \}$$

which lies in \mathfrak{A} is a point of continuity of f . For such a point c lies within each \mathfrak{Q}_n .

Hence

$$\text{Osc } f < \epsilon, \quad \text{in } V_\delta(c),$$

for $\delta > 0$ sufficiently small and n sufficiently great.

Oscillation

460. Let $\omega_\delta = \text{Osc } f(x_1 \cdots x_m)$ in $V_\delta(a)$.

This is a monotone decreasing function of δ . Hence if ω_δ is finite, for some $\delta > 0$,

$$\omega = \lim_{\delta \rightarrow 0} \omega_\delta$$

exists. We call ω the oscillation of f at $x = a$, and write

$$\omega = \text{Osc } f_{x=a}.$$

Should $\omega_\delta = +\infty$, however small $\delta > 0$ is taken, we say $\omega = +\infty$. When $\omega = 0$, f is continuous at $x = a$, if a is a point in the domain of definition of f . When $\omega > 0$, f is discontinuous at this point. It is a measure of the discontinuity of f at $x = a$; we write

$$\omega = \text{Disc } f_{x=a}(x_1 \cdots x_m).$$

461. 1. Let

$$d = \text{Disc } f(x_1 \cdots x_m), \quad e = \text{Disc } g(x_1 \cdots x_m),$$

at $x = a$. Then

$$|d - e| \leq \text{Disc } (f \pm g)_{x=a} \leq d + e.$$

For in $V_\delta(a)$,

$$|\text{Osc } f - \text{Osc } g| \leq \text{Osc } (f \pm g) \leq \text{Osc } f + \text{Osc } g.$$

2. If f is continuous at $x = a$, while $\text{Disc } g = d$, then

$$\text{Disc } (f + g)_{x=a} = d.$$

For f being continuous at a , $\text{Disc } f = 0$.

Hence

$$\text{Disc } g \leq \text{Disc } (f + g) \leq \text{Disc } g = d.$$

3. If c is a constant,

$$\text{Disc } (cf) = |c| \text{Disc } f, \quad \text{at } x = a.$$

For

$$\text{Osc } (cf) = |c| \text{Osc } f, \quad \text{in any } V_\delta(a).$$

4. When the limits

$$f(x-0), \quad f(x+0)$$

exist and at least one of them is different from $f(x)$, the point x is a discontinuity of the first kind, as we have already said. When at least one of the above limits does not exist, the point x is a *point of discontinuity of the second kind*.

462. 1. *The points of infinite discontinuity \mathfrak{S} of f , defined over a limited set \mathfrak{A} , form a complete set.*

For let $\iota_1, \iota_2 \dots$ be points of \mathfrak{S} , having k as limiting point. Then in any $V(k)$ there are an infinity of the points ι_n and hence in any $V(k)$, $\text{Osc } f = +\infty$. The point k does not of course need to lie in \mathfrak{A} .

2. We cannot say, however, that the points of discontinuity of a function form a complete set as is shown by the following

Example. In $\mathfrak{A} = (0, 1)$, let $f(x) = x$ when x is irrational, and $= 0$ when x is rational. Then each point of \mathfrak{A} is a point of discontinuity except the point $x = 0$. Hence the points of discontinuity of f do not form a complete set.

3. *Let f be limited or unlimited in the limited complete set \mathfrak{A} . The points \mathfrak{R} of \mathfrak{A} at which $\text{Osc } f \geq k$ form a complete set.*

For let $a_1, a_2 \dots$ be points of \mathfrak{R} which $\doteq a$. However small $\delta > 0$ is taken, there are an infinity of the a_n lying in $V_\delta(a)$. But at any one of these points, $\text{Osc } f \geq k$. Hence $\text{Osc } f \geq k$ in $V_\delta(a)$, and thus a lies in \mathfrak{R} .

4. *Let $f(x_1 \dots x_m)$ be limited and R -integrable in the limited set \mathfrak{A} . The points \mathfrak{R} at which $\text{Osc } f \geq k$ form a discrete set.*

For let D be a rectangular division of space. Let us suppose $\bar{\mathfrak{R}}_D >$ some constant $c > 0$, however D is chosen. In each cell δ of D ,

$$\text{Osc } f \geq k.$$

Hence the sum of the cells in which the oscillation is $\geq k$ cannot be made small at pleasure, since this sum is $\bar{\mathfrak{R}}_D$. But this contradicts I, 700, 6.

5. *Let $f(x_1 \dots x_m)$ be limited in the complete set \mathfrak{A} . If the points \mathfrak{R} in \mathfrak{A} at which $\text{Osc } f \geq k$ form a discrete set, for each k , then f is R -integrable in \mathfrak{A} .*

For about each point of $\mathfrak{A} - \mathfrak{R}$ as center, we can describe a cube \mathfrak{C} of varying size, such that $\text{Osc } f < 2k$ in \mathfrak{C} . Let D be a cubical division of space of norm d . We may take d so small that $\bar{\mathfrak{R}}_D = \Sigma d_i$ is as small as we please. The points of \mathfrak{A} lie now within the cubes \mathfrak{C} and the set formed of the cubes d_i . By Borel's theorem there are a finite number of cubes, say

$$\eta_1, \eta_2, \dots$$

such that all the points of \mathfrak{A} lie within these η 's. If we prolong the faces of these η 's, we effect a rectangular division such that the sum of those cells in which the oscillation is $> 2k$ is as small as we choose, since this sum is obviously $\leq \bar{\mathfrak{R}}_D$. Hence by I, 700, 5, f is R -integrable.

6. Let $f(x_1 \dots x_n)$ be limited in \mathfrak{A} ; let its points of discontinuity in \mathfrak{A} be \mathfrak{D} . If f is R -integrable, \mathfrak{D} is a null set. If \mathfrak{A} is complete and \mathfrak{D} is a null set, f is R -integrable.

Let f be R -integrable. Then \mathfrak{D} is a null set. For let $\epsilon_1 > \epsilon_2 > \dots \doteq 0$. Let \mathfrak{D}_n denote the points at which $\text{Osc } f \geq \epsilon_n$. Then $\mathfrak{D} = \{\mathfrak{D}_n\}$. But since f is R -integrable, each \mathfrak{D}_n is discrete by 4. Hence \mathfrak{D} is a null set.

Let \mathfrak{A} be complete and \mathfrak{D} a null set. Then each \mathfrak{D}_n is complete by 3. Hence by 365, $\bar{\mathfrak{D}}_n = \mathfrak{D}_n$. As $\bar{\mathfrak{D}} = 0$, we see \mathfrak{D}_n is discrete. Hence by 5, f is R -integrable.

If \mathfrak{A} is not complete, f does not need to be R -integrable when \mathfrak{D} is a null set.

Example. Let $\mathfrak{A}_1 = \left\{ \frac{m}{2^n} \right\}$, $n = 1, 2, \dots$; $m < 2^n$.

$$\mathfrak{A}_2 = \left\{ \frac{r}{3^s} \right\}, \quad s = 1, 2, \dots; \quad r < 3^s.$$

Let $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2$.

$$\begin{aligned} \text{Let } f(x) &= \frac{1}{2^n}, \quad \text{at } x = \frac{m}{2^n} \\ &= 1 \quad \text{in } \mathfrak{A}_2. \end{aligned}$$

Then each point of \mathfrak{A} is a point of discontinuity, and $\mathfrak{A} = \mathfrak{D}$. But $\mathfrak{A}_1, \mathfrak{A}_2$ are null sets, hence \mathfrak{A} is a null set.

On the other hand,

$$\int_{\mathfrak{A}} f = 1 \quad , \quad \int_{\mathfrak{A}} f = 0,$$

and f is not R -integrable in \mathfrak{A} .

Pointwise and Total Discontinuity

463. Let $f(x_1 \dots x_m)$ be defined over \mathfrak{A} . If each point of \mathfrak{A} is a point of discontinuity, we say f is *totally discontinuous* in \mathfrak{A} .

We say f is *pointwise discontinuous* in \mathfrak{A} , if f is not continuous in $\mathfrak{A} = \{a\}$, but has in any $V(a)$ a point of continuity. If f is continuous or pointwise discontinuous, we may say it is *at most pointwise discontinuous*.

Example 1. A function $f(x_1 \dots x_m)$ having only a finite number of points of discontinuity in \mathfrak{A} is pointwise discontinuous in \mathfrak{A} .

Example 2. Let

$$\begin{aligned} f(x) &= 0 \quad , \quad \text{for irrational } x \text{ in } \mathfrak{A} = (0, 1) \\ &= \frac{1}{n} \quad , \quad \text{for } x = \frac{m}{n} \\ &= 1 \quad , \quad \text{for } x = 0, 1. \end{aligned}$$

Obviously f is continuous at each irrational x , and discontinuous at the other points of \mathfrak{A} . Hence f is pointwise discontinuous in \mathfrak{A} .

Example 3. Let \mathfrak{D} be a Harnack set in the unit interval $\mathfrak{A} = (0, 1)$. In the associate set of intervals, end points included, let $f(x) = 1$. At the other points of \mathfrak{A} , let $f = 0$. As \mathfrak{D} is apantactic in \mathfrak{A} , f is pointwise discontinuous.

Example 4. In Ex. 3, let $\mathfrak{D} = \mathfrak{E} + \mathfrak{F}$, where \mathfrak{E} is the set of end points of the associate set of intervals. Let $f = 1/n$ at the end points of these intervals belonging to the n^{th} stage. Let $f = 0$ in \mathfrak{F} . Here f is defined only over \mathfrak{D} . The points \mathfrak{F} are points of continuity in \mathfrak{D} . Hence f is pointwise discontinuous in \mathfrak{D} .

Example 5. Let $f(x)$ be Dirichlet's function, i.e. $f = 0$, for the irrational points \mathfrak{F} in $\mathfrak{A} = (0, 1)$, and $= 1$ for the rational points.

As each point of \mathfrak{A} is a point of discontinuity, f is totally discontinuous in \mathfrak{A} . Let us remove the rational points in \mathfrak{A} ; the deleted domain is \mathfrak{J} . In this domain f is continuous. Thus on removing certain points, a discontinuous function becomes a continuous function in the remaining point set.

This is not always the case. For if in Ex. 4 we remove the points \mathfrak{F} , retaining only the points \mathfrak{C} , we get a function which is *totally* discontinuous in \mathfrak{C} , whereas before f was only pointwise discontinuous.

464. 1. If $f(x_1 \dots x_n)$ is totally discontinuous in the infinite complete set \mathfrak{A} , then the points δ_ω where

$$\text{Disc } f \geq \omega, \quad \omega > 0,$$

form an infinite set, if ω is taken sufficiently small.

For suppose δ_ω were finite however small ω is taken. Let $\omega_1 > \omega_2 > \dots \doteq 0$. Let D_1, D_2, \dots be a sequence of superposed cubical divisions of space of norms $d_n \doteq 0$. We shall only consider cells containing points of \mathfrak{A} . Then if d_1 is taken sufficiently small, D_1 contains a cell δ_1 , containing an infinite number of points of \mathfrak{A} , but no point at which $\text{Disc } f \geq \omega_1$. If d_2 is taken sufficiently small, D_2 contains a cell $\delta_2 < \delta_1$, containing no point at which $\text{Disc } f \geq \omega_2$. In this way we get a sequence of cells,

$$\delta_1 > \delta_2 > \dots$$

which \doteq a point p . As \mathfrak{A} is complete, p lies in \mathfrak{A} . But f is obviously continuous at p . Hence f is not totally discontinuous in \mathfrak{A} .

2. If \mathfrak{A} is not complete, δ_ω does not need to be infinite for any $\omega > 0$.

Example. Let $\mathfrak{A} = \left\{ \frac{m}{2^n} \right\}$, $n = 1, 2, \dots$ and m odd and $< 2^n$. At $\frac{m}{2^n}$, let $f = \frac{1}{2^n}$. Then each point of \mathfrak{A} is a point of discontinuity. But δ_ω is finite, however small $\omega > 0$ is taken.

3. We cannot say f is not pointwise discontinuous in complete \mathfrak{A} , when δ_ω is infinite.

Example. At the points $\left\{\frac{1}{n}\right\} = \mathfrak{N}$, let $f = 0$; at the other points of $\mathfrak{N} = (0, 1)$, let $f = 1$.

Obviously f is pointwise discontinuous in \mathfrak{N} . But \mathfrak{D}_ω is an infinite set for $\omega \leq 1$, as in this case it is formed of \mathfrak{N} , and the point 0.

Examples of Discontinuous Functions

465. In volume I, 330 *seq.* and 348 *seq.*, we have given examples of discontinuous functions. We shall now consider a few more.

Example 1. Riemann's Function.

Let (x) be the difference between x and the nearest integer; and when x has the form $n + \frac{1}{2}$, let $(x) = 0$. Obviously (x) has the period 1.

It can be represented by Fourier's series thus:

$$(x) = \frac{1}{\pi} \left\{ \frac{\sin 2\pi x}{1} - \frac{\sin 2 \cdot 2\pi x}{2} + \frac{\sin 3 \cdot 2\pi x}{3} - \dots \right\}. \quad (1)$$

Riemann's function is now

$$F(x) = \sum_1^{\infty} \frac{(nx)}{n^2}. \quad (2)$$

This series is obviously uniformly convergent in $\mathfrak{N} = (-\infty, \infty)$.

Since (x) has the period 1 and is continuous within $(-\frac{1}{2}, \frac{1}{2})$, we see that (nx) has the period $\frac{1}{n}$, and is continuous within $(-\frac{1}{2n}, \frac{1}{2n})$. The points of discontinuity of (nx) are thus

$$\mathfrak{E}_n = \left\{ \frac{1}{2n} + \frac{s}{n} \right\}, \quad s = 0, \pm 1, \pm 2, \dots$$

Let $\mathfrak{E} = \{\mathfrak{E}_n\}$. Then at any x not in \mathfrak{E} , each term of 2) is a continuous function of x . Hence $F(x)$ is continuous at this point.

On the other hand, F is discontinuous at any point e of \mathfrak{E} . For \sum being uniformly convergent,

$$R \lim_{x=e} F(x) = \sum R \lim_{x=e} \frac{(nx)}{n^2} \quad (3)$$

$$L \lim_{x=e} F(x) = \sum L \lim_{x=e} \frac{(nx)}{n^2}. \quad (4)$$

We show now that 3) has the value

$$F(e) - \frac{\pi^2}{16n^2}, \quad \text{for } e = \frac{2s+1}{2n}, \quad e \text{ irreducible.} \quad (5)$$

and 4) the value

$$F(e) + \frac{\pi^2}{16n^2}. \quad (6)$$

Hence

$$\text{Disc}_{x=e} F(x) = \frac{\pi^2}{8n^2}. \quad (7)$$

To this end let us see when two of the numbers

$$\frac{1}{2m} + \frac{r}{n}, \quad \text{and} \quad \frac{1}{2n} + \frac{s}{n} \quad m \neq n$$

are equal. If equal, we have

$$\frac{2r+1}{m} = \frac{2s+1}{n}. \quad (8)$$

Thus if we take $2s+1$ relatively prime to n , no two of the numbers in \mathfrak{E}_n are equal. Let us do this for each n . Then no two of the numbers in \mathfrak{E} are equal.

Let now $x = e = \frac{1}{2n} + \frac{s}{n}$. Then (mx) is continuous at this point, unless 8) holds; i.e. unless m is a multiple of n , say $m = ln$. In this case, 8) gives

$$2r+1 = l(2s+1).$$

Thus l must be odd; $l = 1, 3, 5 \dots$. In this case $(mx) = 0$ at e , while $R \lim_{x=e} (mx) = -\frac{1}{2}$. When m is not an odd multiple of n , obviously

$$R \lim_{x=e} (mx) = (me).$$

Thus when $m = ln$, l odd,

$$R \lim_{x=e} \frac{(mx)}{m^2} = -\frac{1}{2} \frac{1}{l^2 n^2} = \frac{(mx)}{m^2} - \frac{1}{2} \frac{1}{n^2} \cdot \frac{1}{l^2}.$$

When m is not a multiple of n ,

$$R \lim_{x=e} \frac{(mx)}{m^2} = \frac{(mx)}{m^2}.$$

$$\begin{aligned} R \lim_{x=e} F(x) &= F(e) - \frac{1}{2n^2} \left\{ \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right\} \\ &= F(e) - \frac{\pi^2}{16n^2}, \quad \text{by 218.} \end{aligned}$$

is establishes 5). Similarly we prove 6). Thus $F(x)$ is continuous at each point of \mathfrak{E} . As

$$|F(x)| \leq \sum \frac{1}{n^2}$$

is limited. As the points \mathfrak{E} form an enumerable set, F is integrable in any finite interval.

6. *Example 2.* Let $f(x) = 0$ at the points of a Cantor set $\mathfrak{C} = \{a_1 a_2 \dots\}$; $m = 0$, or a positive or negative integer, and the m th power of 3 is not a factor of m . Let $f(x) = 1$ elsewhere. Since $f(x)$ admits the period $\frac{1}{3}$, $f(3nx)$ admits the period $\frac{1}{3n}$. Let C_1 be the points of \mathfrak{C} which fall in $\mathfrak{A} = (0, 1)$. Let D_1 be the corresponding set of intervals. Let $C_2 = C_1 + \Gamma_1$, where Γ_1 is obtained by putting a $\frac{1}{3}$ in each interval of D_1 . Let D_2 be the intervals corresponding to C_2 . Let $C_3 = C_2 + \Gamma_2$ where Γ_2 is obtained by putting a $\frac{1}{9}$ in each interval of D_2 , etc.

The zeros of $f(3nx)$ are obviously the points of C_n . Let

$$F = \sum \frac{1}{n^2} f(3nx) = \Sigma f_n(x).$$

The zeros of F are the points of $\mathfrak{C} = \{C_n\}$. Since each C_n is a null set, \mathfrak{C} is also a null set. Let $A = \mathfrak{A} - \mathfrak{C}$. The points A, \mathfrak{C} are pantactic in \mathfrak{A} . Obviously F converges uniformly in \mathfrak{A} , $0 \leq f(3nx) \leq 1$. Since $f_n(x)$ is continuous at each point a of A , F is continuous at a , and

$$F(a) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = H.$$

We show now that F is discontinuous at each point of \mathfrak{E} . For let e_m be an end point of one of the intervals of D_{m+1} but not of D_m . Then

$$f_1(e_m) = \frac{1}{1^2}, \quad \dots, f_m(e_m) = \frac{1}{m^2},$$

$$f_{m+p}(e_m) = 0, \quad p = 1, 2, \dots$$

Hence
$$F(e_m) = H_m = \frac{1}{1^2} + \dots + \frac{1}{m^2}.$$

As the points A are pantactic in \mathfrak{A} , there exists a sequence in A which $\doteq e$. For this sequence $F \doteq H$. Hence

$$\text{Disc } F = H - H_m = H_m.$$

$x=e_m$

Similarly, if η_m is not an end point of the intervals D_{m+1} , but a limiting point of such end points,

$$\text{Disc } F = \bar{H}_m.$$

$x=\eta_m$

The function F is R -integrable in \mathfrak{A} since its points of discontinuity \mathfrak{E} form a null set.

467. Let $\mathfrak{E} = \{e_1, \dots, e_s\}$ be an enumerable set of points lying in the limited or unlimited set \mathfrak{A} , which lies in \mathfrak{R}_m . For any x in \mathfrak{A} and any e_i in \mathfrak{E} , let $x - e_i$ lie in \mathfrak{B} . Let $g(x_1 \dots x_m)$ be limited in \mathfrak{B} and continuous, except at $x = 0$, where

$$\text{Disc } g(x) = \mathfrak{d}.$$

Let $C = \Sigma e_1, \dots, e_s$ converge absolutely. Then

$$F(x_1 \dots x_m) = \Sigma e_i g(x - e_i)$$

is continuous in $A = \mathfrak{A} - \mathfrak{E}$, and at $x = e_i$,

$$\text{Disc } F(x) = e_i \mathfrak{d}.$$

For when x ranges over \mathfrak{A} , $x - e_i$ remains in \mathfrak{B} , and g is limited in \mathfrak{B} . Hence F is uniformly and absolutely convergent in \mathfrak{A} .

Now each $g(x - e_i)$ is continuous in A ; hence F is continuous in A by 147, 2.

er hand, F is discontinuous at $x = e_k$. For

$$F(x) = c_k g(x - e_k) + H(x),$$

the series F after removing the term on the right of
tion. But H , as has just been shown, is continuous

ple 1. Let $\mathfrak{E} = \{e_n\}$ denote the rational numbers.

$$g(x) = \sin \frac{\pi}{x}, \quad x \neq 0 \\ = 0, \quad x = 0.$$

$$F(x) = \sum \frac{1}{n_\mu} g(x - e_n), \quad \mu > 1$$

except at the points \mathfrak{E} . At $x = e_n$,

$$\text{Disc } F = \frac{2}{n^\mu}.$$

Let $\mathfrak{E} = \{e_n\}$ denote the rational numbers.

$$g(x) = \lim_{n \rightarrow \infty} \frac{nx}{1 + nx} = 1, \quad x \neq 0 \\ = 0, \quad x = 0,$$

sidered in I, §31.

$$F(x) = \sum \frac{1}{n!} g(x - e_n)$$

except at the rational points, and at $x = e_m$,

$$\text{Disc } F(x) = \frac{1}{m!}.$$

foregoing $g(x)$ is limited. This restriction may be
any cases, as the reader will see from the following
as an example.

... be an enumerable apantactic set in \mathfrak{A} . Let $\mathfrak{E} =$
any x in \mathfrak{A} , and any e_i in E , let $x - e_i$ lie within a
($x_1 \dots x_m$) be continuous in \mathfrak{B} except at $x = 0$, where
 $= 0$. Let $\Sigma c_1, \dots, c_m$ be a positive term convergent series.

Then

$$F(x_1 \dots x_m) = \sum e_i g(x - e_i)$$

is continuous in $A = \mathfrak{X} - \mathfrak{E}$. On the other hand, each point of \mathfrak{E} is a point of infinite discontinuity.

For any given point $x = a$ of A lies at a distance > 0 from \mathfrak{E} . Thus

$$\text{Min } (x - e_i) > 0,$$

as x ranges over some $V_\eta(a)$, and e_i over \mathcal{E} .

Hence

$$|g(x - e_i)| < \text{some } M,$$

for x in $V_\eta(a)$, and e_i in \mathcal{E} . Thus F is uniformly convergent at $x = a$. As each $g(x - e_i)$ is continuous at $x = a$, F is continuous at a .

Let next $x = e_\kappa$. Then there exists a sequence

$$x', x'' \dots \doteq e_\kappa \tag{1}$$

whose points lie in A . Thus the term $g(x - e_\kappa) \doteq +\infty$ as x ranges over 1). Hence a fortiori $F = +\infty$. Thus each point of \mathcal{E} is a point of infinite discontinuity.

Finally any limit point of \mathcal{E} is a point of infinite discontinuity, by 462, 1.

470. Example. Let $g(x) = \frac{1}{x}$, $a_n = -\frac{1}{a^n}$, $a > 1$.

$$e_n = \frac{1}{a^n n!}.$$

Then

$$\begin{aligned} F(x) &= \sum e_n g(x - a_n) \\ &= \sum \frac{1}{n!} \frac{1}{1 + a^n x} \end{aligned}$$

is a continuous function, except at the points

$$0, -\frac{1}{a}, -\frac{1}{a^2}, -\frac{1}{a^3} \dots$$

which are points of infinite discontinuity.

471. Let us show how to construct functions by limiting processes, whose points of discontinuity are any given complete limited apantactic set \mathfrak{E} in an m -way space \mathfrak{X}_m .

1. Let us for simplicity take $m = 2$, and call the coördinates of a point x, y .

Let Q denote the square whose center is the origin, and one of whose vertices is the point $(1, 0)$.

The edge of Q is given by the points x, y satisfying

$$|x| + |y| = 1. \quad (1)$$

Thus

$$Q(x, y) = \lim_{n \rightarrow \infty} \frac{1}{1 + (|x| + |y|)^n} = \begin{cases} \frac{1}{2}, & \text{on the edge} \\ 1, & \text{inside} \\ 0, & \text{outside} \end{cases} \quad (2)$$

of the square Q . Hence

$$L(x, y) = \frac{1}{2} \left[1 - \lim_{n \rightarrow \infty} \frac{n \{1 - |x| - |y|\}}{1 + n \{1 - |x| - |y|\}} \right] = \begin{cases} \frac{1}{2}, & \text{on the edge,} \\ 0, & \text{off the edge.} \end{cases} \quad (3)$$

Thus

$$G(x, y) = Q(x, y) + L(x, y) = \begin{cases} 1, & \text{in } Q, \\ 0, & \text{without } Q. \end{cases}$$

2. We next show how to construct a function g which shall = 0 on one or more of the edges of Q . Let us call these sides e_1, e_2, e_3, e_4 , as we go around the edge of Q beginning with the first quadrant. If $G = 0$ on e_i , let us denote it by G_i ; if $G = 0$ on e_i, e_k let us denote it by G_{ik} , etc. We begin by constructing G_1 . We observe that

$$1 - \lim_{n \rightarrow \infty} \frac{n |t|}{1 + n |t|} = \begin{cases} 1, & \text{for } t = 0, \\ 0, & \text{for } t \neq 0. \end{cases}$$

Now the equation of a right line l may be given the form

$$x \cos \alpha + y \sin \alpha = p$$

where $0 \leq \alpha < 2\pi, p \geq 0$. Hence

$$Z(x, y) = 1 - \lim_{n \rightarrow \infty} \frac{n |x \cos \alpha + y \sin \alpha - p|}{1 + n |x \cos \alpha + y \sin \alpha - p|} = \begin{cases} 1, & \text{on } l, \\ 0, & \text{off } l. \end{cases}$$

If now we make l coincide with e_1 , we see that

$$E_1(x, y) = 2 Z(x, y) L(x, y) = \begin{cases} 1, & \text{on } e_1, \\ 0, & \text{off } e_1. \end{cases}$$

Hence

$$G_1(x, y) = G(x, y) - E_1(x, y) = \begin{cases} 1, & \text{in } Q \text{ except on } e_1, \\ 0, & \text{on } e_1 \text{ and without } Q. \end{cases}$$

In the same way,

$$G_{ij} = G - (E_i + E_j),$$

$$G_{ijk} = G - (E_i + E_j + E_k),$$

$$G_{1234} = G - (E_1 + E_2 + E_3 + E_4).$$

By introducing a constant factor we can replace Q by a square Q_c whose sides are in the ratio $c:1$ to those of Q .

Thus

$$Q(x, y) = \lim_{n \rightarrow \infty} \frac{1}{1 + \left(\frac{|x|}{c} + \frac{|y|}{c} \right)^n} = \begin{cases} \frac{1}{2}, & \text{on the edge of } Q_c, \\ 1, & \text{inside,} \\ 0, & \text{outside.} \end{cases}$$

We can replace the square Q by a similar square whose center is a, b on replacing $|x|, |y|$ by $|x - a|, |y - b|$.

We have thus this result: by a limiting process, we can construct a function $g(x, y)$ having the value 1 inside Q , and on any of its edges, and = 0 outside Q , and on the remaining edges. Q has any point a, b as center, its edges have any length, and its sides are tipped at an angle of 45° to the axes.

We may take them parallel to the axes, if we wish, by replacing $|x|, |y|$ in our fundamental relation 1) by

$$|x - y|, \quad |x + y|.$$

Finally let us remark that we may pass to m -way space, by replacing 1) by

$$|x_1| + |x_2| + \dots + |x_m| = 1.$$

3. Let now $\Omega = \{q_n\}$ be a border set [328], of non-overlapping squares belonging to the complete apantactic set \mathfrak{E} , such that $\Omega + \mathfrak{E} = \mathfrak{R}$ the whole plane. We mark these squares in the plane and note which sides q_n has in common with the preceding q 's. We take the $g_n(xy)$ function so that it is = 1 in q_n , except on these sides, and there 0. Then

$$G(x, y) = \sum g_n(xy)$$

has for each point only one term $\neq 0$, if x, y lies in Ω , and no term $\neq 0$ if it lies in \mathfrak{E} .

Hence

$$G(xy) = \begin{cases} 1, & \text{for each point of } \Omega, \\ 0, & \text{for each point of } \mathfrak{E}. \end{cases}$$

Since \mathfrak{C} is apantactic, each point of \mathfrak{C} is a point of discontinuity of the 2° kind; each point of \mathfrak{Q} is a point of continuity.

4. Let $f(xy)$ be a function defined over \mathfrak{A} which contains the complete apantactic set \mathfrak{C} .

Then

$$F(xy) = \Sigma f(xy) g_n(xy) = \begin{cases} f(xy), & \text{in } \mathfrak{A} - \mathfrak{C}, \\ 0, & \text{in } \mathfrak{C}. \end{cases}$$

472. 1. Let $\mathfrak{A} = (0, 1)$, $\mathfrak{B}_n =$ the points $\frac{2m+1}{2^n}$ in \mathfrak{A} .

Then $\mathfrak{B}_n, \mathfrak{B}_m$ have no points in common.

Let $f_n(x) = 1$ in \mathfrak{B}_n , and $= 0$ in $B_n = \mathfrak{A} - \mathfrak{B}_n$.

Let $\mathfrak{B} = \{\mathfrak{B}_n\}$. Then

$$F(x) = \Sigma f_n(x) = \begin{cases} 1, & \text{in } \mathfrak{B}, \\ 0, & \text{in } B = \mathfrak{A} - \mathfrak{B}. \end{cases}$$

The function F is totally discontinuous in \mathfrak{B} , oscillating between 0 and 1. The series F does not converge uniformly in any subinterval of \mathfrak{A} .

2. Keeping the notation in 1, let

$$G(x) = \Sigma \frac{1}{n} f_n(x).$$

At each point of \mathfrak{B}_n , $G = \frac{1}{n}$, while $G = 0$ in B .

The function G is discontinuous at the points of \mathfrak{B} , but continuous at the points B . The series G converges uniformly in \mathfrak{A} , yet an infinity of terms are discontinuous in any interval in \mathfrak{A} .

473. Let the limited set \mathfrak{A} be the union of an enumerable set of complete sets $\{\mathfrak{A}_n\}$. We show how to construct a function f , which is discontinuous at the points of \mathfrak{A} , but continuous elsewhere in an m -way space.

Let us suppose first that \mathfrak{A} consists of but one set and is complete. A point all of whose coördinates are rational, let us call rational, the other points of space we will call non-rational. If \mathfrak{A} is an inner rational point, let $f = 1$ at this point, on the frontier let $f = 0$ also; at all other points of space let $f = 0$. Then each point a of \mathfrak{A} is a point of discontinuity. For if x is a fron-

tier or an inner rational point of \mathfrak{A} , $f(x) = 1$, while in any $V(x)$ there are points where $f = 0$. If x is not in \mathfrak{A} , all the points of some $D(x)$ are also not in \mathfrak{A} . At these points $f = 0$. Hence f is continuous at such points.

We turn now to the general case. We have

$$\mathfrak{A} = A_1 + A_2 + A_3 + \dots$$

where $A_1 = \mathfrak{A}_1$, $A_2 =$ points of \mathfrak{A}_2 not in \mathfrak{A}_1 , etc. Let $f_1 = 1$ at the rational inner points of A_1 , and at the frontier points of \mathfrak{A}_1 ; at all other points let $f_1 = 0$. Let $f_2 = \frac{1}{2}$ at the rational inner points of A_2 , and at the frontier points of A_2 not in A_1 ; at all other points let $f_2 = 0$. At similar points of A_3 let $f_3 = \frac{1}{3}$, and elsewhere $= 0$, etc.

Consider now

$$F = \Sigma f_n(x_1 \dots x_m).$$

Let $x = a$ be a point of \mathfrak{A} . If it is an inner point of some A_s , it is obviously a point of discontinuity of F . If not, it is a proper frontier point of one of the A 's. Then in any $D(a)$ there are points of space not in \mathfrak{A} , or there are points of an infinite number of the A 's. In either case a is a point of discontinuity. Similarly we see F is continuous at a point not in \mathfrak{A} .

2. We can obviously generalize the preceding problem by supposing \mathfrak{A} to lie in a complete set \mathfrak{B} , such that each frontier point of \mathfrak{A} is a limit point of $A = \mathfrak{B} - \mathfrak{A}$.

For we have only to replace our m -way space by \mathfrak{B} .

Functions of Class 1

474. 1. Baire has introduced an important classification of functions as follows:

Let $f(x_1 \dots x_m)$ be defined over \mathfrak{A} ; f and \mathfrak{A} limited or unlimited. If f is continuous in \mathfrak{A} , we say its class is 0 in \mathfrak{A} , and write

$$\text{Class } f = 0 \quad , \quad \text{or } \text{Cl } f = 0 \quad , \quad \text{Mod } \mathfrak{A}.$$

If

$$f = \lim_{n \rightarrow \infty} f_n(x_1 \dots x_m),$$

each f_n being of class 0 in \mathfrak{A} , we say its class is 1, if f does not lie in class 0, mod \mathfrak{A} .

series

$$F(x) = \sum f_n(x)$$

\mathfrak{A} , each term f_n being continuous in \mathfrak{A} . Since

$$F(x) = \lim_{n \rightarrow \infty} F_n(x),$$

of class 0, or class 1, according as F is continuous, or not in \mathfrak{A} . A similar remark holds for infinite prod-

$$G(x) = \prod g_n(x).$$

derivatives of a function $f(x)$ give rise to functions of

For let $f(x)$ have a unilateral differential coefficient at each point of \mathfrak{A} . Both f and \mathfrak{A} may be unlimited. Ideas, suppose the right-hand differential coefficient $h_1 > h_2 > \dots \doteq 0$. Then

$$q_n(x) = \frac{f(x + h_n) - f(x)}{h_n}, \quad x + h_n \text{ in } \mathfrak{A},$$

is a function of x in \mathfrak{A} . But

$$q(x) = \lim_{n \rightarrow \infty} q_n(x)$$

exists in \mathfrak{A} by hypothesis.

The remark applies to the partial derivatives

$$\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_m}$$

$f(x_1 \dots x_n).$

$$f(x) = \lim_{n \rightarrow \infty} f_n(x_1 \dots x_n),$$

of class 1 in \mathfrak{A} . Then we say, $\text{Cl } f = 2$ if f does not belong to class 1. In this way we may continue. It is of course easy to show that such functions actually exist.

Example 1.

$$f(x) = \lim_{n \rightarrow \infty} \frac{nx}{1 + nx} = \begin{cases} 1, & \text{for } x > 0, \\ 0, & \text{for } x = 0. \end{cases}$$

This function was considered in I, 331. In any interval containing the origin $x = 0$, $\text{Cl } f = 1$; in any interval $x > 0$, not containing the origin, $\text{Cl } f = 0$.

Example 2.

Let

$$f(x) = \lim_{n=\infty} \frac{nx}{e^{nx^2}} = 0, \text{ in } \mathfrak{A} = (-\infty, \infty).$$

The class of $f(x)$ is 0 in \mathfrak{A} . Although each f_n is limited in \mathfrak{A} , the graphs of f_n have peaks near $x = 0$ which $\doteq \infty$, as $n \doteq \infty$.

Example 3. If we combine the two functions in Ex. 1, 2, we get

$$f(x) = \lim_{n=\infty} \left\{ \frac{1}{1+nx} + \frac{1}{e^{nx^2}} \right\} nx = \begin{cases} 1, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$$

Hence $\text{Cl} f(x) = 1$ for any set \mathfrak{B} embracing the origin; $= 0$ for any other set.

Example 4.

Let

$$f(x) = \lim_{n=\infty} x e^{\frac{1}{x+\frac{1}{n}}}, \text{ in } \mathfrak{A} = (0, 1).$$

Then

$$\begin{aligned} f(x) &= 0, \text{ for } x = 0 \\ &= x e^{\frac{1}{x}}, \text{ for } x > 0. \end{aligned}$$

We see thus that f is continuous in $(0^*, 1)$, and has a point of infinite discontinuity at $x = 0$.

Hence

$$\begin{aligned} \text{Class } f(x) &= 1, \text{ in } \mathfrak{A} \\ &= 0, \text{ in } (0^*, 1). \end{aligned}$$

Example 5.

Let

$$f(x) = \lim_{n=\infty} \frac{1}{x + \frac{1}{n}} \text{ in } \mathfrak{A} = (0, \infty).$$

Then

$$\begin{aligned} f(x) &= \frac{1}{x}, \text{ for } x > 0 \\ &= +\infty, \text{ for } x = 0. \end{aligned}$$

Here

$$\lim_{n=\infty} f_n(x)$$

does not exist at $x = 0$. We cannot therefore speak of the class of $f(x)$ in \mathfrak{A} since it is not defined at the point $x = 0$. It is defined in $\mathfrak{B} = (0^*, \infty)$, and its class is obviously 0, mod \mathfrak{B} .

Example 6.

Let

$$f(x) = \sin \frac{1}{x} \quad , \quad \text{for } x \neq 0$$

$$= \text{a constant } c \quad , \quad \text{for } x = 0.$$

We show that $\text{Cl } f = 1$ in $\mathfrak{A} = (-\infty, \infty)$. For let

$$f_n(x) = c \left(1 - \frac{nx}{1+nx} \right) + \frac{nx}{1+nx} \sin \left[\frac{1}{x + \frac{1}{n}} \right]$$

$$= g_n(x) + h_n(x).$$

Now by Ex. 1,

$$\lim g_n(x) = \begin{cases} 0, & \text{for } x > 0, \\ c, & \text{for } x = 0; \end{cases}$$

while

$$\lim h_n(x) = \begin{cases} \sin \frac{1}{x}, & \text{for } x > 0, \\ 0, & \text{for } x = 0. \end{cases}$$

As each f_n is continuous in \mathfrak{A} , and

$$\lim f_n(x) = f(x) \text{ in } \mathfrak{A},$$

we see its class is ≤ 1 . As f is discontinuous at $x = 0$, its class is not 0 in \mathfrak{A} .

Example 7. Let

$$f(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \sin \frac{1}{x}.$$

Here the functions $f_n(x)$ under the limit sign are not defined for $x = 0$. Thus f is not defined at this point. We cannot therefore speak of the class of f with respect to any set embracing the point $x = 0$. For any set \mathfrak{B} not containing this point, $\text{Cl } f = 0$, since $f(x) = 0$ in \mathfrak{B} .

Let us set

$$\phi(x) = \sin \frac{1}{x} \quad , \quad \text{for } x \neq 0$$

$$= \text{a constant } c \quad , \quad \text{for } x = 0.$$

Let

$$g(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \phi(x) = \lim_{n \rightarrow \infty} \phi_n(x).$$

Here g is a continuous function in $\mathfrak{A} = (-\infty, \infty)$. Its class is thus 0 in \mathfrak{A} . On the other hand, the functions ϕ_n are each of class 1 in \mathfrak{A} .

Example 8.

$$\Gamma(x) = \frac{1}{x} \prod_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^x}{1 + \frac{x}{n}}$$

is defined at all the points of $(-\infty, \infty)$ except $0, -1, -2, \dots$. These latter are points of infinite discontinuity. In its domain of definition, Γ is a continuous function. Hence $\text{Cl } \Gamma(x) = 0$ with respect to this domain.

476. 1. If \mathfrak{A} , limited or unlimited, is the union of an enumerable set of complete sets, we say \mathfrak{A} is *hypercomplete*.

Example 1. The points S^* within a unit sphere S , form a hypercomplete set. For let Σ_r have the same center as S , and radius $r < 1$. Obviously each Σ_r is complete, while $\{\Sigma_r\} = S^*$, r ranging over $r_1 < r_2 < \dots \doteq 1$.

Example 2. An enumerable set of points a_1, a_2, \dots form a hypercomplete set. For each a_n may be regarded as a complete set, embracing but a single point.

2. If $\mathfrak{A}_1, \mathfrak{A}_2, \dots$ are limited hypercomplete sets, so is their union $\{\mathfrak{A}_n\} = \mathfrak{A}$.

For each \mathfrak{A}_n is the union of an enumerable set of complete sets $\mathfrak{A}_{m,n}$. Thus $\mathfrak{A} = \{\mathfrak{A}_{m,n}\} \ m, n = 1, 2, \dots$ is hypercomplete.

Let \mathfrak{A} be complete. If \mathfrak{B} is a complete part of \mathfrak{A} , $A = \mathfrak{A} - \mathfrak{B}$ is hypercomplete.

For let $\Omega = \{q_n\}$ be a border set of \mathfrak{B} , as in 328. The points A_n of A in each q_n are complete, since \mathfrak{A} is complete. Thus $A = \{A_n\}$, and A is hypercomplete.

Let $\mathfrak{A} = \{\mathfrak{A}_n\}$ be hypercomplete, each \mathfrak{A}_n being complete. If \mathfrak{B} is a complete part of \mathfrak{A} , $A = \mathfrak{A} - \mathfrak{B}$ is hypercomplete.

For let A_n denote the points of \mathfrak{A}_n not in \mathfrak{B} . Then as above, A_n is hypercomplete. As $A = \{A_n\}$, A is also hypercomplete.

477. 1. \mathfrak{E}_ϵ Sets. If the limited or unlimited set \mathfrak{N} is the union of an enumerable set of limited complete sets, in each of which $\text{Osc } f < \epsilon$, we shall say \mathfrak{N} is an \mathfrak{E}_ϵ set. If, however small $\epsilon > 0$ is taken, \mathfrak{N} is an \mathfrak{E}_ϵ set, we shall say \mathfrak{N} is an \mathfrak{E}_ϵ set, $\epsilon \doteq 0$, which we may also express by $\mathfrak{E}_{\epsilon \doteq 0}$.

2. Let $f(x_1 \dots x_m)$ be continuous in the limited complete set \mathfrak{N} . Then \mathfrak{N} is an \mathfrak{E}_ϵ set, $\epsilon \doteq 0$.

For let $\epsilon > 0$ be taken small at pleasure and fixed. By I, 353, there exists a cubical division of space D , such that if \mathfrak{N}_n denote the points of \mathfrak{N} in one of the cells of D , $\text{Osc } f < \epsilon$ in \mathfrak{N}_n . As \mathfrak{N} is complete, since \mathfrak{N} is, \mathfrak{N} is an \mathfrak{E}_ϵ set.

3. An enumerable set of points $\mathfrak{N} = \{a_n\}$ is an $\mathfrak{E}_{\epsilon \doteq 0}$ set.

For each a_n may be regarded as a complete set, embracing but single point. But in a set embracing but one point, $\text{Osc } f = 0$.

4. The union of an enumerable set of \mathfrak{E}_ϵ sets $\mathfrak{N} = \{\mathfrak{N}_m\}$ is an \mathfrak{E}_ϵ set.

For each \mathfrak{N}_m is the union of an enumerable set of limited sets $\mathfrak{N}_m = \{\mathfrak{N}_{m,n}\}$, $n = 1, 2, \dots$ and $\text{Osc } f < \epsilon$ in each $\mathfrak{N}_{m,n}$.

Thus

$$\mathfrak{N} = \{\mathfrak{N}_{mn}\} \quad , \quad m, n = 1, 2, \dots$$

But an enumerable set of enumerable sets is an enumerable set. Hence \mathfrak{N} is an \mathfrak{E}_ϵ set.

5. Let $f(x_1 \dots x_m)$ be continuous in the complete set \mathfrak{N} , except at the points $\mathfrak{D} = d_1, d_2 \dots d_s$. Then \mathfrak{N} is an $\mathfrak{E}_{\epsilon \doteq 0}$ set.

For let $\epsilon > 0$ be taken small at pleasure and fixed. About each point of \mathfrak{D} we describe a sphere of radius ρ . Let \mathfrak{N}_ρ denote the points of \mathfrak{N} not within one of these spheres. Obviously \mathfrak{N}_ρ is complete. Let ρ range over $r_1 > r_2 > \dots = 0$. If we set $\mathfrak{N} = A + \mathfrak{D}$, obviously $A = \{\mathfrak{N}_{r_n}\}$. As f is continuous in \mathfrak{N}_{r_n} , it is an \mathfrak{E}_ϵ set. Hence \mathfrak{N} , being the union of A and \mathfrak{D} , is an \mathfrak{E}_ϵ set.

478. 1. Let \mathfrak{N} be an \mathfrak{E}_ϵ set. The points \mathfrak{D} of \mathfrak{N} common to the limited complete set \mathfrak{B} form an \mathfrak{E}_ϵ set.

For \mathfrak{N} is the union of the complete sets \mathfrak{N}_n , in each of which $\text{Osc } f < \epsilon$. But the points of \mathfrak{N}_n in \mathfrak{B} form a complete set A_n , and of course $\text{Osc } f < \epsilon$ in A_n . As $\mathfrak{D} = \{A_n\}$, it is an \mathfrak{E}_ϵ set.

2. Let \mathfrak{A} be a limited \mathfrak{E}_ϵ set. Let \mathfrak{B} be a complete part of \mathfrak{A} . Then $A = \mathfrak{A} - \mathfrak{B}$ is an \mathfrak{E}_ϵ set.

For \mathfrak{A} is the union of the complete sets \mathfrak{A}_n , in each of which $\text{Osc } f < \epsilon$. The points of \mathfrak{A}_n not in \mathfrak{B} form a set A_n , such that $\text{Osc } f < \epsilon$ in A_n also. But $A = \{A_n\}$, and each A_n being hyper-complete, is an \mathfrak{E}_ϵ set.

3. Let $f(x_1 \cdots x_m)$ be defined over \mathfrak{A} , either f or \mathfrak{A} being limited or unlimited. The points of \mathfrak{A} at which

$$\alpha \leq f \leq \beta \quad (1)$$

may be denoted by

$$(\alpha \leq f \leq \beta). \quad (2)$$

If in 1) one of the equality signs is missing, it will of course be dropped in 2).

479. 1. Let f_1, f_2, \dots be continuous in the limited complete set \mathfrak{A} . If at each point of \mathfrak{A} , $\lim_{n \rightarrow \infty} f_n$ exists, \mathfrak{A} is an $\mathfrak{E}_{\epsilon \pm 0}$ set and so is any complete $\mathfrak{B} < \mathfrak{A}$.

For let $\lim_{n \rightarrow \infty} f_n(x_1 \cdots x_m) = f(x_1 \cdots x_m)$ in \mathfrak{A} . Let us effect a division of norm $\epsilon/2$ of the interval $(-\infty, \infty)$ by interpolating the points $\dots m_{-2}, m_{-1}, m_0 = 0, m_1, m_2 \dots$

Let $\mathfrak{A}_i = (m_i < f < m_{i+2})$, then $\mathfrak{A} = \{\mathfrak{A}_i\}$.

Next let $\mathfrak{D}_{n,p} = Dv \left\{ m_i + \frac{1}{n} \leq f_q \leq m_{i+2} - \frac{1}{n} \right\}$.

Then $\mathfrak{A}_i = \{\mathfrak{D}_{n,p}\} \quad , \quad n, p = 1, 2 \dots \quad (1)$

For let α be a point of \mathfrak{A}_i , and say $f(\alpha) = \alpha$. Then

$$m_i < \alpha < m_{i+2}.$$

But $\alpha - \epsilon < f_q(\alpha) < \alpha + \epsilon \quad , \quad q > \text{some } p,$

and we may take ϵ and n so that

$$m_i + \frac{1}{n} \leq f_q(\alpha) \leq m_{i+2} - \frac{1}{n}.$$

Hence α is in $\mathfrak{D}_{n,p}$.

Conversely, let α be a point of $\{\mathfrak{D}_{n,p}\}$. Then α lies in some $\mathfrak{D}_{n,p}$. Hence,

$$m_i + \frac{1}{n} \leq f_q(\alpha) \leq m_{i+2} - \frac{1}{n} \quad , \quad q \geq p.$$

But as $f_n(a) \doteq f(a)$, we have

$$|f(a) - f_q(a)| < \epsilon \quad , \quad q > \text{some } p'.$$

Hence if ϵ is sufficiently small,

$$m_i < f(a) < m_{i+2},$$

and thus a is in \mathfrak{A}_i .

Thus 1) is established. But \mathfrak{D}_{np} is a divisor of complete sets, and is therefore complete. Thus \mathfrak{A} is the union of an enumerable set of complete sets $\{\mathfrak{B}_i\}$, in each of which $\text{Osc } f < \epsilon$, ϵ small at pleasure.

Let now \mathfrak{B} be any complete part of \mathfrak{A} . Let $a_i = Dv \{ \mathfrak{B}, \mathfrak{B}_i \}$. Then a_i is complete, and $\text{Osc } f < \epsilon$, in a_i . Moreover, $\mathfrak{B} = \{a_i\}$.

Hence \mathfrak{B} is an $\mathfrak{G}_{\epsilon=0}$ set.

2. If $\text{Class } f \leq 1$ in limited complete \mathfrak{A} , f limited or unlimited, \mathfrak{A} is an \mathfrak{G}_ϵ set.

This is an obvious result from 1.

3. Let $f(x_1 \dots x_m)$ be a totally discontinuous function in the non-enumerable set \mathfrak{A} . Then $\text{Class } f$ is not 0 or 1 in \mathfrak{A} , if $b = \text{Disc } f$ at each point is $\leq k > 0$.

For in any subset \mathfrak{B} of \mathfrak{A} containing the point x , $\text{Osc } f \geq k$. Hence $\text{Osc } f$ is not $\leq \epsilon$, in any part of \mathfrak{A} , if $\epsilon < k$. Thus \mathfrak{A} cannot be an \mathfrak{G}_ϵ set.

4. If $\text{Class } f(x_1 \dots x_m) \leq 1$ in the limited complete set \mathfrak{A} , the set $\mathfrak{A} = (a < f < b)$ is a hypercomplete set, a, b being arbitrary numbers.

For we have only to take $a = m_i$, $b = m_{i+2}$. Then $\mathfrak{B} = \mathfrak{A}_i$, which, in 1, is hypercomplete.

480. (Lebesgue.) Let the limited or unlimited function $f(x_1 \dots x_m)$ defined over the limited set \mathfrak{A} . If \mathfrak{A} may be regarded as an \mathfrak{A}_0 set with respect to f , the class of f is ≤ 1 .

For let $\omega_1 > \omega_2 > \dots \doteq 0$. By hypothesis \mathfrak{A} is the union of a sequence of complete sets

$$\mathfrak{A}_{11} \quad , \quad \mathfrak{A}_{12} \quad , \quad \mathfrak{A}_{13} \dots \quad (\mathcal{S}_1)$$

each of which $\text{Osc } f \leq \omega_1$. \mathfrak{A} is also the union of a sequence of complete sets

$$\mathfrak{B}_{11} \quad , \quad \mathfrak{B}_{12} \quad , \quad \mathfrak{B}_{13} \dots \quad (\mathcal{I}_1)$$

in each of which $\text{Osc } f \leq \omega_2$. If we superpose the division 1) of \mathfrak{A} on the division S_1 , each $\mathfrak{A}_{i\kappa}$ will fall into an enumerable set of complete sets, and together they will form an enumerable sequence

$$\mathfrak{A}_{21}, \mathfrak{A}_{22}, \mathfrak{A}_{23} \dots \quad (S_2)$$

in each of which $\text{Osc } f \leq \omega_2$. Continuing in this way we see that \mathfrak{A} is the union of the complete sets

$$\mathfrak{A}_{n1}, \mathfrak{A}_{n2}, \mathfrak{A}_{n3} \dots \quad (S'_n)$$

such that in each set of S'_n , $\text{Osc } f < \omega_n$, and such that each set lies in some set of the preceding sequence S'_{n-1} .

With each \mathfrak{A}_{ns} we associate a constant C_{ns} , such that

$$|f(x) - C_{ns}| \leq \omega_n, \quad \text{in } \mathfrak{A}_{ns}, \quad (2)$$

and call C_{ns} the corresponding *field constant*.

We show now how to define a sequence of continuous functions $f_1, f_2 \dots$ which $\doteq f$. To this end we effect a sequence of superimposed divisions of space $D_1, D_2 \dots$ of norms $\doteq 0$. The vertices of the cubes of D_n we call the *lattice points* L_n . The cells of D_n containing a given lattice point l of L_n form a cube Ω . Let \mathfrak{A}_{1l} be the first set of S_1 containing a point of Ω . Let \mathfrak{A}_{2l} be the first set of S_2 containing a point of Ω lying in \mathfrak{A}_{1l} . Continuing in this way we get

$$\mathfrak{A}_{1l} \supseteq \mathfrak{A}_{2l} \supseteq \dots \supseteq \mathfrak{A}_{n_l l}.$$

To $\mathfrak{A}_{n_l l}$ belongs the field constant $C_{n_l l}$; this we associate with the lattice point l and call it the corresponding *lattice constant*.

Let now \mathfrak{C} be a cell of D_n containing a point of \mathfrak{A} . It has 2^n vertices or lattice points. Let P_s denote any product of s different factors $x_{r_1}, x_{r_2}, \dots x_{r_s}$. We consider the polynomial

$$\phi = AP_n + \Sigma BP_{n-1} + \Sigma CP_{n-2} + \dots + \Sigma KP_1 + L,$$

the summation in each case extending over all the distinct products of that type. The number of terms in ϕ is, by I, 96,

$$\binom{n}{n} + \binom{n}{1} + \binom{n}{2} + \dots + 1 = 2^n.$$

We can thus determine the 2^n coefficients of ϕ so that the values of ϕ at the lattice points of \mathfrak{C} are the corresponding lattice constants. Thus ϕ is a continuous function in \mathfrak{C} , whose greatest and least values are the greatest and least lattice constants belonging to \mathfrak{C} . Each cube \mathfrak{C} containing a point of \mathfrak{A} has associated with it a ϕ function.

We now define $f_n(x_1 \dots x_m)$ by stating that its value in any cube \mathfrak{C} of D_n , containing a point of \mathfrak{A} , is that of the corresponding ϕ function. Since ϕ is linear in each variable, two ϕ 's belonging to adjacent cubes have the same values along their common points.

We show now that $f_n(x) \doteq f(x)$ at any point x of \mathfrak{A} , or that

$$\epsilon > 0, \quad \nu, \quad |f(x) - f_n(x)| < \epsilon, \quad n > \nu. \quad (3)$$

Let $\omega_\epsilon < \epsilon/8$. Let $\mathfrak{A}_{1,1}$ be the first set in S_1 containing the point x , $\mathfrak{A}_{2,1}$ the first set of S_2 lying in $\mathfrak{A}_{1,1}$ and containing x . Continuing we get

$$\mathfrak{A}_{1,1} \supseteq \mathfrak{A}_{2,1} \supseteq \mathfrak{A}_{3,1} \supseteq \dots \supseteq \mathfrak{A}_{e,1}.$$

Let \mathfrak{B}_e be the union of the sets in S_1 preceding $\mathfrak{A}_{1,1}$; of the sets in S_2 preceding $\mathfrak{A}_{2,1}$ and lying in $\mathfrak{A}_{1,1}$, and so on, finally the sets of S_e preceding $\mathfrak{A}_{e,1}$, and lying in $\mathfrak{A}_{e-1,1}$. Their number being finite, $\delta = \text{Dist}(\mathfrak{A}_{e,1}, \mathfrak{B}_e)$ is obviously > 0 . We may therefore take $\nu > e$ so large that cubes of D_ν about the point x lie wholly in $D_\eta(x)$, $\eta < \delta$.

Consider now $f_n(x)$, $n > \nu$, and let us suppose first that x is not a lattice point of D_n . Let it lie within the cell \mathfrak{C} of D_n . Then $f_n(x)$ is a mean of the values of

$$f_n(l) = C_{n,j_n},$$

where l is any one of the 2^n vertices of \mathfrak{C} , and C_{n,j_n} is the corresponding lattice constant, which we know is associated with the set \mathfrak{A}_{n,j_n} .

We observe now that each of the

$$\mathfrak{A}_{n,j_n} \leq \mathfrak{A}_{e,1}. \quad (4)$$

For each set in S_n is a part of some set in any of the preceding sequences. Now \mathfrak{A}_{n,j_n} cannot be a part of $\mathfrak{A}_{1,k}$, $k < 1$, for none of

these points lie in $D_n(x)$. Hence \mathfrak{N}_{nj_n} is a part of \mathfrak{N}_{14} . For the same reason it is a part of \mathfrak{N}_{24} , etc., which establishes 4).

Let now x' be a point of \mathfrak{N}_{nj_n} . Then

$$|C_{nj_n} - C_{c_n}| \leq |C_{nj_n} - f(x')| + |f(x') - C_{c_n}|.$$

$$\leq \omega_n + \omega_o < \frac{\epsilon}{4}, \quad \text{by 2).}$$

From this follows, since $f_n(x)$ is a mean of these C_{nj_n} , that

$$|f_n(x) - C_{nj_n}| < \frac{\epsilon}{2}.$$

But now

$$|f(x) - f_n(x)| \leq |f(x) - C_{nj_n}| + |C_{nj_n} - f_n(x)|.$$

As x lies in \mathfrak{N}_{c_n} ,

$$|f(x) - C_{nj_n}| < |f(x) - C_{c_n}| + |C_{c_n} - C_{nj_n}|$$

$$\leq \omega_o + \frac{\epsilon}{4} < \frac{\epsilon}{2},$$

by 2), 5). From 6), 8) we have 3) for the present case.

The case that x is a lattice point for some division and hence for all following, has really been established by the foregoing reasoning.

481. 1. Let f be defined over the limited set \mathfrak{N} . If for arbitrary a, b , the sets $\mathfrak{B} = (a < f < b)$ are hypercomplete, then $\text{Class } f \leq 1$:

For let us effect a division of norm $\epsilon/2$ of $(-\infty, \infty)$ as in 479, 1. Then $\mathfrak{N} = \{\mathfrak{N}_i\}$, where as before $\mathfrak{N}_i = (m_i < f < m_{i+2})$. But as $\text{Osc } f < \epsilon$ in \mathfrak{N}_i , and as each \mathfrak{N}_i is hypercomplete by hypothesis, our theorem is a corollary of 480.

2. For $f(x_1 \dots x_n)$ to be of class ≤ 1 in the limited complete \mathfrak{N} , it is necessary and sufficient that the sets $(a < f < b)$ are hypercomplete, a, b being arbitrary.

This follows from 1 and 479, 2.

3. Let limited \mathfrak{N} be the union of an enumerable set of complete sets $\{\mathfrak{N}_n\}$, such that $\text{Cl } f \leq 1$ in each \mathfrak{N}_n , then $\text{Cl } f \leq 1$ in \mathfrak{N} .

For by 479, 1, \mathfrak{N}_n is the union of an enumerable set of complete sets in each of which $\text{Osc } f \leq \epsilon$. Thus \mathfrak{N} is also such a set, i.e. an \mathfrak{E} -set. We now apply 480, 1.

4. If $\text{Class } f \leq 1$ in the limited complete set \mathfrak{N} , its class is ≤ 1 , any complete part \mathfrak{B} of \mathfrak{N} .

This follows from 479, 1 and 480, 1.

482. 1. Let $f(x_1 \dots x_m)$ be defined over the complete set \mathfrak{N} , and have only an enumerable set \mathfrak{E} of points of discontinuity in \mathfrak{N} . Then $\text{Class } f = 1$ in \mathfrak{N} .

For the points E of \mathfrak{N} at which $\text{Osc } f \geq \epsilon/2$ form a complete part of \mathfrak{N} , by 462, 3. But E , being a part of \mathfrak{E} , is enumerable and is hence an \mathfrak{E}_c set by 477, 3. Let us turn to $\mathfrak{B} = \mathfrak{N} - E$. For each of its points b , there exists a $\delta > 0$, such that $\text{Osc } f < \epsilon$ in the set \mathfrak{b} of points of \mathfrak{B} lying in $D_\delta(b)$. As \mathfrak{N} is complete, so is \mathfrak{B} . As E is complete, there is an enumerable set of these \mathfrak{b} , call them $\mathfrak{b}_1, \mathfrak{b}_2, \dots$, such that $\mathfrak{B} = \{\mathfrak{b}_s\}$. As $\mathfrak{N} = \mathfrak{B} + E$, it is the union of an enumerable set of complete sets, in each of which $\text{Osc } f < \epsilon$. This is true however small $\epsilon > 0$ is taken. We apply now 480, 1.

2. We can now construct functions of class 2.

Example. Let $f_n(x_1 \dots x_m) = 1$ at the rational points in the unit cube \mathfrak{Q} , whose coördinates have denominators $\leq n$. Elsewhere let $f_n = 0$. Since f_n has only a finite number of discontinuities in \mathfrak{Q} , $\text{Cl } f_n = 1$ in \mathfrak{Q} . Let now

$$f(x_1 \dots x_m) = \lim_{n \rightarrow \infty} f_n.$$

At a non-rational point, each $f_n = 0$, $\therefore f = 0$. At a rational point, $f_n = 1$ for all $n > \text{some } s$. Hence at such a point $f = 1$. Thus each point of \mathfrak{Q} is a point of discontinuity and $\text{Disc } f = 1$. Hence $\text{Cl } f$ is not 1. As f is the limit of functions of class 1, its class is 2.

483. Let $f(x_1 \dots x_m)$ be continuous with respect to each x_i , at each point of a limited set \mathfrak{N} , each of whose points is an inner point. Then $\text{Class } f \leq 1$.

For let \mathfrak{A} lie within a cube Ω . Then $A = \Omega - \mathfrak{A}$ is complete. We may therefore regard \mathfrak{A} as a border set of A ; that is, a set of non-overlapping cubes $\{q_n\}$. We show now that $\text{Cl}f \leq 1$ in any one of these cubes as q . To this end we show that the points \mathfrak{B}_m of q at which

$$a + \frac{1}{m} \leq f \leq b - \frac{1}{m}$$

form a complete set. For let $b_1, b_2 \dots$ be points of \mathfrak{B}_m , which $\doteq \beta$. We wish to show that β lies in \mathfrak{B}_m . Suppose first that $b_s, b_{s+1} \dots$ have all their coördinates except one, say x , the same as the coördinates of β . Since

$$a + \frac{1}{m} \leq f(b_{s+p}) \leq b - \frac{1}{m},$$

therefore

$$a + \frac{1}{m} < \lim_{p \rightarrow \infty} f(b_{s+p}) \leq b - \frac{1}{m}.$$

As f is continuous in x_1 , and as only the coördinate x_1 varies in b_{s+p} , we have

$$a + \frac{1}{m} \leq f(\beta) \leq b - \frac{1}{m}.$$

Hence β lies in \mathfrak{B}_m .

We suppose next that $b_s, b_{s+1} \dots$ have all their coördinates the same as β except two, say x_1, x_2 .

We may place each b_n at the center of an interval i of length δ , parallel to the x_1 axis, such that

$$a + \frac{1}{m} - \epsilon \leq f(x) \leq b - \frac{1}{m} + \epsilon,$$

since f is uniformly continuous in x_1 , by I, 352. These intervals cut an ordinate in the x_1, x_2 plane through β , in a set of points c_{s+p} which $\doteq \beta$. Then as before,

$$a + \frac{1}{m} - \epsilon \leq f(\beta) \leq b - \frac{1}{m} + \epsilon.$$

As ϵ is small at pleasure, β lies in \mathfrak{B}_m . In this way we may continue.

As $\text{Cl}f \leq 1$ in each q_n , it is in \mathfrak{A} , by 481, 3.

484. (Volterra.) Let $f_1, f_2 \dots$ be at most pointwise discontinuous in the limited complete set \mathfrak{A} . Then there exists a point of \mathfrak{A} at which all the f_n are continuous.

For if \mathfrak{A} contains an isolated point, the theorem is obviously true, since every function is continuous at an isolated point. Let us therefore suppose that \mathfrak{A} is perfect.

Let $\epsilon_1 > \epsilon_2 > \dots \doteq 0$. Let a_1 be a point of continuity of f_1 . Then $\text{Osc } f_1 < \epsilon$, in some $\mathfrak{A}_1 = V_{\delta_1}(a_1)$.

In \mathfrak{A}_1 there is a point b of continuity of f_1 . Hence $\text{Osc } f_1 < \epsilon_2$ in some $V_\eta(b)$, and we may take b so that $V_\eta(b) \subset \mathfrak{A}_1$. But in $V_\eta(b)$ there is a point a_2 at which f_2 is continuous. Hence

$$\text{Osc } f_1 < \epsilon_2 \quad , \quad \text{Osc } f_2 < \epsilon_1 \quad , \quad \text{in some } \mathfrak{A}_2 = V_{\delta_2}(a_2),$$

and we may take a_2 such that $\mathfrak{A}_2 \subset \mathfrak{A}_1$. Similarly there exists a point a_3 in \mathfrak{A}_2 , such that

$$\text{Osc } f_1 < \epsilon_3 \quad , \quad \text{Osc } f_2 < \epsilon_2 \quad , \quad \text{Osc } f_3 < \epsilon_1 \quad , \quad \text{in some } \mathfrak{A}_3 = V_{\delta_3}(a_3),$$

and we may take a_3 so that $\mathfrak{A}_3 \subset \mathfrak{A}_2$.

In this way we may continue. As the sets \mathfrak{A}_n are obviously complete, $Dv\{\mathfrak{A}_n\}$ contains at least one point α of \mathfrak{A} . But at this point each f_m is continuous.

485. 1. Let $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$ be complete, let $\mathfrak{B}, \mathfrak{C}$ be pantactic with reference to \mathfrak{A} . Then there exists no pair of functions f, g defined over \mathfrak{A} , such that if \mathfrak{B} are the points of discontinuity of f in \mathfrak{A} , then \mathfrak{C} shall be the points of continuity of g in \mathfrak{A} .

This is a corollary of Volterra's theorem. For in any $V_\delta(a)$ of a point of \mathfrak{A} , there are points of \mathfrak{B} and of \mathfrak{C} . Hence there are points of continuity of f and g . Hence f, g are at most pointwise discontinuous in \mathfrak{A} . Then by 484, there is a point in \mathfrak{A} where f and g are both continuous, which contradicts the hypothesis.

2. Let $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$ be complete, and let $\mathfrak{B}, \mathfrak{C}$ each be pantactic with reference to \mathfrak{A} . If \mathfrak{B} is hypercomplete, \mathfrak{C} is not.

For if $\mathfrak{B}, \mathfrak{C}$ were the union of an enumerable set of complete sets, 473 shows that there exists a function f defined over \mathfrak{A} which has \mathfrak{B} as its points of discontinuity; and also a function g

which has \mathfrak{E} as its points of discontinuity. But no such pair of functions can exist by 1.

3. *The non-rational points \mathfrak{F} in any cube Ω cannot be hypercomplete.*

For the rational points in Ω are hypercomplete.

4. As an application of 2 we can state:

The limited function $f(x_1 \dots x_m)$ which is ≤ 0 at the irrational points of a cube Ω , and > 0 at the other points \mathfrak{F} of Ω , cannot be of class 0 or 1 in Ω .

For if $\text{Cl} f \leq 1$, the points of Ω where $f > 0$ must form a hypercomplete set, by 479, 4. But these are the points \mathfrak{F} .

486. 1. (Baire.) *If the class of $f(x_1 \dots x_m)$ is 1 in the complete set \mathfrak{X} , it is at most pointwise discontinuous in any complete $\mathfrak{B} \leq \mathfrak{X}$.*

If $\text{Cl} f = 1$ in \mathfrak{X} , it is ≤ 1 in any complete $\mathfrak{B} < \mathfrak{X}$ by 481, 4; we may therefore take $\mathfrak{B} = \mathfrak{X}$. Let a be any point of \mathfrak{X} . We shall show that in any $V = V_\delta(a)$ there is a point c of continuity of f . Let $\epsilon_1 > \epsilon_2 > \dots = 0$. Using the notation of 479, 1, we saw that the sets $\mathfrak{X}_i = (m_i < f < m_{i+2})$ are hypercomplete. By 473, we can construct a function $\phi_i(x_1 \dots x_m)$, defined over the m -way space \mathfrak{X}_m which is discontinuous at the points \mathfrak{X}_i , and continuous elsewhere in \mathfrak{X}_m . These functions $\phi_1, \phi_2 \dots$ are not all at most pointwise discontinuous in V . For then, by 484, there exists in V a point of continuity b , common to all the ϕ 's. This point b must lie in some \mathfrak{X}_i , whose points are points of discontinuity of ϕ_i .

Let us therefore suppose that ϕ_j is not at most pointwise discontinuous in V . Then there exists a point c_1 in V , and an η_1 such that $V_1 = V_{\eta_1}(c_1)$ contains no point of continuity of ϕ_j . Thus $V_1 \leq \mathfrak{X}_j$. But in \mathfrak{X}_j and hence in V_1 , $\text{Osc } f < \epsilon_1$. The same reasoning shows that in V_1 there exists a $V_2 = V_{\eta_2}(c_2)$, such that $\text{Osc } f < \epsilon_2$ in V_2 . As \mathfrak{X} is complete, $V_1 \geq V_2 \geq \dots$ defines a point c in V at which f is continuous.

2. *If the class of $f(x_1 \dots x_m)$ is 1 in the complete set \mathfrak{X} , its points of discontinuity \mathfrak{D} form a set of the first category.*

For by 462, 3, the points \mathfrak{D}_n of \mathfrak{D} at which $\text{Osc } f \geq \frac{1}{n}$ form a complete set. Each \mathfrak{D}_n is apantactic, since f is at most pointwise discontinuous, and \mathfrak{D}_n is complete. Hence $\mathfrak{D} = \{\mathfrak{D}_n\}$ is the union of an enumerable set of apantactic sets, and is therefore of the 1^o category.

487. 1. Let f be defined over the limited complete set \mathfrak{X} . If $\text{Osc } f$ is not ≤ 1 , there exists a perfect set \mathfrak{D} in \mathfrak{X} , such that f is totally discontinuous in \mathfrak{D} .

For if $\text{Cl } f$ is not ≤ 1 there exists, by 480, an ϵ such that for $\text{Osc } f < \epsilon$, \mathfrak{X} is not an \mathfrak{E}_ϵ set. Let now c be a point of \mathfrak{X} such that the points a of \mathfrak{X} which lie within some cube q , whose center is c , form an \mathfrak{E}_ϵ set. Let $\mathfrak{B} = \{a\}$, $\mathfrak{C} = \{c\}$.

Then $\mathfrak{B} = \mathfrak{C}$. For obviously $\mathfrak{C} < \mathfrak{B}$, since each c is in some \mathfrak{E}_ϵ set. On the other hand, $\mathfrak{B} \leq \mathfrak{C}$. For any point b of \mathfrak{B} lies within some q . Thus b is the center of a cube q' within q . Obviously the points of \mathfrak{X} within q' form an \mathfrak{E}_ϵ set.

By Borel's theorem, each point c lies within an enumerable set of cubes $\{c_n\}$, such that each c lies within some q . Thus the points a_n of \mathfrak{X} in c_n , form an \mathfrak{E}_ϵ set. As $\mathfrak{C} = \{a_n\}$, \mathfrak{C} is an \mathfrak{E}_ϵ set. Let $\mathfrak{D} = \mathfrak{X} - \mathfrak{C}$. If \mathfrak{D} were \emptyset , $\mathfrak{X} = \mathfrak{C}$ and \mathfrak{X} would be an \mathfrak{E}_ϵ set contrary to hypothesis. Thus $\mathfrak{D} > \emptyset$.

\mathfrak{D} is complete. For if l were a limiting point of \mathfrak{D} in \mathfrak{C} , l must lie in some c . But every point of \mathfrak{X} in c is a point of \mathfrak{C} as we saw. Thus l cannot lie in \mathfrak{C} .

We show finally that at any point d of \mathfrak{D} ,

$$\text{Osc } f \geq \epsilon, \text{ with respect to } \mathfrak{D}.$$

If not, $\text{Osc } f < \epsilon$ with respect to the points b of \mathfrak{D} within some cube q whose center is d . Then b is an \mathfrak{E}_ϵ set. Also the points c of \mathfrak{C} in q form an \mathfrak{E}_ϵ set. Thus the points $b + c$, that is, the points of \mathfrak{X} in q form an \mathfrak{E}_ϵ set. Hence d belongs to \mathfrak{C} , and not to \mathfrak{D} . As $\text{Osc } f \geq \epsilon$ at each point of \mathfrak{D} , each point of \mathfrak{D} is a point of discontinuity with respect to \mathfrak{D} . Thus f is totally discontinuous in \mathfrak{D} .

This shows that \mathfrak{D} can contain no isolated points. Hence \mathfrak{D} is perfect.

2. Let f be defined over the limited complete set \mathfrak{A} . If f is at most pointwise discontinuous in any perfect $\mathfrak{B} \subseteq \mathfrak{A}$, its class is ≤ 1 in \mathfrak{A} .

This is a corollary of 1. For if Class f were not 0, or 1, there exists a perfect set \mathfrak{D} such that f is totally discontinuous in \mathfrak{D} .

488. If the class of $f, g \leq 1$ in the limited complete set \mathfrak{A} , the class of their sum, difference, or product is ≤ 1 . If $f > 0$ in \mathfrak{A} , the class of $\phi = 1/f$ is ≤ 1 .

For example, let us consider the product $h = fg$. If Cl h is not ≤ 1 , there exists a perfect set \mathfrak{D} in \mathfrak{A} , as we saw in 487, 1, such that h is totally discontinuous in \mathfrak{D} . But f, g being of class ≤ 1 , are at most pointwise discontinuous in \mathfrak{D} by 486. Then by 484, there exists a point of \mathfrak{D} at which f, g are both continuous. Then h is continuous at this point, and is therefore not totally discontinuous in \mathfrak{D} .

Let us consider now the quotient ϕ . If Cl ϕ is not ≤ 1 , ϕ is totally discontinuous in some perfect set \mathfrak{D} in \mathfrak{A} . But since $f > 0$ in \mathfrak{D} , f must also be totally discontinuous in \mathfrak{D} . This contradicts 486.

489. 1. Let $F = \sum f_{i_1 \dots i_n}(x_1 \dots x_m)$ converge uniformly in the complete set \mathfrak{A} . Let the class of each term f_i be ≤ 1 , then Class $F \leq 1$ in \mathfrak{A} .

For setting as usual [117],

$$F = F_\lambda + \bar{F}_\lambda \quad (1)$$

there exists for each $\epsilon > 0$, a fixed rectangular cell R_λ , such that

$$|\bar{F}_\lambda| < \epsilon, \quad \text{as } x \text{ ranges over } \mathfrak{A}. \quad (2)$$

As the class of each term in F_λ is ≤ 1 , Cl $F_\lambda \leq 1$ in \mathfrak{A} . Hence \mathfrak{A} is an \mathfrak{E}_ϵ set with respect to F_λ .

From 1), 2) it follows that \mathfrak{A} is an \mathfrak{E}_ϵ set with respect to F .

2. Let $F = \prod f_{i_1 \dots i_n}(x_1 \dots x_m)$ converge uniformly in the complete set \mathfrak{A} . If the class of each f_i is ≤ 1 , then Cl $F \leq 1$ in \mathfrak{A} .

Semicontinuous Functions

490. Let $f(x_1 \dots x_m)$ be defined over \mathfrak{A} . If a is a point of \mathfrak{A} , $\text{Max} f$ in $V_\delta(a)$ exists, finite or infinite, and may be regarded as a function of δ . When finite, it is a monotone decreasing function of δ . Thus its limit as $\delta \doteq 0$ exists, finite or infinite. We call this limit the *maximum of f at $x = a$* , and we denote it by

$$\text{Max}_{x=a} f.$$

Similar remarks apply to the minimum of f in $V_\delta(a)$. Its limit, finite or infinite, as $\delta \doteq 0$, we call the *minimum of f at $x = a$* , and we denote it by

$$\text{Min}_{x=a} f.$$

The maximum and minimum of f in $V_\delta(a)$ may be denoted by

$$\text{Max}_{a, \delta} f, \quad \text{Min}_{a, \delta} f.$$

Obviously,

$$\text{Max}_{x=a} (-f) = -\text{Min}_{x=a} f,$$

$$\text{Min}_{x=a} (-f) = -\text{Max}_{x=a} f.$$

491. *Example 1.*

$$f(x) = \frac{1}{x} \text{ in } (-1, 1) \quad , \quad \text{for } x \neq 0 \\ = 0 \quad , \quad \text{for } x = 0.$$

Then

$$\text{Max}_{x=0} f = +\infty \quad , \quad \text{Min}_{x=0} f = -\infty.$$

Example 2.

$$f(x) = \sin \frac{1}{x} \text{ in } (-1, 1) \quad , \quad \text{for } x \neq 0 \\ = 0 \quad , \quad \text{for } x = 0.$$

Then

$$\text{Max}_{x=0} f = 1 \quad , \quad \text{Min}_{x=0} f = -1.$$

Example 3.

$$f(x) = 1 \text{ in } (-1, 1) \quad , \quad \text{for } x \neq 0 \\ = 2 \quad , \quad \text{for } x = 0.$$

Then

$$\text{Max}_{x=0} f = 2 \quad , \quad \text{Min}_{x=0} f = 1.$$

We observe that in Exs. 1 and 2,

$$\lim_{x \rightarrow 0} f = \text{Max}_{x=0} f, \quad \lim_{x \rightarrow 0} f = \text{Min}_{x=0} f;$$

while in Ex. 3,

$$\lim_{x \rightarrow 0} f = 1, \quad \text{and hence } \text{Max}_{x=0} f > \lim_{x \rightarrow 0} f.$$

Also

$$\lim_{x \rightarrow 0} f = \text{Min}_{x=0} f.$$

Example 4.

$$f(x) = (x^2 + 1) \sin \frac{1}{x} \text{ in } (-1, 1), \quad \text{for } x \neq 0$$

$$= -2, \quad \text{for } x = 0.$$

Here

$$\text{Max}_{x=0} f = 1, \quad \text{Min}_{x=0} f = -2,$$

$$\lim_{x \rightarrow 0} f = 1, \quad \lim_{x \rightarrow 0} f = -1.$$

Example 5.

$$\text{Let } f(x) = x, \quad \text{for rational } x \text{ in } (0, 1)$$

$$= 1, \quad \text{for irrational } x.$$

Here

$$\text{Max}_{x=0} f = 1, \quad \text{Min}_{x=0} f = 0,$$

$$\lim_{x \rightarrow 0} f = 1.$$

492. 1. For M to be the maximum of f at $x = a$, it is necessary and sufficient that

$$1^\circ \epsilon > 0, \quad \delta > 0, \quad f(x) < M + \epsilon, \quad \text{for any } x \text{ in } V_\delta(a);$$

2° there exists for each $\epsilon > 0$, and in any $V_\delta(a)$, a point a such that

$$M - \epsilon < f(a).$$

These conditions are necessary. For M is the limit of $\text{Max}_{x \in V_\delta(a)} f$, as $\delta \rightarrow 0$. Hence

$$\epsilon > 0, \quad \delta > 0, \quad \text{Max}_{a, \delta} f < M + \epsilon.$$

But for any x in $V_\delta(a)$,

$$f(x) \leq \text{Max}_{a, \delta} f.$$

Hence

$$f(x) < M + \epsilon, \quad x \text{ in } V_\delta(a),$$

which is condition 1°.

As to 2°, we remark that for each $\epsilon > 0$, and in any $V_\delta(a)$, there is a point α , such that

$$- \epsilon + \text{Max}_{a, \delta} f < f(\alpha).$$

But

$$M \leq \text{Max}_{a, \delta} f.$$

Hence

$$- \epsilon + M < f(\alpha),$$

which is 2°.

These conditions are *sufficient*. For from 1° we have

$$\text{Max}_{a, \delta} f \leq M + \epsilon,$$

and hence letting $\delta \doteq 0$,

$$\text{Max}_{x=a} f \leq M, \tag{1}$$

since $\epsilon > 0$ is small at pleasure.

From 2° we have

$$\text{Max}_{a, \delta} f \geq M - \epsilon,$$

and hence letting $\delta \doteq 0$,

$$\text{Max}_{x=a} f \geq M. \tag{2}$$

From 1), 2) we have $M = \text{Max}_{x=a} f$.

2. For m to be the minimum of f at $x = a$, it is necessary and sufficient that

1° $\epsilon > 0$, $\delta > 0$, $m - \epsilon < f(x)$, for any x in $V_\delta(a)$;

2° that there exists for each $\epsilon > 0$, and in any $V_\delta(a)$, a point α such that

$$f(\alpha) < m + \epsilon.$$

493. When $\text{Max}_{x=a} f = f(a)$, we say f is *supracontinuous* at $x = a$.

When $\text{Min}_{x=a} f = f(a)$, we say f is *infracontinuous* at a . When f is *supra* (infra) continuous at each point of \mathfrak{A} , we say f is *supra* (infra) continuous in \mathfrak{A} . When f is either *supra* or *infracontinuous* at a and we do not care to specify which, we say it is *semi-continuous* at a .

The function which is equal to $\text{Max } f$ at each point x of \mathfrak{A} we call the *maximal* function of f , and denote it by a dash above, viz. $\bar{f}(x)$. Similarly the *minimal* function $\underline{f}(x)$ is defined as the value of $\text{Min } f$ at each point of \mathfrak{A} .

Obviously
$$\text{Osc } f = \text{Max}_{x=a} f - \text{Min}_{x=a} f = \text{Disc } f.$$

We call
$$\omega(x) = \bar{f}(x) - \underline{f}(x)$$

the *oscillatory function*.

We have at once the theorem :

For f to be continuous at $x = a$, it is necessary and sufficient that

$$f(a) = \bar{f}(a) = \underline{f}(a).$$

For
$$\text{Min}_{a, \delta} f \leq f(a) \leq \text{Max}_{a, \delta} f.$$

Passing to the limit $x = a$, we have

$$\text{Min}_{x=a} f \leq f(a) \leq \text{Max}_{x=a} f,$$

or
$$\underline{f}(a) \leq f(a) \leq \bar{f}(a).$$

But for f to be continuous at $x = a$, it is necessary and sufficient that

$$\omega(a) = \text{Osc } f = 0.$$

494. 1. *For f to be supracontinuous at $x = a$, it is necessary and sufficient that for each $\epsilon > 0$, there exists a $\delta > 0$, such that*

$$f(x) < f(a) + \epsilon \quad , \quad \text{for any } x \text{ in } V_\delta(a). \quad (1)$$

Similarly the condition for infracontinuity is

$$f(a) - \epsilon < f(x) \quad , \quad \text{for any } x \text{ in } V_\delta(a). \quad (2)$$

Let us prove 1). It is *necessary*. For when f is supracontinuous at a ,

$$f(a) = \text{Max}_{x=a} f(x).$$

Then by 492, 1,

$$\epsilon > 0 \quad , \quad \delta > 0 \quad , \quad f(x) < f(a) + \epsilon \quad , \quad \text{for any } x \text{ in } V_\delta(a),$$

which is 1).

It is sufficient. For 1) is condition 1° of 492, 1. The condition 2° is satisfied, since for a we may take the point a .

2. The maximal function $\bar{f}(x)$ is supracontinuous; the minimal function $\underline{f}(x)$ is infracontinuous, in \mathfrak{A} .

To prove that \bar{f} is supracontinuous we use 1, showing that

$$\bar{f}(x) < \bar{f}(a) + \epsilon, \quad \text{for any } x \text{ in some } V_\delta(a).$$

Now by 492, 1, .

$$\epsilon' > 0, \delta > 0, \quad f(x) < \bar{f}(a) + \epsilon', \quad \text{for any } x \text{ in } V_\delta(a).$$

Thus if $\epsilon' < \epsilon$

$$\bar{f}(x) < \bar{f}(a) + \epsilon, \quad \text{for any } x \text{ in } V_\eta(a), \quad \eta = \frac{\delta}{2}.$$

3. The sum of two supra (infra) continuous functions in \mathfrak{A} is a supra (infra) continuous function in \mathfrak{A} .

For let f, g be supracontinuous in \mathfrak{A} ; let $f + g = h$. Then by 1,

$$f(x) < f(a) + \frac{\epsilon}{2},$$

$$g(x) < g(a) + \frac{\epsilon}{2},$$

for any x in some $V_\delta(a)$; hence

$$h(x) < h(a) + \epsilon.$$

This, by 1, shows that h is supracontinuous at $x = a$.

4. If $f(x)$ is supra (infra) continuous at $x = a$, $g(x) = -f(x)$ is infra (supra) continuous.

Let us suppose that f is supracontinuous. Then by 1,

$$f(x) < f(a) + \epsilon, \quad \text{for any } x \text{ in some } V_\delta(a).$$

Hence

$$-f(a) - \epsilon < -f(x),$$

or

$$g(a) - \epsilon < g(x), \quad \text{for any } x \text{ in } V_\delta(a).$$

Thus by 1, g is infracontinuous at a .

495. If $f(x_1 \dots x_m)$ is supracontinuous in the limited complete set \mathfrak{A} , the points \mathfrak{B} of \mathfrak{A} at which $f \geq c$ an arbitrary constant form complete set.

For let $f \geq c$ at $b_1, b_2 \dots$ which $\equiv b$; we wish to show that b lies in \mathfrak{B} .

Since f is supracontinuous, by 494, 1,

$$f(x) < f(b) + \epsilon, \quad \text{for any } x \text{ in some } V_\delta(b) = V.$$

But $c \leq f(b_n)$, by hypothesis; and b_n lies in V , for $n > \text{some } \tau$. Hence

$$c \leq f(b_n) < f(b) + \epsilon,$$

or

$$c - \epsilon < f(b).$$

As $\epsilon > 0$ is small at pleasure,

$$f(b) \geq c,$$

and b lies in \mathfrak{B} .

496. 1. The oscillatory function $\omega(x)$ is supracontinuous.

For by 493,

$$\begin{aligned} \omega(x) &= \text{Max } f - \text{Min } f \\ &= \text{Max } f + \text{Max } (-f). \end{aligned}$$

But these two maximal functions are supracontinuous by 494, Hence by 494, 3, their sum ω is supracontinuous.

2. The oscillatory function ω is not necessarily infracontinuous, as is shown by the following

Example. $f = 1$ in $(-1, 1)$, except for $x = 0$, where $f = 0$. Then $\omega(x) = 0$, except at $x = 0$, where $\omega = 1$. Thus

$$\lim_{x \rightarrow 0} \omega(x) = 0, \quad \text{while } \omega(0) = 1.$$

Hence $\omega(x)$ is not infracontinuous at $x = 0$.

3. Let $\omega(x)$ be the oscillatory function of $f(x_1 \dots x_m)$ in \mathfrak{A} . If f to be at most pointwise discontinuous in \mathfrak{A} , it is necessary that $\text{Min } \omega = 0$ at each point of \mathfrak{A} . If \mathfrak{A} is complete, this condition is sufficient.

It is *necessary*. For let a be a point of \mathfrak{A} . As f is at most pointwise discontinuous, there exists a point of continuity in $V_\delta(a)$. Hence $\text{Min}_{x=a} \omega(x) = 0$, in $V_\delta(a)$. Hence $\text{Min}_{x=a} \omega(x) = 0$.

It is *sufficient*. For let $\epsilon_1 > \epsilon_2 > \dots \doteq 0$. Since $\text{Min}_{x=a} \omega(x) = 0$ there exists in any $V_\delta(a)$ a point α_1 such that $\omega(\alpha_1) < \frac{1}{2}$. Hence $\omega(x) < \epsilon_1$ in some $V_{\delta_1}(\alpha_1) < V_\delta$. In V_{δ_1} there exists a point α_2 such that $\omega(x) < \epsilon_2$ in some $V_{\delta_2}(\alpha_2) < V_{\delta_1}$, etc. Since \mathfrak{A} is complete and since we may let $\delta_n \doteq 0$,

$$V_{\delta_1} > V_{\delta_2} > \dots \doteq \text{a point } \alpha \text{ of } \mathfrak{A},$$

at which f is obviously continuous. Thus in each $V_\delta(a)$ is a point of continuity of f . Hence f is at most pointwise discontinuous.

497. 1. At each point x of \mathfrak{A} ,

$$\phi = \text{Min} \{ \bar{f}(x) - f(x) \}, \text{ and } \psi = \text{Min} \{ f(x) - \underline{f}(x) \}$$

are both $= 0$.

Let us show that $\phi = 0$ at an arbitrary point a of \mathfrak{A} . By 494, 1, $\bar{f}(x)$ is supracontinuous; hence by 494, 1,

$$\bar{f}(x) < \bar{f}(a) + \epsilon, \text{ for any } x \text{ in some } V_\delta(a) = V.$$

Also there exists a point α in V such that

$$-\epsilon + \bar{f}(\alpha) < f(\alpha).$$

Also by definition

$$f(\alpha) \leq \bar{f}(\alpha).$$

If in 1) we replace x by α we get

$$\bar{f}(\alpha) < \bar{f}(a) + \epsilon.$$

From 2), 3), 4) we have

$$-\epsilon + \bar{f}(\alpha) < f(\alpha) \leq \bar{f}(\alpha) < \bar{f}(a) + \epsilon,$$

or

$$0 \leq \bar{f}(a) - f(\alpha) < 2\epsilon.$$

As $\epsilon > 0$ is small at pleasure, this gives

$$\phi(a) = 0.$$

2. If f is semicontinuous in the complete set \mathfrak{A} , it is at most pointwise discontinuous in \mathfrak{A} .

For

$$\begin{aligned}\omega(x) &= \bar{f}(x) - \underline{f}(x) \\ &= [\bar{f}(x) - f(x)] + [f(x) - \underline{f}(x)] \quad (1) \\ &= \phi(x) + \psi(x).\end{aligned}$$

To fix the ideas let f be supracontinuous. Then $\phi = 0$ in \mathfrak{A} . Hence 1) gives

$$\text{Min } \omega(x) = \text{Min } \psi(x) = 0, \quad \text{by 1.}$$

Thus by 496, 3, f is at most pointwise discontinuous in \mathfrak{A} .

CHAPTER XV

DERIVATES, EXTREMES, VARIATION

Derivates

498. Suppose we have given a one-valued continuous function $f(x)$ spread over an interval $\mathfrak{A} = (a < b)$. We can state various properties which it enjoys. For example, it is limited, it takes on its extreme values, it is integrable. On the other hand, we do not know 1° how it oscillates in \mathfrak{A} , or 2° if it has a differential coefficient at each point of \mathfrak{A} . In this chapter we wish to study the behavior of continuous functions with reference to these last two properties. In Chapters VIII and XI of volume I this subject was touched upon; we wish here to develop it farther.

499. In I, 363, 364, we have defined the terms difference quotient, differential coefficient, derivative, right- and left-hand differential coefficients and derivatives, unilateral differential coefficients and derivatives. The corresponding symbols are

$$\frac{\Delta f}{\Delta x} \quad , \quad f'(a) \quad , \quad f'(x) \quad , \quad Rf'(a) \quad , \\ Lf'(a) \quad , \quad Rf'(x) \quad , \quad Lf'(x).$$

The unilateral differential coefficient and derivative may be denoted by

$$Uf'(a) \quad , \quad Uf'(x). \quad (1)$$

When

$$\lim_{h \rightarrow 0} \frac{\Delta f}{\Delta x}$$

does not exist, finite or infinite, we may introduce its upper and lower limits. Thus

$$\bar{f}'(a) = \overline{\lim}_{h \rightarrow 0} \frac{\Delta f}{\Delta x} \quad , \quad \underline{f}'(a) = \underline{\lim}_{h \rightarrow 0} \frac{\Delta f}{\Delta x} \quad (2)$$

always exist, finite or infinite. We call them the upper and lower differential coefficients at the point $x = a$. The aggregate of values

that 2) take on define the upper and lower derivatives of $f(x)$, as in I, 363.

In a similar manner we introduce the upper and lower right- and left-hand differential coefficients and derivatives,

$$\bar{R}f' \quad , \quad \underline{R}f' \quad , \quad \bar{L}f' \quad , \quad \underline{L}f'. \quad (3)$$

Thus, for example,

$$\bar{R}f'(a) = R \lim_{h=0} \frac{f(a+h) - f(a)}{h},$$

finite or infinite. Cf. I, 336 *seq.*

If $f(x)$ is defined *only* in $\mathfrak{A} = (a < \beta)$, the points $a, a+h$ must lie in \mathfrak{A} . Thus there is no upper or lower right-hand differential coefficient at $x = \beta$; also no upper or lower left-hand differential coefficient at $x = a$. This fact must be borne in mind. We call the functions 3) *derivates* to distinguish them from the derivatives Rf', Lf' . When $\bar{R}f'(a) = \underline{R}f'(a)$, finite or infinite, $Rf'(a)$ exists also finite or infinite, and has the same value. A similar remark applies to the left-hand differential coefficient.

To avoid such repetition as just made, it is convenient to introduce the terms upper and lower unilateral differential coefficients and derivatives, which may be denoted by

$$\bar{U}f' \quad , \quad \underline{U}f'. \quad (4)$$

The symbol U should of course refer to the same side, if it is used more than once in an investigation.

When no ambiguity can arise, we may abbreviate the symbols 3), 4) thus:

$$\bar{R} \quad , \quad \underline{R} \quad , \quad \bar{L} \quad , \quad \underline{L} \quad , \quad \bar{U} \quad , \quad \underline{U}.$$

The value of one of these derivates as \bar{R} at a point $x = a$ may similarly be denoted by

$$\bar{R}(a).$$

The difference quotient

$$\frac{f(a) - f(b)}{a - b}$$

may be denoted by

$$\Delta(a, b).$$

$$\begin{aligned}\text{Example 1.} \quad f(x) &= x \sin \frac{1}{x}, \quad x \neq 0 \text{ in } (-1, 1) \\ &= 0, \quad x = 0.\end{aligned}$$

$$\text{Here for } x = 0, \quad \frac{\Delta f}{\Delta x} = \frac{h \sin \frac{1}{h}}{h} = \sin \frac{1}{h}.$$

$$\begin{aligned}\text{Hence} \quad \bar{R}f'(0) &= +1, \quad \underline{R}f'(0) = -1, \\ \bar{L}f'(0) &= +1, \quad \underline{L}f'(0) = -1, \\ \bar{f}'(0) &= +1, \quad \underline{f}'(0) = -1.\end{aligned}$$

$$\begin{aligned}\text{Example 2.} \quad f(x) &= x^{\frac{1}{3}} \sin \frac{1}{x}, \quad x \neq 0 \text{ in } (-1, 1) \\ &= 0, \quad x = 0.\end{aligned}$$

$$\text{Here for } x = 0, \quad \frac{\Delta f}{\Delta x} = \frac{\sin \frac{1}{h}}{h^{\frac{2}{3}}}.$$

$$\begin{aligned}\text{Hence} \quad \bar{R}f'(0) &= +\infty, \quad \underline{R}f'(0) = -\infty, \\ \bar{L}f'(0) &= +\infty, \quad \underline{L}f'(0) = -\infty, \\ \bar{f}'(0) &= +\infty, \quad \underline{f}'(0) = -\infty.\end{aligned}$$

$$\begin{aligned}\text{Example 3.} \quad f(x) &= x \sin \frac{1}{x}, \quad \text{for } 0 < x \leq 1 \\ &= x^{\frac{1}{3}} \sin \frac{1}{x}, \quad \text{for } -1 \leq x < 0 \\ &= 0, \quad \text{for } x = 0.\end{aligned}$$

$$\begin{aligned}\text{Here} \quad \bar{R}f'(0) &= +1, \quad \underline{R}f'(0) = -1, \\ \bar{L}f'(0) &= +\infty, \quad \underline{L}f'(0) = -\infty, \\ \bar{f}'(0) &= +\infty, \quad \underline{f}'(0) = -\infty.\end{aligned}$$

500. 1. Before taking up the general theory it will be well for the reader to have a few examples in mind to show him how complicated matters may get. In I, 367 *seq.*, we have exhibited functions which oscillate infinitely often about the points of a set

of the 1° species, and which may or may not have differential coefficients at these points.

The following theorem enables us to construct functions which do not possess a differential coefficient at the points of an enumerable set.

2. Let $\mathfrak{E} = \{e_n\}$ be an enumerable set lying in the interval \mathfrak{A} . For each x in \mathfrak{A} , and e_n in \mathfrak{E} , let $x - e_n$ lie in an interval \mathfrak{B} containing the origin. Let $g(x)$ be continuous in \mathfrak{B} . Let $g'(x)$ exist and be numerically $\leq M$ in \mathfrak{B} , except at $x = 0$, where the difference quotients are numerically $\leq M$. Let $A = \sum a_n$ converge absolutely. Then

$$F(x) = \sum a_n g(x - e_n)$$

is a continuous function in \mathfrak{A} , having a derivative in $\mathfrak{E} = \mathfrak{A} - \mathfrak{E}$. At the points of \mathfrak{E} , the difference quotient of F behaves essentially as that of g at the origin.

For $g(x)$ being continuous in \mathfrak{B} , it is numerically $<$ some constant in \mathfrak{A} . Thus F converges uniformly in \mathfrak{A} . As each term $g(x - e_n)$ is continuous in \mathfrak{A} , F is continuous in \mathfrak{A} .

Let us consider its differential coefficient at a point x of \mathfrak{E} . Since $g'(x - e_n)$ exists and is numerically $\leq M$,

$$F'(x) = \sum a_n g'(x - e_n) \quad , \quad \text{by 156, 2.}$$

Let now $x = e_m$, a point of \mathfrak{E} ,

$$\begin{aligned} F(x) &= a_m g(x - e_m) + \sum^* a_n g(x - e_n) \\ &= a_m g(x - e_m) + G(x). \end{aligned}$$

The summation in \sum^* extends over all $n \neq m$. Hence by what has just been shown, G has a differential coefficient at $x = e_m$.

Thus $\frac{\Delta F}{\Delta x}$ behaves at $x = e_m$, essentially as $\frac{\Delta g}{\Delta x}$ at $x = 0$. Hence

$$\underline{U}F'(e_m) = a_m \underline{U}g'(0) + G'(e_m). \quad (1)$$

501. Example 1. Let

$$\begin{aligned} g(x) &= ax \quad , \quad x \geq 0 \\ &= bx \quad , \quad x < 0, \end{aligned} \quad b < 0 < a.$$

Then

$$F(x) = \sum \frac{1}{n^2} g(x - e_n)$$

continuous in any interval \mathfrak{A} , and has a derivative

$$F'(x) = \sum \frac{1}{n^2} g'(x - e_n)$$

the points of \mathfrak{A} not in \mathfrak{E} . At the point e_m ,

$$RF'(x) = a_m a + \sum^* \frac{1}{m^2} g'(e_m - e_n),$$

$$LF'(x) = a_m b + \sum^* \frac{1}{m^2} g'(e_m - e_n).$$

Let \mathfrak{E} denote the rational points in \mathfrak{A} . The graph of $F(x)$ is a continuous curve having tangents at a pantactic set of points; and at another pantactic set, viz. the set \mathfrak{E} , angular points (I, 366). A simple example of a g function is

$$g(x) = |x| = +\sqrt{x^2}.$$

Example 2. Let $g(x) = x^2 \sin \frac{\pi}{x}$, $x \neq 0$
 $= 0$, $x = 0$.

This function has a derivative

$$g'(x) = 2x \sin \frac{\pi}{x} - \pi \cos \frac{\pi}{x}, \quad x \neq 0$$

$$= 0, \quad x = 0.$$

Thus if $\sum e_n$ is an absolutely convergent series, and $\mathfrak{E} = \{e_n\}$ an enumerable set in the interval $\mathfrak{A} = (0, 1)$,

$$F(x) = \sum e_n g(x - e_n)$$

a continuous function whose derivative in \mathfrak{A} is

$$F'(x) = \sum e_n g'(x - e_n).$$

Thus F has a derivative which is continuous in $\mathfrak{A} - \mathfrak{E}$, and at the point $x = e_m$

$$\text{Disc } F' = 2 e_m \pi,$$

so

$$\text{Disc } g'(x) = 2 \pi.$$

If \mathfrak{E} is the set of rational points in \mathfrak{A} , the graph of $F(x)$ is a continuous curve having at each point of \mathfrak{A} a tangent which does not turn continuously as the point of contact ranges over the curve; indeed the points of abrupt change in the direction of the tangent are pantactic in \mathfrak{A} .

Example 3. Let $y(x) = x \sin \log x^2$, $x \neq 0$
 $= 0$, $x = 0$.

Then $y'(x) = \sin \log x^2 + 2 \cos \log x^2$, $x \neq 0$.

At $x = 0$, $\frac{\Delta y}{\Delta x} = \sin \log h^2$

which oscillates infinitely often between ± 1 , as $h = \Delta x \rightarrow 0$. Let $\mathfrak{E} = \{e_n\}$ denote the rational points in an interval \mathfrak{A} . The series

$$F = \sum \frac{1}{n^2} (x - e_n) \sin \log (x - e_n)^2$$

satisfies the condition of our theorem. Hence $F(x)$ is a continuous function in \mathfrak{A} which has a derivative in $\mathfrak{A} - \mathfrak{E}$. At $x = e_m$,

$$\overline{UF}'(x) = \frac{1}{m^2} + G'(e_m) \quad , \quad \underline{UF}'(x) = -\frac{1}{m^2} + G'(e_m).$$

Thus the graph of F is a continuous curve which has tangents at a pantactic set of points in \mathfrak{A} , and at another pantactic set it has neither right- nor left-hand tangents.

502. Weierstrass' Function. For a long time mathematicians thought that a continuous function of x must have a derivative, at least after removing certain points. The examples just given show that these exceptional points may be pantactic. Weierstrass called attention to a continuous function which has at no point a differential coefficient. This celebrated function is defined by the series

$$F(x) = \sum_0 a^n \cos b^n \pi x = \cos \pi x + a \cos b \pi x + a^2 \cos b^2 \pi x + \dots \quad (1)$$

where $0 < a < 1$; b is an odd integer so chosen that

$$ab > 1 + \frac{3}{2} \pi. \quad (2)$$

The series F converges absolutely and uniformly in any interval \mathfrak{A} , since

$$|a^n \cos b^n \pi x| \leq a^n.$$

Hence F is a continuous function in \mathfrak{A} . Let us now consider the series obtained by differentiating 1) termwise,

$$G(x) = -\pi \Sigma (ab)^n \sin b^n \pi x.$$

If $ab < 1$, this series also converges absolutely and uniformly, and

$$F'(x) = G(x),$$

by 155, 1. In this case the function has a finite derivative in \mathfrak{A} . Let us suppose, however, that the condition 2) holds. We have

$$\frac{\Delta F}{\Delta x} = Q = \sum_0^{\infty} \frac{a^n}{h} \{ \cos b^n \pi (x+h) - \cos b^n \pi x \} = Q_m + \bar{Q}_m. \quad (3)$$

Now

$$\begin{aligned} Q_m &= \sum_0^{m-1} \frac{a^n}{h} \{ \cos b^n \pi (x+h) - \cos b^n \pi x \} \\ &= -\pi \sum_0^{m-1} \frac{(ab)^n}{h} \int_x^{x+h} \sin b^n \pi u du. \end{aligned}$$

Since

$$\left| \int_x^{x+h} \sin b^n \pi u du \right| < \left| \int_x^{x+h} du \right| = |h|,$$

$$|Q_m| < \pi \sum_0^{m-1} (ab)^n = \pi \frac{1 - (ab)^m}{1 - ab} < \pi \frac{(ab)^m}{ab - 1}, \text{ if } ab > 1.$$

Consider now

$$\bar{Q}_m = \sum_m^{\infty} \frac{a^n}{h} \{ \cos b^n \pi (x+h) - \cos b^n \pi x \}.$$

Up to the present we have taken h arbitrary. Let us now take it as follows; the reason for this choice will be evident in a moment.

Let

$$b^m x = \iota_m + \xi_m,$$

where ι_m is the nearest integer to $b^m x$. Thus

$$-\frac{1}{2} \leq \xi_m \leq \frac{1}{2}.$$

Then

$$b^m(x+h) = \iota_m + \xi_m + hb^m = \iota_m + \eta_m.$$

We choose h so that

$$\eta_m = \xi_m + h b^m \text{ is } \pm 1, \text{ at pleasure.}$$

Then

$$h = \frac{\eta_m - \xi_m}{b^m} \doteq 0, \text{ as } m \doteq \infty;$$

moreover

$$\operatorname{sgn} h = \operatorname{sgn} \eta_m, \quad \text{and } |\eta_m - \xi_m| \leq \frac{2}{3}.$$

This established, we note that

$$\begin{aligned} \cos b^n \pi (x + h) &= \cos b^{n-m} \pi \cdot b^m (x + h) = \cos b^{n-m} (\iota_m + \eta_m) \pi \\ &= \cos (\iota_m + \eta_m) \pi, \quad \text{since } b \text{ is odd} \\ &= (-1)^{\iota_m+1}, \quad \text{since } \eta_m \text{ is odd.} \end{aligned}$$

Also

$$\begin{aligned} \cos b^n \pi x &= \cos b^{n-m} (\iota_m + \xi_m) \pi \\ &= (-1)^{\iota_m} \cos b^{n-m} \xi_m \pi. \end{aligned}$$

Thus

$$\bar{Q}_m = e_m \sum_{n=m}^{\infty} \frac{a^n}{h} \{1 + \cos b^{n-m} \xi_m \pi\},$$

where

$$e_m = (-1)^{\iota_m+1}.$$

Now each $\{ \} \geq 0$ and in particular the first is > 0 . Thus

$$\operatorname{sgn} \bar{Q}_m = \operatorname{sgn} \frac{e_m}{h} = \operatorname{sgn} e_m \eta_m,$$

and

$$|\bar{Q}_m| > \frac{a^m}{h} = \frac{(ab)^m}{\eta_m - \xi_m} \geq \frac{2}{3} (ab)^m.$$

Thus if 2) holds, $|\bar{Q}_m| > |Q_m|$. Hence from 3),

$$\operatorname{sgn} Q = \operatorname{sgn} \bar{Q}_m = \operatorname{sgn} e_m \eta_m,$$

and

$$|Q| > (ab)^m \left(\frac{2}{3} - \frac{\pi}{ab-1} \right).$$

Let now $m \doteq \infty$. Since $\eta_m = \pm 1$ at pleasure, we can make $Q \doteq +\infty$, or to $-\infty$, or oscillate between $\pm \infty$, without becoming definitely infinite. Thus $F(x)$ has at no point a finite or infinite differential coefficient. This does not say that the graph of F does not have tangents; but when they exist, they must be cuspidal tangents.

503. 1. *Volterra's Function.*

In the interval $\mathfrak{X} = (0, 1)$, let $\mathfrak{S} = \{\eta\}$ be a Harnack set of measure $0 < h < 1$. Let $\Delta = \{\delta_n\}$ be the associate set of black intervals. In each of the intervals $\delta_n = (\alpha < \beta)$, we define an auxiliary function f_n as follows:

$$f_n(x) = (x - \alpha)^2 \sin \frac{1}{x - \alpha} \quad , \quad \text{in } (\alpha^*, \gamma), \quad (1)$$

where γ is the largest value of x corresponding to a maximum of the function on the right of 1), such that γ lies to the left of the middle point μ of δ_n . If the value of $f_n(x)$ at γ is g , we now make

$$f_n(x) = g \quad , \quad \text{in } (\gamma, \mu).$$

Finally $f_n(\alpha) = 0$. This defines $f_n(x)$ for one half of the interval δ_n . We define $f_n(x)$ for the other half of δ_n by saying that if $x < x'$ are two points of δ_n at equal distances from the middle point μ , then

$$f_n(x) = f_n(x').$$

With Volterra we now define a function $f(x)$ in \mathfrak{X} as follows:

$$\begin{aligned} f(x) &= f_n(x) \quad , \quad \text{in } \delta_n \quad , \quad n = 1, 2, \dots \\ &= 0 \quad , \quad \text{in } \mathfrak{S}. \end{aligned}$$

Obviously $f(x)$ is continuous in \mathfrak{X} .

At a point x of \mathfrak{X} not in \mathfrak{S} , $f'(x)$ behaves as

$$2x \sin \frac{1}{x} - \cos \frac{1}{x},$$

as is seen from 1). Thus as x converges in δ_n toward one of its end points α, β , we see that $f'(x)$ oscillates infinitely often between limits which $\doteq \pm 1$. Thus

$$R \overline{\lim}_{x=\alpha} f'(x) = +1 \quad , \quad R \underline{\lim}_{x=\alpha} f'(x) = -1;$$

similar limits exist for the points β .

Let us now consider the differential coefficient at a point η of \mathfrak{S} . We have

$$\frac{\Delta f}{\Delta x} = \frac{f(\eta + k) - f(\eta)}{k} = \frac{f(\eta + k)}{k} \quad , \quad \text{since } f(\eta) = 0.$$

If $\eta + k$ is a point of \mathfrak{S} , $f(\eta + k) = 0$. If not, $\eta + k$ lies in some interval δ_m . Let $x = e$ be the end point of δ_m nearest $\eta + k$. Then

$$\left| \frac{\Delta f}{\Delta x} \right| \leq \frac{|\eta + k - e|^2}{|k|} \leq |k| \doteq 0 \quad , \quad \text{as } k \doteq 0.$$

Thus $f'(\eta) = 0$. Hence Volterra's function $f(x)$ has a differential coefficient at each point of \mathfrak{A} ; moreover $f'(x)$ is limited in \mathfrak{A} . Each point η of \mathfrak{S} is a point of discontinuity of $f'(x)$, and

$$\text{Disc}_{x=\eta} f'(x) \geq 2.$$

Hence $f'(x)$ is not R -integrable, as $\mathfrak{S} = h > 0$.

We have seen, in I, 549, that not every limited R -integrable function has a primitive. Volterra's function illustrates conversely the remarkable fact that *Not every limited derivative is R -integrable.*

2. It is easy to show, however, that *The derivative of Volterra's function is L -integrable.*

For let \mathfrak{A}_λ denote the points of \mathfrak{A} at which $f'(x) \geq \lambda$. Then when $\lambda > 1/m$, $m = 1, 2, \dots$ \mathfrak{A}_λ consists of an enumerable set of intervals. Hence in this case \mathfrak{A}_λ is measurable. Hence \mathfrak{A}_λ , $\lambda > 0$, is measurable. Now \mathfrak{A} , $\lambda \geq 0$, differs from the foregoing by adding the points \mathfrak{B}_n in each δ_n at which $f'(x) = 0$, and the points \mathfrak{S} . But each \mathfrak{B}_n is enumerable, and hence a null set, and \mathfrak{S} is measurable, as it is perfect. Thus \mathfrak{A}_λ , $\lambda \geq 0$, is measurable. In the same way we see \mathfrak{A}_λ is measurable when λ is negative. Thus \mathfrak{A}_λ is measurable for any λ , and hence L -integrable.

504. 1. We turn now to general considerations and begin by considering the upper and lower limits of the sum, difference, product, and quotient of two functions at a point $x = a$.

Let us note first the following theorem :

Let $f(x_1 \dots x_m)$ be limited or not in \mathfrak{A} which has $x = a$ as a limiting point. Let $\Phi_\delta = \text{Max } f$, $\phi_\delta = \text{Min } f$ in $V_\delta^(a)$. Then*

$$\lim_{x \rightarrow a} f = \lim_{\delta \rightarrow 0} \phi_\delta \quad , \quad \overline{\lim}_{x \rightarrow a} f = \lim_{\delta \rightarrow 0} \Phi_\delta.$$

This follows at once from I, 338.

2. Let $f(x_1 \dots x_m)$, $g(x_1 \dots x_m)$ be limited or not in \mathfrak{A} which has $x = a$ as limiting point.

Let

$$\underline{\lim} f = \alpha \quad , \quad \underline{\lim} g = \beta$$

$$\overline{\lim} f = A \quad , \quad \overline{\lim} g = B$$

as $x \rightarrow a$. Then, these limits being finite,

$$\alpha + \beta \leq \underline{\lim} (f + g) \leq A + B, \quad (1)$$

$$\alpha - B \leq \overline{\lim} (f - g) \leq A - \beta. \quad (2)$$

For in any $V_\delta^*(a)$,

$$\text{Min } f + \text{Min } g \leq \text{Min } (f + g) \leq \text{Max } (f + g) \leq \text{Max } f + \text{Max } g.$$

Letting $\delta \rightarrow 0$, we get 1).

Also in $V_\delta^*(a)$,

$$\text{Min } f - \text{Max } g \leq \text{Min } (f - g) \leq \text{Max } (f - g) \leq \text{Max } f - \text{Min } g.$$

Letting $\delta \rightarrow 0$, we get 2).

3. If

$$f(x) \geq 0 \quad , \quad g(x) \geq 0,$$

$$\alpha\beta \leq \underline{\lim} fg \leq AB. \quad (3)$$

If

$$f(x) \geq 0 \quad , \quad \beta \leq 0 \leq B,$$

$$A\beta \leq \underline{\lim} fg \leq AB. \quad (4)$$

4. If

$$f(x) \geq 0 \quad , \quad g(x) \geq k > 0,$$

$$\frac{\alpha}{B} \leq \underline{\lim} \frac{f}{g} \leq \frac{A}{\beta}. \quad (5)$$

If

$$\alpha < 0 < A \quad , \quad g(x) \geq k > 0,$$

$$\frac{\alpha}{\beta} \leq \underline{\lim} \frac{f}{g} \leq \frac{A}{\beta}. \quad (6)$$

The relations 3), 4), 5), 6) may be proved as in 2. For example, to prove 5), we observe that in $V_\delta^*(a)$,

$$\frac{\text{Min } f}{\text{Max } g} \leq \text{Min } \frac{f}{g} \leq \text{Max } \frac{f}{g} \leq \frac{\text{Max } f}{\text{Min } g}.$$

2. Let $f(x)$ be continuous in the interval $\mathfrak{A} = (a < b)$. Then $Uf'(x)$ cannot be constantly $+\infty$, or constantly $-\infty$ in \mathfrak{A} .

For consider

$$\phi(x) = f(x) - f(a) - \frac{x-a}{b-a} \{f(b) - f(a)\},$$

which is continuous, and vanishes for $x=a$, $x=b$. We observe that $\phi(x)$ differs from $f(x)$ only by a linear function. If now $Uf'(x) = +\infty$ constantly, obviously $U\phi'(x) = +\infty$ also. Thus ϕ is a univariant function in \mathfrak{A} . This is not possible, since ϕ has the same value at a and b .

3. Let $f(x)$ be continuous in $\mathfrak{A} = (a < b)$, and have a derivative, finite or infinite, in $\mathfrak{A} = (a^*, b)$. Then

$$\text{Min} f'(x) \leq \bar{R}f'(a) \leq \text{Max} f'(x) \quad , \quad \text{in } \mathfrak{A}.$$

For the Law of the Mean holds, hence

$$\frac{f(a+h) - f(a)}{h} = f'(\alpha) \quad , \quad a < \alpha < a+h.$$

Letting now $h \doteq 0$, we get the theorem.

Remark. This theorem answers the question: Can a continuous curve have a vertical tangent at a point $x=a$, if the derivatives remain $< M$ in $V^*(a)$? The answer is, No.

4. Let $f(x)$ be continuous in $\mathfrak{A} = (a < b)$, and have a derivative, finite or infinite, in $\mathfrak{A}^* = (a^*, b)$. If $f'(a)$ exists, finite or infinite, there exists a sequence $\alpha_1 > \alpha_2 > \dots \doteq a$ in \mathfrak{A} , such that

$$f'(a) = \lim_{n \rightarrow \infty} f'(\alpha_n). \quad (1)$$

For

$$\frac{f(a+h) - f(a)}{h} = f'(\alpha_h) \quad , \quad a < \alpha_h < a+h. \quad (2)$$

Let now h range over $h_1 > h_2 > \dots \doteq 0$. If we set $\alpha_n = \alpha_{h_n}$, the relation 1) follows at once from 2), since $f'(a)$ exists by hypothesis.

510. 1. A right-hand derivate of a continuous function $f(x)$ cannot have a discontinuity of the 1° kind on the right. A similar statement holds for the other derivates.

For let $R(x)$ be one of the right-hand derivatives. If it has a discontinuity of the 1° kind on the right at $x=a$, there exists a number l such that

$$l - \epsilon \leq R(x) \leq l + \epsilon \quad , \quad \text{in some } (a < a + \delta).$$

Then by 506, 1,

$$l - \epsilon \leq \frac{f(a+h) - f(a)}{h} \leq l + \epsilon \quad , \quad 0 < h < \delta.$$

Hence

$$R(a) = l,$$

and $R(x)$ is continuous on the right at $x=a$, which is contrary to hypothesis.

2. It can, however, have a discontinuity of the 1° kind on the left, as is shown by the following

Example. Let $f(x) = |x| = +\sqrt{x^2}$, in $\mathfrak{A} = (-1, 1)$.

Here

$$\begin{aligned} R(x) &= +1 \quad , \quad \text{for } x \geq 0 \text{ in } \mathfrak{A} \\ &= -1 \quad , \quad \text{for } x < 0. \end{aligned}$$

Thus at $x=0$, R is continuous on the right, but has a discontinuity of the 1° kind on the left.

3. Let $f(x)$ be continuous in $\mathfrak{A} = (a, b)$, and have a derivative, finite or infinite, in $\mathfrak{A}^* = (a^*, b^*)$. Then the discontinuities of $f'(x)$ in \mathfrak{A} , if any exist, must be of the second kind.

This follows from 1.

Example. $f(x) = x^2 \sin \frac{1}{x}$, for $x \neq 0$ in $\mathfrak{A} = (0, 1)$
 $= 0$, for $x = 0$.

Then

$$\begin{aligned} f'(x) &= 2x \sin \frac{1}{x} - \cos \frac{1}{x} \quad , \quad x \neq 0 \\ &= 0 \quad , \quad x = 0. \end{aligned}$$

The discontinuity of $f'(x)$ at $x=0$, is in fact of the 2° kind.

4. Let $f(x)$ be continuous in $\mathfrak{A} = (a < b)$, except at $x=a$, which is a point of discontinuity of the 2° kind. Let $f'(x)$ exist, finite or infinite, in (a^*, b) . Then $x=a$ is a point of infinite discontinuity of $f'(x)$.

We introduce the auxiliary function

$$\phi(x) = f(x) - (M + c)x, \quad (2)$$

where

$$0 < c < e = c + \delta.$$

$$\text{Then} \quad \frac{\phi(\beta) - \phi(\alpha)}{\beta - \alpha} = \frac{f(\beta) - f(\alpha)}{\beta - \alpha} - (M + c) = \delta.$$

Hence

$$\phi(\beta) - \phi(\alpha) = \delta(\beta - \alpha) = \eta.$$

Consider now the equation

$$\phi(\beta) - \phi(x) = \eta.$$

It is satisfied for $x = \alpha$. If it is satisfied for any other x in the interval $(\alpha\beta)$, there is a last point, say $x = \gamma$, where it is satisfied, by 458, 3.

$$\text{Thus for } x > \gamma, \quad \phi(x) \text{ is } > \phi(\alpha).$$

Hence

$$\bar{R}\phi'(\gamma) \geq 0. \quad (3)$$

Now from 2) we have

$$\begin{aligned} \bar{R}f'(\gamma) &= \bar{R}\phi'(\gamma) + M + c \\ &> M. \end{aligned}$$

Hence M is not the maximum of $\bar{R}f'(x)$ in \mathfrak{A} . Similarly the other half of 1) is established. The case that m or M is infinite is obviously true.

2. Let $f(x)$ be defined over $\mathfrak{A} = (a < b)$. Let $a_1 < a_2 < \dots < a_n$ lie in \mathfrak{A} . Let m and M denote the minimum and maximum of the difference quotients

$$\Delta(a_1, a_2), \quad \Delta(a_2, a_3), \quad \dots, \quad \Delta(a_{n-1}, a_n).$$

Then

$$m \leq \Delta(a_1, a_n) \leq M. \quad (1)$$

. For let us first take three points $\alpha < \beta < \gamma$ in \mathfrak{A} . We have identically

$$\Delta(\alpha, \gamma) = \frac{\alpha - \beta}{\alpha - \gamma} \cdot \Delta(\alpha, \beta) + \frac{\beta - \gamma}{\alpha - \gamma} \cdot \Delta(\beta, \gamma).$$

Now the coefficients of Δ on the right lie between 0 and 1. Hence 1) is true in this case. The general case is now obvious.

507. 1. Let $f(x)$ be continuous in $\mathfrak{A} = (a < b)$. The four derivatives of f have the same extremes in \mathfrak{A} .

To fix the ideas let

$$\text{Min } \underline{L} = m, \quad \text{Min } \bar{R} = \mu, \quad \text{in } \mathfrak{A}.$$

We wish to show that $m = \mu$. To this end we first show that

$$\mu \leq m. \quad (1)$$

For there exists an α in \mathfrak{A} , such that

$$\underline{L}(\alpha) < m + \epsilon.$$

There exists therefore a $\beta < \alpha$ in \mathfrak{A} , such that

$$q = \frac{f(\alpha) - f(\beta)}{\alpha - \beta} < m + \epsilon', \quad 0 < \epsilon' < \epsilon.$$

Now by 506, 1,

$$\mu = \text{Min } \bar{R} \leq q.$$

Hence

$$\mu \leq m,$$

as $\epsilon > 0$ is small at pleasure.

We show now that

$$m \leq \mu. \quad (2)$$

For there exists an α in \mathfrak{A} , such that

$$\bar{R}(\alpha) < \mu + \epsilon.$$

There exists therefore a $\beta > \alpha$ in \mathfrak{A} , such that

$$q = \frac{f(\alpha) - f(\beta)}{\alpha - \beta} < \mu + \epsilon', \quad 0 < \epsilon' < \epsilon.$$

Thus by 506, 1,

$$m = \text{Min } \underline{L} \leq q.$$

Hence as before $m \leq \mu$. From 1), 2) we have $m = \mu$.

2. In 499, we emphasized the fact that the left-hand derivatives are not defined at the left-hand end point of an interval, and the right-hand derivatives at the right-hand end point of an interval for which we are considering the values of a function. The following example shows that our theorems may be at fault if this fact is overlooked.

Example. Let $f(x) = |x|$.

If we restrict x to lie in $\mathfrak{A} = (0, 1)$, the four derivatives = 1 when they are defined. Thus the theorem 1 holds in this case. If, however, we regarded the left-hand derivatives as defined at $x = 0$, and to have the value

$$Lf'(0) = -1,$$

as they would have if we considered values of f to the left of \mathfrak{A} , the theorem 1 would no longer be true.

For then $\text{Min } \underline{L} = -1$, $\text{Min } \overline{R} = +1$,

and the four derivatives do not have the same minimum in \mathfrak{A} .

3. Let $f(x)$ be continuous about the point $x = c$. If one of its four derivatives is continuous at $x = c$, all the derivatives defined at this point are continuous, and all are equal.

For their extremes in any $V_\delta(c)$ are the same. If now \bar{R} is continuous at $x = c$,

$$\bar{R}(c) - \epsilon < \bar{R}(x) < \bar{R}(c) + \epsilon,$$

for any x in some $V_\delta(c)$.

4. Let $f(x)$ be continuous about the point $x = c$. If one of its four derivatives is continuous at $x = c$, the derivative exists at this point.

This follows at once from 3.

Remark. We must guard against supposing that the derivative is continuous at $x = c$, or even exists in the vicinity of this point.

Example. Let $F(x)$ be as in 501, Ex. 1. Let

$$\mathfrak{A} = (0, 1) \text{ and } \mathfrak{E} = \left\{ \frac{1}{n} \right\}.$$

Let

$$H(x) = x^2 F(x).$$

Then

$$RH'(x) = 2x F(x) + x^2 RF'(x),$$

$$LH'(x) = 2x F(x) + x^2 LF'(x).$$

Obviously both RH' and LH' are continuous at $x = 0$ and $H'(0) = 0$. But H' does not exist at the points of \mathfrak{E} , and hence

does not exist in any vicinity $(0, \delta)$ of the origin, however small $\delta > 0$ is taken.

5. *If one of the derivatives of the continuous function $f(x)$ is continuous in an interval \mathfrak{A} , the derivative $f'(x)$ exists, and is continuous in \mathfrak{A} .*

This follows from 3.

6. *If one of the four derivatives of the continuous function $f(x)$ is $= 0$ in an interval \mathfrak{A} , $f(x) = \text{const}$ in \mathfrak{A} .*

This follows from 3.

508. 1. *If one of the derivatives of the continuous function $f(x)$ is ≥ 0 in $\mathfrak{A} = (a < b)$, $f(x)$ is monotone increasing in \mathfrak{A} .*

For then $m = \text{Min } \bar{R}f' \geq 0$, in $(a < x)$. Thus by 506, 1,

$$f(x) - f(a) \geq 0.$$

2. *If one of the derivatives of the continuous function $f(x)$ is ≥ 0 in \mathfrak{A} , $f(x)$ is monotone decreasing.*

3. *If one of the derivatives of the continuous function $f(x)$ is ≥ 0 in \mathfrak{A} , without being constantly 0 in any little interval of \mathfrak{A} , $f(x)$ is an increasing function in \mathfrak{A} . Similarly f is a decreasing function in \mathfrak{A} , if one of the derivatives is ≤ 0 , without being constantly 0 in any little interval of \mathfrak{A} .*

The proof is analogous to I, 403.

509. 1. *Let $f(x)$ be continuous in the interval \mathfrak{A} , and have a derivative, finite or infinite, within \mathfrak{A} . Then the points where the derivative is finite form a pantactic set in \mathfrak{A} .*

For let $\alpha < \beta$ be two points of \mathfrak{A} . Then by the Law of the Mean,

$$f'(\gamma) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha}, \quad \alpha < \gamma < \beta.$$

As the right side has a definite value, the left side must have. Thus in any interval (α, β) in \mathfrak{A} , there is a point γ where the differential coefficient is finite.

For if

$$p = R \overline{\lim}_{x=a} f(x) \quad , \quad q = R \underline{\lim}_{x=a} f(x),$$

there exists a sequence of points $\alpha_1 > \alpha_2 > \dots \doteq a$, such that $f(\alpha_n) \doteq p$; and another sequence $\beta_1 > \beta_2 > \dots \doteq a$, such that $f(\beta_n) \doteq q$. We may suppose

$$\alpha_n > \beta_n \quad , \quad \text{or } \alpha_n < \beta_n \quad , \quad n = 1, 2, \dots$$

Then the Law of the Mean gives

$$Q_n = \frac{f(\alpha_n) - f(\beta_n)}{\alpha_n - \beta_n} = f'(\gamma_n),$$

where γ_n lies between α_n, β_n . Now the numerator $\doteq p - q$, while the denominator $\doteq 0$. Hence $Q_n \doteq +\infty$, or $-\infty$, as we choose.

5. Let $f(x)$ have a finite unilateral differential coefficient U at each point of the interval \mathfrak{A} . Then U is at most pointwise discontinuous in \mathfrak{A} .

For by 474, 3, U is a function of class 1. Hence, by 486, 1, it is at most pointwise discontinuous in \mathfrak{A} .

511. Let $f(x)$ be continuous in the interval $(a < b)$. Let $R(x)$ denote one of the right-hand derivatives of $f(x)$. If R is not continuous on the right at a , then

$$l \leq R(a) \leq m, \tag{1}$$

where

$$l = R \underline{\lim} R(x) \quad , \quad m = R \overline{\lim} R(x) \quad , \quad x \doteq a.$$

To fix the ideas let R be the upper right-hand derivate. Let us suppose that $\alpha = Rf'(a)$ were $> m$. Let us choose η , and c such that

$$m + \eta < c < \alpha. \tag{2}$$

We introduce the auxiliary function

$$\phi(x) = cx - f(x).$$

Then

$$\bar{R}\phi'(x) = c - \underline{R}f'(x) \quad , \quad \underline{R}\phi'(x) = c - \bar{R}f'(x). \tag{3}$$

Now if $\delta > 0$ is sufficiently small,

$$\bar{R}f'(x) < m + \eta \quad , \quad \text{for any } x \text{ in } \mathfrak{A}^* = (a^*, a + \delta).$$

Thus 2), 3), show that

$$\underline{R}\phi'(x) \geq \sigma, \quad \sigma > 0.$$

Hence $\phi(x)$ is an increasing function in \mathfrak{A}^* . But, on the other hand,

$$\bar{R}f'(a) = \underline{R}f'(a),$$

since $a > m$. Hence

$$\bar{R}\phi'(a) = c - \bar{R}f'(a) = c - a < 0.$$

Hence ϕ is a decreasing function at $x = a$. This is impossible since ϕ is continuous at a . Thus $a \leq m$.

Similarly we may show that $l \leq a$.

512. 1. Let $f(x)$ be continuous in $\mathfrak{A} = (a < b)$, and have a derivative, finite or infinite. If $\alpha = f'(a)$, $\beta = f'(b)$, then $f'(x)$ takes on all values between α , β , as x ranges over \mathfrak{A} .

For let $\alpha < \gamma < \beta$, and let

$$Q(x, h) = \frac{f(x+h) - f(x)}{h}, \quad h > 0.$$

We can take h so small that

$$Q(a, h) < \gamma, \quad \text{and} \quad Q(b, -h) > \gamma.$$

Now

$$Q(b, -h) = Q(b-h, h).$$

Hence

$$Q(b-h, h) > \gamma.$$

If now we fix h , $Q(x, h)$ is a continuous function of x . As Q is $< \gamma$, for $x = a$, and $> \gamma$, for $x = b-h$, it takes on the value γ for some x , say for $x = \xi$, between a , $b-h$. Thus

$$Q(\xi, h) = \gamma.$$

But by the Law of the Mean,

$$Q(\xi, h) = f'(\eta),$$

where

$$a < \xi < \eta < \xi + h < b.$$

Thus $f'(x) = \gamma$, at $x = \eta$ in \mathfrak{A} .

2. Let $f(x)$ be continuous in the interval \mathfrak{A} , and admit a derivative, finite or infinite. If $f'(x) = 0$ in \mathfrak{A} , except possibly at an enumerable set \mathfrak{E} , then $f' = 0$ also in \mathfrak{E} .

For if $f'(a) = 0$, and $f'(\beta) = b \neq 0$, then $f'(x)$ ranges over all values in $(0, b)$, as x passes from a to β . But this set of values has the cardinal number c . Hence there is a set of values in (a, β) whose cardinal number is c , where $f'(x) \neq 0$. This is contrary to the hypothesis.

3. Let $f(x)$, $g(x)$ be continuous and have derivatives, finite or infinite, in the interval \mathfrak{A} . If in \mathfrak{A} there is an a for which

$$f'(a) > g'(a),$$

and a β for which

$$f'(\beta) < g'(\beta),$$

then there is a γ for which

$$f'(\gamma) = g'(\gamma),$$

provided

$$\delta(x) = f(x) - g(x)$$

has a derivative, finite or infinite.

For by hypothesis

$$\delta'(a) > 0 \quad , \quad \delta'(\beta) < 0.$$

Hence by 1 there is a point where $\delta' = 0$.

513. 1. If one of the four derivatives of the continuous function $f(x)$ is limited in the interval \mathfrak{A} , all four are, and they have the same upper and lower R -integrals.

The first part of the theorem is obvious from 507, 1. Let us effect a division of \mathfrak{A} of norm d . Then

$$\int_{\mathfrak{A}} \bar{R} = \lim_{d \rightarrow 0} \sum M_i d_i \quad , \quad M_i = \text{Max } \bar{R}_i \text{ in } d_i.$$

But the maximum of the three other derivatives in d_i is also M_i by 507, 1. Hence the last part of the theorem.

2. Let $f(x)$ be continuous and have a limited unilateral derivative as \bar{R} in $\mathfrak{A} = (a < b)$. Then

$$\int_a^b \bar{R} dx \geq f(b) - f(a) \leq \int_a^b \bar{R} dx. \quad (1)$$

For let $a < a_1 < a_2 < \dots < b$ determine a division of \mathfrak{A} , of norm d .

hen by 506, 1,

$$\text{Min } \bar{R} \leq \frac{f(a_{m+1}) - f(a_m)}{a_{m+1} - a_m} < \text{Max } \bar{R},$$

the interval $(a_m, a_{m+1}) = d_m$.

Hence

$$\Sigma d_m \text{Min } \bar{R} \leq f(b) - f(a) \leq \Sigma d_m \text{Max } \bar{R}.$$

Letting $d \doteq 0$, we get 1).

3. If $f(x)$ is continuous, and $\bar{U}f'$ is limited and R -integrable in $(a < b)$, then

$$\int_a^b \bar{U}f' = f(b) - f(a).$$

514. 1. Let $f(x)$ be limited in $\mathfrak{A} = (a < b)$, and

$$F(x) = \int_a^x f dx, \quad a \leq x \leq b.$$

Then

$$U \lim_{x=u} f \leq \bar{U}F'(u) \leq U \lim_{x=u} f, \quad (1)$$

for any u within \mathfrak{A} .

To fix the ideas let us take a right-hand derivate at $x = u$. Then

$$h \text{Min } f \leq \int_u^{u+h} f dx \leq h \text{Max } f, \quad \text{in } (u^*, u + h), h > 0.$$

Thus

$$\text{Min } f \leq \frac{\Delta F}{\Delta x} \leq \text{Max } f.$$

Letting $h \doteq 0$, we get

$$R \lim_{x=u} f \leq \bar{R}F'(u) \leq R \lim_{x=u} f,$$

which is 1) for this case.

2. Let $f(x)$ be limited in the interval $\mathfrak{A} = (a < b)$. If $f(x+0)$ exists,

$$R \text{ derivative } \int_a^x f dx = f(x+0);$$

and if $f(x-0)$ exists, $a < x < b$

$$L \text{ derivative } \int_a^x f dx = f(x-0).$$

3. Let $f(x)$ be limited and R -integrable in $\mathfrak{A} = (a < b)$. The points where

$$F(x) = \int_a^x f dx \quad , \quad a \leq x \leq b$$

does not have a differential coefficient in \mathfrak{A} form a null set.

For

$$F'(x) = f(x) \quad \text{by I, 537, 1,}$$

when f is continuous at x . But by 462, 6, the points where f is not continuous form a null set.

515. In I, 400, we proved the theorem :

Let $f(x)$ be continuous in $\mathfrak{A} = (a < b)$, and let its derivative $= 0$ within \mathfrak{A} . Then f is a constant in \mathfrak{A} . This theorem we have extended in 507, 6, to a derivate of $f(x)$. It can be extended still farther as follows :

1. (*L. Scheefer*). If $f(x)$ is continuous in $\mathfrak{A} = (a < b)$, and if one of its derivatives $= 0$ in \mathfrak{A} except possibly at the points of an enumerable set \mathfrak{E} , then $f = \text{constant}$ in \mathfrak{A} .

If f is a constant, the theorem is of course true. We show that the contrary case leads to an absurdity, by showing that $\text{Card } \mathfrak{E}$ would $= c$, the cardinal number of an interval.

For if f is not a constant, there is a point c in \mathfrak{A} where $p = f(c) - f(a)$ is $\neq 0$. To fix the ideas let $p > 0$; also let us suppose the given derivate is $\bar{R} = \bar{R}_t f'(x)$.

Let

$$g(x, t) = f(x) - f(a) - t(x - a) \quad , \quad t > 0.$$

Obviously $|g|$ is the distance f is above or below the secant line,

$$y = t(x - a) + f(a).$$

Thus in particular for any t ,

$$g(a, t) = 0 \quad , \quad g(c, t) = p - t(c - a).$$

Let $q > 0$ be an arbitrary but fixed number $< p$. Then

$$\begin{aligned} g(c, t) - q &= p - q - t(c - a) \\ &= (p - q) \left\{ 1 - t \frac{c - a}{p - q} \right\} > 0, \end{aligned}$$

if $t < T$, where

$$T = \frac{p - q}{c - a}.$$

Hence

$$g(e, t) > q$$

any t in the interval $\mathfrak{T} = (\tau, T)$, $0 < \tau < T$. We note that

$$\text{Card } \mathfrak{T} = c.$$

Since for any t in \mathfrak{T} , $g(a, t) = 0$, and $g(e, t) > q$, let $x = e_t$ be a maximum of the points $< e$ where $g(x, t) = q$. Then $e < e_t$, and for any h such that $e + h$ lies in (e, e_t) ,

$$0 < \frac{g(e+h) - g(e)}{h} = \frac{f(e+h) - f(e)}{h} - t.$$

Hence

$$\bar{R}f'(e) > 0.$$

Thus for any t in \mathfrak{T} , e_t lies in \mathfrak{E} . As t ranges over \mathfrak{T} , let e_t range over $\mathfrak{E}_1 \leq \mathfrak{E}$. To each point e of \mathfrak{E}_1 corresponds but one point t of \mathfrak{T} . For

$$0 = g(e, t) - g(e, t') = (t - t')(e - a).$$

Hence

$$t = t', \text{ as } e > a.$$

Thus

$$\text{Card } \mathfrak{T} = \text{Card } \mathfrak{E}_1 \leq \text{Card } \mathfrak{E},$$

which is absurd.

2. Let $f(x)$ be continuous in $\mathfrak{A} = (a < b)$. Let \mathfrak{E} denote the set of \mathfrak{A} where one of the derivatives has one sign. If \mathfrak{E} exists,

Card $\mathfrak{E} = c$, the cardinal number of the continuum.

The proof is entirely similar to that in 1. For let e be a point of \mathfrak{E} . Then there exists a $d > e$ such that

$$f(d) - f(e) = p > 0.$$

We now introduce the function

$$g(x, t) = f(x) - f(e) - t(x - e), \quad t > 0,$$

reason on this as we did on the corresponding g in 1, using the interval (e, d) instead of (a, b) . We get

$$\text{Card } \mathfrak{E}_1 = \text{Card } \mathfrak{T} = c.$$

3. Let $f(x), g(x)$ be continuous in the interval \mathfrak{A} . Let a pair of corresponding derivatives as $\bar{R}f', \bar{R}g'$ be finite and equal, except possibly at an enumerable set \mathfrak{E} . Then $f' = g' + C$, in \mathfrak{A} , where C is a constant.

For let

$$\phi = f - g, \quad \psi = g - f.$$

Then in

$$A = \mathfrak{A} - \mathfrak{E},$$

$$\bar{R}\phi' \geq \bar{R}f' - \bar{R}g' = 0, \quad \bar{R}\psi' \geq 0.$$

But if $\bar{R}\phi' < 0$ at one point in \mathfrak{A} , it is < 0 at a set of points \mathfrak{B} whose cardinal number is c . But \mathfrak{B} lies in \mathfrak{E} . Hence $\bar{R}\phi$ is never < 0 , in \mathfrak{A} . The same holds for ψ . Hence, by 508, ϕ and ψ are both monotone increasing. This is impossible unless $\phi =$ a constant.

516. The preceding theorem states that the continuous function $f(x)$ in the interval \mathfrak{A} is known in \mathfrak{A} , aside from a constant, when $f'(x)$ is finite and known in \mathfrak{A} , aside from an enumerable set.

Thus $f(x)$ is known in \mathfrak{A} when f' is finite and known at each *irrational* point of \mathfrak{A} .

This is not the case when f' is finite and known at each *rational* point only in \mathfrak{A} .

For the rational points in \mathfrak{A} being enumerable, let them be

$$r_1, r_2, r_3 \dots \tag{1}$$

Let

$$l = l_1 + l_2 + l_3 + \dots$$

be a positive term series whose sum l is $< \mathfrak{A}$. Let us place r_1 within an interval δ_1 of length $\leq l_1$. Let r_2 be the first number in 1) not in δ_1 . Let us place it within a non-overlapping interval δ_2 of length $\leq l_2$, etc.

We now define a function $f(x)$ in \mathfrak{A} such that the value of f at any x is the length of all the intervals and part of an interval lying to the left of x . Obviously $f(x)$ is a continuous function of x in \mathfrak{A} . At each rational point $f'(x) = 1$. But $f(x)$ is not determined aside from a constant. For $\sum \delta_n \leq l$. Therefore when l is small enough we may vary the position and lengths of the δ -intervals, so that the resulting f 's do not differ from each other only by a constant.

517. 1. Let $f(x)$ be continuous in $\mathfrak{A} = (a < b)$ and have a finite derivate, say $\bar{R}f'$, at each point of \mathfrak{A} . Let \mathfrak{E} denote the points of \mathfrak{A} where \bar{R} has one sign, say > 0 . If \mathfrak{E} exists, it cannot be a null set.

For let c be a point of \mathfrak{E} , then there exists a point $d > c$ such that

$$f(d) - f(c) = p > 0. \quad (1)$$

Let \mathfrak{E}_n denote the points of \mathfrak{E} where

$$n - 1 \leq \bar{R}f' < n. \quad (2)$$

Then $\mathfrak{E} = \mathfrak{E}_1 + \mathfrak{E}_2 + \dots$. Let $0 < q < p$. We take the positive constants $q_1, q_2 \dots$ such that

$$q_1 + 2q_2 + 3q_3 + \dots \leq q.$$

If now \mathfrak{E} is a null set, each \mathfrak{E}_m is also. Hence the points of \mathfrak{E}_m can be inclosed *within* a set of intervals δ_{mn} such that $\sum_n \delta_{mn} < q_m$.

Let now $q_m(x)$ be the sum of the intervals and parts of intervals δ_{mn} , $n = 1, 2 \dots$ which lie in the interval $(a \leq x)$. Let

$$Q(x) = \sum_m q_m(x).$$

Obviously $Q(x)$ is a monotone increasing function, and

$$0 \leq Q(x) \leq q. \quad (3)$$

Consider now

$$P(x) = f(x) - f(a) - Q(x).$$

We have at a point of $\mathfrak{H} - \mathfrak{E}$,

$$\frac{\Delta P}{\Delta x} = \frac{\Delta f}{\Delta x} - \frac{\Delta Q}{\Delta x} \leq \frac{\Delta f}{\Delta x}, \quad \Delta x > 0.$$

Hence at such a point

$$\bar{R}P' \leq \bar{R}f' \leq 0.$$

But at a point x of \mathfrak{E} , $\bar{R}P' < 0$ also. For x must lie in some \mathfrak{E}_m , and hence within some δ_{mn} . Thus $q_m(x)$ increases by at least δ_{mn} when x is increased to $x + \Delta x$. Hence $m q_m(x)$, and thus $P(x)$ is increased at least $m \Delta x$. Thus

$$\frac{\Delta Q}{\Delta x} \geq m.$$

Thus

$$\bar{R}P' \leq \bar{R}f' - m < 0, \text{ by 2),}$$

since x lies in \mathfrak{E}_m . Thus $\bar{R}P' \leq 0$ at any point of \mathfrak{A} . Thus P is a monotone decreasing function in \mathfrak{A} , by 508, 2. Hence

$$P(c) - P(d) \geq 0.$$

Hence

$$f(c) - f(d) - \{Q(c) - Q(d)\} \geq 0,$$

or using 1), 3)

which is not so, as p is $> q$.

2. (Lebesgue.) Let $f(x), g(x)$ be continuous in the interval \mathfrak{A} , and have a pair of corresponding derivates as $\bar{R}f', \bar{R}g'$ which are finite at each point of \mathfrak{A} , and also equal, the equality holding except possibly at a null set. Then $f(x) - g(x) = \text{constant}$ in \mathfrak{A} .

The proof is entirely similar to that of 515, 3, the enumerable set \mathfrak{E} being here replaced by a null set. We then make use of 1.

518. Let $f'(x)$ be continuous in some interval $\Delta = (u - \delta, u + \delta)$. Let $f''(x)$ exist, finite or infinite, in Δ , but be finite at the point $x = u$. Then

$$f'(u) = \lim_{h \rightarrow 0} Qf, \quad (1)$$

where

$$Qf(u) = \frac{f(u+h) + f(u-h) - 2f(u)}{h^2}, \quad h > 0.$$

Let us first suppose that $f''(u) = 0$. We have for $0 < h \leq \eta < \delta$,

$$\begin{aligned} Qf &= \frac{1}{h} \left\{ \frac{f(u+h) - f(u)}{h} - \frac{f(u-h) - f(u)}{-h} \right\} \\ &= \frac{1}{h} \{f'(x') - f'(x'')\}, \quad u < x' < u+h, \quad u-h < x'' < u \\ &= \frac{1}{h} [(x' - u)\{f''(u) + \epsilon'\} - (x'' - u)\{f''(u) + \epsilon''\}], \end{aligned}$$

where $|\epsilon'|, |\epsilon''|$ are $< \epsilon/2$ for η sufficiently small.

Now

$$\frac{x' - u}{h} \leq 1, \quad \frac{|x'' - u|}{h} \leq 1,$$

while

$$f''(u) = 0, \quad \text{by hypothesis.}$$

Hence

$$|Qf| < \epsilon, \quad \text{for } 0 < h \leq \eta,$$

and 1) holds in this case.

Suppose now that $f''(u) = a \neq 0$. Let

$$g(x) = f(x) - q(x) \quad , \quad \text{where } q(x) = \frac{1}{2} ax^2 + bx + c.$$

Since $q''(u) = a \quad , \quad g''(u) = 0$.

Thus we are in the preceding case, and $\lim Qg = 0$.

But $Qg = Qf - Qq$.

Hence $\lim Qf = a$.

Maxima and Minima

519. 1. In I, 466 and 476, we have defined the terms $f(x)$ as maximum or a minimum at a point. Let us extend these terms follows. Let $f(x_1 \cdots x_n)$ be defined over \mathfrak{A} , and let $x = a$ be an inner point of \mathfrak{A} .

We say f has a maximum at $x = a$ if 1°, $f(a) - f(x) \geq 0$, for any x in some $V(a)$, and 2°, $f(a) - f(x) > 0$ for some x in any $V(a)$. the sign \geq can be replaced by $>$ in 1°, we will say f has a proper maximum at a , when we wish to emphasize this fact; and when \geq cannot be replaced by $>$, we will say f has an improper maximum. A similar extension of the old definition holds for the minimum. A common term for maximum and minimum is extreme.

2. If $f(x)$ is a constant in some segment \mathfrak{B} , lying in the interval \mathfrak{A} , \mathfrak{B} is called a segment of invariability, or a constant segment of f in \mathfrak{A} .

Example. Let $f(x)$ be continuous in $\mathfrak{A} = (0, 1^*)$.

Let
$$x = \cdot \alpha_1 \alpha_2 \alpha_3 \cdots \quad (1)$$

the expression of a point of \mathfrak{A} in the normal form in the dyadic system. Let

$$\xi = \cdot \alpha_1 \alpha_2 \alpha_3 \cdots \quad (2)$$

expressed in the triadic system, where $\alpha_n = a_n$, when $a_n = 0$, and $\alpha_n = 2$ when $a_n = 1$. The points $\mathfrak{C} = \{\xi\}$ form a Cantor set.

272. Let $\{\mathfrak{I}_n\}$ be the adjoint set of intervals. We associate

now the point 1) with the point 2), which we indicate as usual by $x \sim \xi$. We define now a function $g(x)$ as follows:

$$g(\xi) = f(x) \quad , \quad \text{when } x \sim \xi.$$

This defines g for all the points of \mathfrak{E} . In the interval \mathfrak{I}_n , let g have a constant value. Obviously g is continuous, and has a pantactic set of intervals in each of which g is constant.

3. We have given criteria for maxima and minima in I, 468 seq., to which we may add the following:

Let $f(x)$ be continuous in $(a - \delta, a + \delta)$. If $\underline{R}f'(a) > 0$ and $\bar{L}f'(a) < 0$, finite or infinite, $f(x)$ has a minimum at $x = a$.

If $\bar{R}f'(a) < 0$ and $\underline{L}f'(a) > 0$, finite or infinite, $f(x)$ has a maximum at $x = a$.

For on the 1° hypothesis, let us take α such that $\underline{R} - \alpha > 0$. Then there exists a $\delta' > 0$ such that

$$\frac{f(a+h) - f(a)}{h} > \underline{R} - \alpha > 0 \quad , \quad 0 < h \leq \delta'.$$

Hence

$$f(a+h) > f(a) \quad , \quad a+h \text{ in } (a^*, a + \delta').$$

Similarly if β is chosen so that $\bar{L} + \beta < 0$, there exists a $\delta'' > 0$, such that

$$\frac{f(a-h) - f(a)}{-h} < \bar{L} + \beta.$$

Hence

$$f(a-h) > f(a) \quad , \quad a+h \text{ in } (a - \delta'', a^*).$$

520. Example 1. Let $f(x)$ oscillate between the x -axis and the two lines $y = x$ and $y = -x$, similar to

$$y = \left| x \sin \frac{\pi}{x} \right|.$$

In any interval about the origin, y oscillates infinitely often, having an infinite number of proper maxima and minima. At the point $x = 0$, f has an improper minimum.

Example 2. Let us take two parabolas P_1, P_2 defined by $y = x^2$, $y = 2x^2$. Through the points $x = \pm \frac{1}{2}, \pm \frac{1}{3}, \dots$ let us erect ordinates, and join the points of intersection with P_1, P_2 , alternately by straight lines, getting a broken line oscillating between the

arabolas P_1, P_2 . The resulting graph defines a continuous function $f(x)$ which has proper extremes at the points $\mathfrak{E} = \left\{ \pm \frac{1}{n} \right\}$. However, unlike Ex. 1, the limit point $x = 0$ of these extremes is not a point at which $f(x)$ has a proper extreme.

Example 3. Let $\{\delta\}$ be a set of intervals which determine a Lebesgue set \mathfrak{S} lying in $\mathfrak{A} = (0, 1)$. Over each interval $\delta = (\alpha, \beta)$ belonging to the n^{th} stage, let us erect a curve, like a segment of a sine curve, of height $h_n \doteq 0$, as $n \doteq \infty$, and having horizontal tangents at α, β , and at γ , the middle point of the interval δ . At the points $\{\xi\}$ of \mathfrak{A} not in any interval δ , let $f(x) = 0$. The function f is now defined in \mathfrak{A} and is obviously continuous. At the points $\{\gamma\}$, f has a proper maximum; at points of the type α, β , f has an improper minimum. These latter points form the set whose cardinal number is c . The function is increasing in each interval (α, γ) , and decreasing in each (γ, β) . It oscillates infinitely often in the vicinity of any point of \mathfrak{S} .

We note that while the points where f has a proper extreme form an enumerable set, the points of improper extreme may form a set whose cardinal number is c .

Example 4. We use the same set of intervals $\{\delta\}$ but change the curve over δ , so that it has a constant segment $\eta = (\lambda, \mu)$ in its middle portion. As before $f = 0$, at the points ξ not in the intervals δ .

The function $f(x)$ has now no proper extremes. At the points \mathfrak{S} , f has an improper minimum; at the points of the type λ, μ , it has an improper maximum.

Example 5. Weierstrass' Function. Let \mathfrak{E} denote the points in interval \mathfrak{A} of the type

$$x = \frac{r}{b^s} \quad , \quad r, s, \text{ positive integers.}$$

For such an x we have, using the notation of 502,

$$b^m x = \iota_m + \xi_m = b^{m-s} r.$$

Hence $\xi_m = 0$, for $m \geq s$.

Thus $e_m = (-1)^{\iota_m+1} = (-1)^{r+1}$.

Hence $\operatorname{sgn} \frac{\Delta F}{\Delta x} = \operatorname{sgn} Q = \operatorname{sgn} e_m \eta_m = \operatorname{sgn} (-1)^r h.$

Thus $\operatorname{sgn} Rf'(x) = +1$, $\operatorname{sgn} Lf'(x) = -1,$

if r is even, and reversed if r is odd. Thus at the points \mathfrak{E} , the curve has a vertical cusp. By 519, 3, F has a maximum at the points \mathfrak{E} , when r is odd, and a minimum when r is even. The points \mathfrak{E} are pantactic in \mathfrak{A} .

Weierstrass' function has no constant segment δ , for then $f'(x) = 0$ in δ . But F' does not exist at any point.

521. 1. *Let $f(x_1 \dots x_m)$ be continuous in the limited or unlimited set \mathfrak{A} . Let \mathfrak{E} denote the points of \mathfrak{A} where f has a proper extreme. Then \mathfrak{E} is enumerable.*

Let us first suppose that \mathfrak{A} is limited. Let $\delta > 0$ be a fixed positive number. There can be but a finite number of points α in \mathfrak{A} such that

$$f(\alpha) > f(x) \quad , \quad \text{in } V_\delta^*(\alpha). \quad (1)$$

For if there were an infinity of such points, let β be a limiting point and $\eta < \frac{1}{2}\delta$. Then in $V_\eta(\beta)$ there exist points α', α'' such that $V_\delta(\alpha'), V_\delta(\alpha'')$ overlap. Thus in one case

$$f(\alpha') > f(\alpha''),$$

and in the other

$$f(\alpha') < f(\alpha''),$$

which contradicts the first.

Let now $\delta_1 > \delta_2 > \dots \doteq 0$. There are but a finite number of points α for which 1) holds for $\delta = \delta_1$, only a finite number for $\delta = \delta_2$, etc. Hence \mathfrak{E} is enumerable. The case that \mathfrak{A} is unlimited follows now easily.

2. We have seen that Weierstrass' function has a pantactic set of proper extremes. However, according to 1, they must be enumerable. In Ex. 3, the function has a minimum at each point of the non-enumerable set \mathfrak{S} ; but these minima are improper. On the other hand, the function has a proper maximum at the points $\{\gamma\}$, but these form an enumerable set.

522. 1. Let $f(x)$ be continuous in the interval \mathfrak{A} . Let f have a proper maximum at $x = \alpha$, and $x = \beta$ in \mathfrak{A} . Then there is a point γ between α, β where f has a minimum, which need not however be a proper minimum.

For say $\alpha < \beta$. In the vicinity of α , $f(x)$ is $< f(\alpha)$; also in the vicinity of β , $f(x)$ is $< f(\beta)$. Thus there are points \mathfrak{B} in (α, β) where f is $<$ either $f(\alpha)$ or $f(\beta)$. Let μ be the minimum of the values of $f(x)$, as x ranges over \mathfrak{B} . There is a least value x in (α, β) for which $f(x) = \mu$. We may take this as the point in question. Obviously γ is neither α nor β .

2. That at the point γ , f does not need to have a proper minimum is illustrated by Exs. 1, or 3.

3. In $\mathfrak{A} = (\alpha, b)$ let $f'(x)$ exist, finite or infinite. The points within \mathfrak{A} at which f has an extreme proper or improper, lie among the zeros of $f'(x)$.

This follows from the proof used in I, 468, 2, if we replace there 0, by ≤ 0 , and > 0 , by ≥ 0 .

4. Let $f'(x)$ be continuous in the interval \mathfrak{A} , and let $f(x)$ have no constant segments in \mathfrak{A} . The points \mathfrak{E} of \mathfrak{A} where f has an extreme, form an apantactic set in \mathfrak{A} . Let \mathfrak{B} denote the zeros of $f'(x)$ in \mathfrak{A} . If $\mathfrak{B} = \{b_n\}$ is the border set of intervals lying in \mathfrak{A} corresponding to \mathfrak{B} , $f(x)$ is univariant in each b_n .

For by 3, the points \mathfrak{E} lie in \mathfrak{B} . As $f'(x)$ is continuous, \mathfrak{B} is complete and determines the border set \mathfrak{B} . Within each b_n , $f(x)$ has one sign. Hence $f(x)$ is univariant in b_n .

5. Let $f(x)$ be a continuous function having no constant segment in the interval \mathfrak{A} . If the points \mathfrak{E} where f has an extreme form an apantactic set in \mathfrak{A} , then the points \mathfrak{B} where $f'(x)$ does not exist or is discontinuous, form also a pantactic set in \mathfrak{A} .

For if \mathfrak{B} is not pantactic in \mathfrak{A} , there is an interval \mathfrak{C} in \mathfrak{A} containing no point of \mathfrak{B} . Thus $f'(x)$ is continuous in \mathfrak{C} . But the points of \mathfrak{E} in \mathfrak{C} form an apantactic set in \mathfrak{C} by 4. This, however, contradicts our hypothesis.

Example. Weierstrass' function satisfies the condition of the theorem 5. Hence the points where $f'(x)$ does not exist or is

discontinuous form a pantactic set. This is indeed true, since F' exists at no point.

6. Let $f(x)$ be continuous and have no constant segment in the interval \mathfrak{A} . Let $f'(x)$ exist, finite or infinite. The points where $f'(x)$ is finite and is $\neq 0$ form a pantactic set in \mathfrak{A} .

For let $\alpha < \beta$ be any two points in \mathfrak{A} . If $f(\alpha) = f(\beta)$, there is a point $\alpha < \gamma < \beta$ such that $f(\alpha) \neq f(\gamma)$, since f has no constant segment in \mathfrak{A} . Then the Law of the Mean gives

$$f'(\xi) = \frac{f(\alpha) - f(\gamma)}{\alpha - \gamma} \neq 0.$$

Thus in the arbitrary interval (α, β) there is a point ξ , where $f'(x)$ exists and is $\neq 0$.

7. Let $f'(x)$ be continuous in the interval \mathfrak{A} . Then any interval \mathfrak{B} in \mathfrak{A} which is not a constant segment contains a segment \mathfrak{C} in which f is univariant.

For since f is not constant in \mathfrak{B} , there are two points a, b in \mathfrak{B} at which f has different values. Then by the Law of the Mean

$$f(a) - f(b) = (a - b)f'(c), \quad c \text{ in } \mathfrak{B}.$$

Hence $f'(c) \neq 0$. As $f'(x)$ is continuous, it keeps its sign in some interval $(c - \delta, c + \delta)$, and f is therefore univariant.

523. Let $f(x)$ be continuous in the interval \mathfrak{A} , and have in any interval in \mathfrak{A} a constant segment or a point at which f has an extreme. If $f'(x)$ exists, finite or infinite, it is discontinuous infinitely often in any interval in \mathfrak{A} , not a constant segment. At a point of continuity of the derivative, $f'(x) = 0$.

For if $f'(x)$ were continuous in an interval \mathfrak{B} , not a constant segment, f would be univariant in some interval $\mathfrak{C} \leq \mathfrak{B}$, by 522, 7. But this contradicts the hypothesis, which requires that any interval as \mathfrak{C} has a constant segment. Hence $f'(x)$ is discontinuous in any interval, however small.

Let now $x = c$ be a point of continuity. Then if c lies in a constant segment, $f'(c) = 0$ obviously. If not, there is a sequence of points $e_1, e_2, \dots \rightarrow c$ such that $f(x)$ has an extreme at e_n . But then $f'(e_n) = 0$, by 522, 3. As $f'(x)$ is continuous at $x = c$, $f'(c) = 0$ also.

524. (König.) Let $f(x)$ be continuous in \mathfrak{A} and have a pantactic set of cuspidal points \mathfrak{C} . Then for any interval \mathfrak{B} of \mathfrak{A} , there exists a β such that $f(x) = \beta$ at an infinite set of points in \mathfrak{B} . Moreover, there is a pantactic set of points $\{\xi\}$ in \mathfrak{B} , such that k being taken at pleasure,

$$\underline{f'}(x) \leq k \leq \bar{f'}(x). \quad (1)$$

For among the points \mathfrak{C} there is an infinite pantactic set c of proper maxima, or of proper minima. To fix the ideas, suppose the former. Let $x = c$ be one of these points within \mathfrak{B} . Then there exists an interval $\mathfrak{b} \subseteq \mathfrak{B}$, containing c , such that

$$f(c) > f(x) \quad , \quad \text{for any } x \text{ in } \mathfrak{b}.$$

Let $\mu = \text{Min } f(x)$, in \mathfrak{b} .

Then there is a point \bar{x} where f takes on this minimum value. The point c divides the interval \mathfrak{b} into two intervals. Let \mathfrak{l} be that one of these intervals which contains \bar{x} , the other interval we denote by \mathfrak{m} . Within \mathfrak{m} let us take a point c_1 of c . Then in \mathfrak{l} there is a point c'_1 such that

$$f(c_1) = f(c'_1).$$

The point c_1 determines an interval \mathfrak{b}_1 , just as c determined \mathfrak{b} . Obviously $\mathfrak{b}_1 \subseteq \mathfrak{m}$, and \mathfrak{b}_1 falls into two segments $\mathfrak{l}_1, \mathfrak{m}_1$ as before \mathfrak{b} did. Within \mathfrak{m}_1 we take a point of c . Then in \mathfrak{l} there is a point c'_2 , and in \mathfrak{l}_1 a point c''_2 , such that

$$f(c_2) = f(c'_2) = f(c''_2).$$

In this way we may continue indefinitely. Let

$$c'_1 \quad , \quad c'_2 \quad , \quad c'_3 \quad \dots$$

be the points obtained in this way which fall in \mathfrak{l} . Let c' be a limit point of this set. Let

$$c''_1 \quad , \quad c''_2 \quad , \quad c''_3 \quad \dots$$

be the points obtained above which fall in \mathfrak{l}_1 , and let c'' be a limit point of this set. Continuing in this way we get a sequence of limiting points

$$c' \quad , \quad c'' \quad , \quad c''' \quad \dots \quad (2)$$

lying respectively in $\mathfrak{l}, \mathfrak{l}_1, \mathfrak{l}_2 \dots$

Since f is continuous,

$$f(c') = f(c'') = f(c''') = \dots \quad (3)$$

Thus if we set $f(c') = \beta$ we see that $f(x)$ takes on the value β at the infinite set of points 2), which lie in \mathfrak{B} .

Let $\gamma_1, \gamma_2 \dots$ be a set of points in 2) which $\doteq \gamma$.

Then

$$\frac{f(\gamma) - f(\gamma_1)}{\gamma - \gamma_1} = \frac{f(\gamma) - f(\gamma_2)}{\gamma - \gamma_2} = \dots = 0. \quad (4)$$

Thus if $f'(x)$ exists at $x = \gamma$, the equations 3) show that $f'(\gamma) = 0$. If f' does not exist at γ , they show that

$$\underline{f'} \leq 0 \leq \bar{f'} \quad , \quad \text{at } x = \gamma.$$

Let now k be taken at pleasure. Then

$$g(x) = f(x) - kx$$

is constituted as f , and

$$\underline{g'}(x) = \underline{f'}(x) - k.$$

This gives 1).

525. 1. Lineo-Oscillating Functions. The oscillations of a continuous function fall into two widely different classes, according as $f(x)$ becomes monotone on adding a linear function $l(x) = ax + b$, or does not.

The former are called lineo-oscillating functions. A continuous function which does not oscillate in \mathfrak{A} , or if it does is lineo-oscillating, we say is *at most* a lineo-oscillating function.

Example 1. Let $f(x) = \sin x \quad , \quad l(x) = x.$

If we set

$$y = f(x) + l(x)$$

and plot the graph, we see at once that y is an increasing function. At the point $x = \pi$, the slope of the tangent to $f(x) = \sin x$ is greatest negatively, i.e. $\sin x$ is decreasing here fastest. But the angle that the tangent to $\sin x$ makes at this point is -45° , while the slope of the line $l(x)$ is constantly 45° . Thus at $x = \pi$, y has a point of inflection with horizontal tangent.

If we take $l(x) = ax$, $a > 1$, y is an increasing function, increasing still faster than before.

All this can be verified by analysis. For setting

$$y = \sin x + ax, \quad a > 1,$$

get

$$y' = a + \cos x,$$

$$y' > 0.$$

Thus y is a lineo-oscillating function in any interval.

Example 2.
$$f(x) = x^2 \sin \frac{1}{x}, \quad x \neq 0$$

$$= 0, \quad x = 0.$$

$$l(x) = ax + b, \quad y = f(x) + l(x).$$

Then

$$y' = 2x \sin \frac{1}{x} - \cos \frac{1}{x} + a, \quad x \neq 0$$

$$= a, \quad x = 0.$$

Hence, if $a > 1 + 2\pi$, y is an increasing function in $\mathfrak{A} = (-\pi, \pi)$. The function f oscillates infinitely often in \mathfrak{A} , but is a lineo-oscillating function.

Example 3.
$$f(x) = x \sin \frac{1}{x}, \quad x \neq 0$$

$$= 0, \quad x = 0.$$

$$l(x) = ax + b, \quad y = f(x) + l(x).$$

Here

$$y' = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x} + a, \quad x \neq 0.$$

For $x = 0$, y' does not exist, finitely or infinitely.

Obviously, however great a is taken, y has an infinity of oscillations in any interval about $x = 0$. Hence f is not a lineo-oscillating function in such an interval.

If one of the four derivatives of the continuous function $f(x)$ is limited in the interval \mathfrak{A} , $f(x)$ is at most lineo-oscillating in \mathfrak{A} .

For say $\underline{R}f' > -\alpha$ in \mathfrak{A} . Let $0 < \alpha < \beta$,

$$g(x) = f(x) + \beta x.$$

Then

$$\underline{g}'(x) = \beta + \underline{f}'(x) > 0.$$

Hence g is monotone increasing by 508, 1.

3. Let $f(x)$ be at most lineo-oscillating in the interval \mathfrak{A} . If Uf' does not exist finitely at a point x in \mathfrak{A} , it is definitely infinite at the point. Moreover, the sign of the ∞ is the same throughout \mathfrak{A} .

For if f is monotone in \mathfrak{A} , the theorem is obviously true. If not, let

$$g(x) = f(x) + ax$$

be monotone. Then

$$Uf' = Ug' - a,$$

and this case is reduced to the preceding.

Remark. This shows that no continuous function whose graph has a vertical cusp can be lineo-oscillating. All its vertical tangents correspond to points of inflection, as in

$$y = x^{\frac{1}{3}}.$$

Variation

526. 1. Let $f(x)$ be continuous in the interval \mathfrak{A} , and have limited variation. Let D be a division of \mathfrak{A} of norm d . Then using the notation of 443,

$$\lim V_D f = Vf, \quad \lim P_D f = Pf, \quad \lim N_D f = Nf. \quad (1)$$

For there exists a division Δ such that

$$V - \frac{\epsilon}{2} < V_\Delta \leq V,$$

where for brevity we have dropped f after the symbol V . Let now Δ divide \mathfrak{A} into ν segments whose minimum length call λ . Let D be a division of \mathfrak{A} of norm $d \leq d_0 < \lambda$. Then not more than one point of Δ , say a_κ , can lie in any interval as (a_i, a_{i+1}) of D . Let $E = D + \Delta$, the division obtained by superposing Δ on D . Then μ denoting some integer $\leq \nu$,

$$V_E - V_D = \sum_{\kappa=1}^{\mu} \{ |f(a_\kappa) - f(a_i)| + |f(a_{i+1}) - f(a_\kappa)| - |f(a_{i+1}) - f(a_i)| \}.$$

If now d_0 is taken sufficiently small, $\text{Osc} f$ in any interval of D is small as we choose, say $< \frac{\epsilon}{6\nu}$. Then

$$|V_E - V_D| \leq \frac{\mu\epsilon}{2\nu} \leq \frac{\epsilon}{2}.$$

But since E is got by superposing Δ on D ,

$$V_\Delta < V_E \leq V.$$

Hence for any D of norm $< d_0$,

$$|V_D - V| < \epsilon,$$

which proves the first relation in 1. The other two follow at once now from 443.

27. If $f(x)$ is continuous and has limited variation in the interval $\mathfrak{A} = (a < b)$, then

$$P(x) \quad , \quad N(x) \quad , \quad V(x)$$

are also continuous functions of x in \mathfrak{A} .

Let us show that $V(x)$ is continuous; the rest of the theorem follows at once by 443.

By 526, there exists a d_0 , such that for any division D of norm $< d_0$,

$$V(b) = V_D(b) + \epsilon' \quad , \quad 0 \leq \epsilon' < \epsilon/3.$$

Then *a fortiori*, for any $x < b$ in \mathfrak{A} ,

$$V(x) = V_D(x) + \epsilon_1 \quad , \quad 0 \leq \epsilon_1 < \epsilon/3. \quad (1)$$

In the division D , we may take x as one of the end points of an interval, and $x + h$ as the other end point. Then

$$V(x + h) = V_D(x) + |f(x + h) - f(x)| + \epsilon_2 \quad , \quad 0 \leq \epsilon_2 < \epsilon/3. \quad (2)$$

On the other hand, if d_0 is taken sufficiently small,

$$|f(x + h) - f(x)| < \frac{\epsilon}{3} \quad , \quad \text{for } 0 < h < \delta. \quad (3)$$

From 1), 2), 3) we have

$$0 \leq V(x + h) - V(x) < \epsilon \quad , \quad \text{for any } 0 \leq h < \delta. \quad (4)$$

But in the division D , x is the right-hand end point of some interval as $(x-k, x)$. The same reasoning shows that

$$|V(x-k) - V(x)| < \epsilon, \quad \text{for any } 0 \leq k < \delta. \quad (5)$$

From 4), 5) we see $V(x)$ is continuous.

528. 1. *If one of the derivatives of the continuous function $f(x)$ is numerically $\leq M$ in the interval \mathfrak{A} , the variation V of f is $\leq M\bar{\mathfrak{A}}$.*

For by definition $V = \text{Max } V_D$,

with respect to all divisions $D = \{d_i\}$ of \mathfrak{A} . Here

$$V_D = \Sigma |f(a_i) - f(a_{i+1})|.$$

Now by 506, 1,

$$-M \leq \frac{f(a_i) - f(a_{i+1})}{a_i - a_{i+1}} \leq M,$$

or

$$|f(a_i) - f(a_{i+1})| \leq M d_i.$$

Hence

$$V_D \leq M \Sigma d_i \leq M\bar{\mathfrak{A}}.$$

2. *Let $f(x)$ be limited and R -integrable in $\mathfrak{A} = (a < b)$. Then*

$$F(x) = \int_a^x f dx, \quad a \leq x \leq b$$

has limited variation in \mathfrak{A} .

For let D be a division of \mathfrak{A} into the intervals $d_i = (a_i, a_{i+1})$.

Then

$$\begin{aligned} V_D \cdot F &= \Sigma |F(a_{i+1}) - F(a_i)| = \Sigma \left| \int_{a_i}^{a_{i+1}} f dx \right| \\ &\leq \Sigma \int_{a_i}^{a_{i+1}} |f| dx \leq M \Sigma \int_{a_i}^{a_{i+1}} dx = M \Sigma d_i = M\bar{\mathfrak{A}}. \end{aligned}$$

Thus

$$\text{Max } V_D \cdot F \leq M\bar{\mathfrak{A}},$$

and F has limited variation.

529. 1. *If $f(x)$ has limited variation in the interval \mathfrak{A} , the points \mathfrak{R} where $\text{Osc } f \geq k$, are finite in number.*

For suppose they were not. Then however large G is taken, we may take n so large that $nk > G$. There exists a division D

of \mathfrak{A} , such that there are at least n intervals, each containing a point of \mathfrak{P} within it. Thus for the division D ,

$$\Sigma \text{Osc } f \geq nk > G.$$

Thus the variation of f is large at pleasure, and therefore is not limited.

2. If f has limited variation in the interval \mathfrak{A} , its points of discontinuity form an enumerable set.

This follows at once from 1.

530. 1. Let $D_1, D_2 \dots$ be a sequence of superposed divisions, of norms $d_n \doteq 0$, of the interval \mathfrak{A} . Let Ω_{D_n} be the sum of the oscillations of f in the intervals of D_n . If $\text{Max}_n \Omega_{D_n}$ is finite, $f(x)$ has limited variation in \mathfrak{A} .

For suppose f does not have limited variation in \mathfrak{A} . Then there exists a sequence of divisions $E_1, E_2 \dots$ such that if Ω_{E_n} is the sum of the oscillations of f in the intervals of E_n , then

$$\Omega_{E_1} < \Omega_{E_2} < \dots \doteq +\infty. \quad (1)$$

Let us take ν so large that no interval of D_ν contains more than one interval of E_n or at most parts of two E_n intervals. Let $F_n = E_n + D_\nu$. Then an interval δ of D_ν is split up into at most two intervals δ', δ'' in F_n . Let $\omega, \omega', \omega''$ denote the oscillation of f in $\delta, \delta', \delta''$. Then the term ω in D_ν goes over into

$$\omega' + \omega'' \leq 2\omega$$

in Ω_{F_n} . Hence if $\text{Max } \Omega_{D_n} = M$,

$$\Omega_{E_n} \leq 2\Omega_{D_\nu} \leq 2M,$$

which contradicts 1).

2. Let $V_{D_n} = \Sigma |f(a_i) - f(a_{i+1})|$, the summation extended over the intervals (a_i, a_{i+1}) of the division D_n . If $\text{Max}_n V_{D_n}$ is finite with respect to a sequence of superposed divisions $\{D_n\}$, we cannot say that f has limited variation.

Example. For let $f(x) = 0$, at the rational points in the interval $\mathfrak{A} = (0, 1)$, and $= 1$, at the irrational points. Let D_n be

obtained by interpolating the points $\frac{2m+1}{2^n}$ in \mathfrak{A} . Then $f=0$ at the end points a_i, a_{i+1} of the intervals of D_n . Hence $V_{D_n}=0$. On the other hand, $f(x)$ has not limited variation in \mathfrak{A} as is obvious.

531. Let $F(x) = \lim_{t \rightarrow \tau} f(x, t)$, τ finite or infinite, for x in the interval \mathfrak{A} . Let $\text{Var } f(x, t) \leq M$ for each t near τ .

Then $F(x)$ has limited variation in \mathfrak{A} .

To fix the ideas let τ be finite. Let

$$F = f(x, t) + g(x, t).$$

Then for a division D of \mathfrak{A} ,

$$V_D F \leq V_D f + V_D g.$$

But

$$V_D g = \sum |g(a_m) - g(a_{m+1})|,$$

where (a_m, a_{m+1}) are the intervals of D .

But for some $t = t'$ near τ , each

$$g(a_i, t') < \frac{\eta}{2s},$$

where s is the number of intervals in the division D .

Thus

$$V_D g < \eta.$$

Hence

$$V_D F < M + \eta,$$

and F has limited variation.

532. Let $f(x), g(x)$ have limited variation in the interval \mathfrak{A} , then their sum, difference, and product have limited variation.

If also $|g| \geq \gamma > 0$, in \mathfrak{A}

then f/g has limited variation.

Let us show, for example, that $h = fg$ has limited variation. For let

$$\begin{aligned} \text{Min } f = m \quad , \quad \text{Min } g = n \\ \text{Osc } f = \omega \quad , \quad \text{Osc } g = \tau \end{aligned} \quad \text{in the interval } \mathfrak{A}.$$

Then

$$f = m + \alpha\omega \quad , \quad g = n + \beta\tau \quad , \quad \text{in } d_n,$$

$$0 < \alpha < 1 \quad , \quad 0 \leq \beta \leq 1.$$

Thus

$$fg = mn + m\beta\tau + n\alpha\omega + \alpha\beta\omega\tau.$$

Now

$$mn - |m|\tau - |n|\omega - \omega\tau < fg \leq mn + |m|\tau + |n|\omega + \omega\tau.$$

Hence

$$\eta = \text{Osc } h \leq 2\{\tau|m| + \omega|n| + \omega\tau\}.$$

But

$$|m|, |n|, \tau \leq \text{some } K.$$

Thus

$$V_D h \leq 4K\Sigma\omega + 2K\Sigma\tau,$$

$$< \text{some } G,$$

and h has limited variation.

533. 1. Let us see what change will be introduced if we replace the finite divisions D employed up to the present by divisions E , which divide the interval $\mathfrak{A} = (a < b)$ into an infinite numerable set of intervals (a_i, a_{i+1}) .

Let

$$W_E = \sum_1^{\infty} |f'(a_m) - f'(a_{m+1})|, \quad (1)$$

and

$$W = \text{Max } V_E,$$

or the class of finite or infinite enumerable divisions $\{E\}$.

Obviously

$$W \geq V;$$

hence if W is finite, so is V .

We show that if V is finite, so is W . For suppose W were finite. Then for any $G > 0$, there exists a division E , and an n such that the sum of the first n terms in 1) is $\geq G$, or

$$W_{E,n} \geq G. \quad (2)$$

Let now D be the finite division determined by the points a_1, \dots, a_{n+1} which figure in 2).

Then

$$V_D \geq G,$$

hence $V = \infty$, which is contrary to our hypothesis.

We show now that V and W are equal, when finite. For let E be so chosen that

$$W - \frac{\epsilon}{2} < W_E \leq W.$$

Now

$$W_E = W_{E,n} + \epsilon' \quad , \quad |\epsilon'| < \epsilon/2$$

if n is sufficiently large.

Let D correspond to the points $a_1 a_2 \dots$ in $W_{E,n}$. Then

$$V_D \geq W_{E,n},$$

and hence

$$V_D + \epsilon' \geq W_{E,n} + \epsilon' = W_E.$$

Hence

$$W - V_D < \epsilon.$$

We may therefore state the theorem :

2. *If f has limited variation in the interval \mathfrak{A} with respect to the class of finite divisions D , it has with respect to the class of enumerable divisions E , and conversely. Moreover*

$$\text{Max } V_D = \text{Max } V_E.$$

534. Let us show that *Weierstrass' function F* , considered in 502, *does not have limited variation in any interval $\mathfrak{A} = (\alpha < \beta)$ when $ab > 1$. Since F is periodic, we may suppose $\alpha > 0$. Let*

$$\frac{k}{b^m}, \frac{k+1}{b^m}, \dots, \frac{k+\mu}{b^m}$$

be the fractions of denominator b^m which lie in \mathfrak{A} .

These points effect a division D_m of \mathfrak{A} , and

$$\begin{aligned} V_{D_m} = & \left| F\left(\frac{k}{b^m}\right) - F(\alpha) \right| + \sum_{j=0}^{\mu-1} \left| F\left(\frac{k+j+1}{b^m}\right) - F\left(\frac{k+j}{b^m}\right) \right| \\ & + \left| F(\beta) - F\left(\frac{k+\mu}{b^m}\right) \right|. \end{aligned}$$

If l is the minimum of the terms F_j under the Σ sign,

$$V_{D_m} \geq \mu l. \tag{1}$$

Now

$$\frac{k-1}{b^m} < \alpha \quad , \quad \frac{k+\mu+1}{b^m} > \beta.$$

Hence

$$\bar{\mathfrak{A}} < \frac{\mu+2}{b^m} \quad , \quad \mu > b^m \bar{\mathfrak{A}} - 2. \tag{2}$$

On the other hand, using the notation and results of 502,

$$b^m x = \iota_m + \xi_m, \quad h = \frac{\eta_m - \xi_m}{b^m};$$

$$\text{also} \quad \left| \frac{F(x+h) - F(x)}{h} \right| \geq a^m b^m \left(\frac{2}{3} - \frac{\pi}{ab-1} \right). \quad (3)$$

Let us now take

$$\xi_m = 0, \quad \eta_m = +1, \quad \iota_m = k + j.$$

$$\text{Then} \quad x = \frac{k+j}{b^m}, \quad h = \frac{1}{b^m}.$$

$$\text{Hence from 3),} \quad F_j \geq a^m \left(\frac{2}{3} - \frac{\pi}{ab-1} \right).$$

$$\text{Thus} \quad V_{D_m} \geq a^m \left(\frac{2}{3} - \frac{\pi}{ab-1} \right) (b^m \bar{\mathfrak{A}} - 2), \quad \text{by 1), 2).}$$

As $a < 1$, and $ab > 1$, we see that

$$V_{D_m} \doteq +\infty, \text{ as } m \doteq \infty.$$

Non-intuitional Curves

35. 1. Let $f(x)$ be continuous in the interval \mathfrak{A} . The graph of f is a continuous curve C . If f has only a finite number of oscillations in \mathfrak{A} , and has a tangent at each point, we would call C an ordinary or intuitional curve. It might even have a finite number of angle points, *i.e.* points where the right-hand tangent is different from the left-hand one [cf. I, 366]. But if there were an infinity of such points, or an infinity of points in the vicinity of each of which f oscillates infinitely often, the curve grows less and less clear to the intuition as these singularities increase in number and complexity. Just where the dividing point lies between curves whose peculiarities can be clearly seen by the intuition, and those which cannot, is hard to say. Probably different persons would set this point at different places.

For example, one might ask: Is it possible for a continuous curve to have tangents at a pantactic set of points, and no tangent at another pantactic set? If one were asked to picture such a curve to the imagination, it would probably prove an impossibility.

Yet such curves exist, as Ex. 3 in 501 shows. Such curves might properly be called *non-intuitional*.

Again we might ask of our intuition: Is it possible for a continuous curve to have a tangent at every point of an interval \mathfrak{A} , which moreover turns abruptly at a pantactic set of points? Again the answer would not be forthcoming. Such curves exist, however, as was shown in Ex. 2 in 501.

We wish now to give other examples of non-intuitional curves. Since their singularity depends on their derivatives or the nature of their oscillations, they may be considered in this chapter.

Let us first show how to define curves, which, like Weierstrass' curve, have a pantactic set of cusps. To effect this we will extend the theorem of 500, 2, so as to allow $g(x)$ to have a cusp at $x = 0$.

536. Let $\mathfrak{E} = \{e_n\}$ denote the rational points in the interval $\mathfrak{A} = (-a, a)$. Let $g(x)$ be continuous in $\mathfrak{B} = (-2a, 2a)$, and $= 0$, at $x = 0$. Let \mathfrak{B}^* denote the interval \mathfrak{B} after removing the point $x = 0$. Let g have a derivative in \mathfrak{B}^* , such that

$$|g'(x)| \leq \frac{M}{|x|^\alpha}, \quad \alpha > 0. \quad (1)$$

Then

$$F(x) = \sum \frac{1}{n^{2+\alpha+\beta}} g(x - e_n) = \sum a_n g(x - e_n), \quad \beta > 0$$

is a continuous function in \mathfrak{A} , and $\frac{\Delta F}{\Delta x}$ behaves at $x = e_m$ essentially as $\frac{\Delta g}{\Delta x}$ does at the origin.*

To simplify matters, let us suppose that \mathfrak{E} does not contain the origin. Having established this case, it is easy to dispose of the general case. We begin by ordering the e_n as in 233. Then obviously if

$$e_n = \frac{p}{q}, \quad q > 0, \quad p \text{ positive or negative,}$$

we have

$$n \geq q.$$

Let

$$\begin{aligned} e_{mn} &= e_m - e_n. & \text{If } e_m = \frac{r}{s}, \\ |e_{mn}| &= \left| \frac{p}{q} - \frac{r}{s} \right| \geq \frac{1}{qs} \geq \frac{1}{mn}. \end{aligned} \quad (2)$$

* Cf. Dini, *Theorie der Functionen*, etc., p. 102 seq. Leipzig, 1802.

Let $E(x)$ be the F series after deleting the m^{th} term. Then

$$F(x) = a_m g(x - e_m) + E(x).$$

We show that E has a differential coefficient at $x = e_m$, obtained by differentiating E termwise. To this end we show that as $h \rightarrow 0$,

$$D(h) = \sum a_n \frac{g(e_{mn} + h) - g(e_{mn})}{h}, \quad m \neq n \quad (3)$$

converges to

$$G = \sum a_n g'(e_{mn}), \quad m \neq n. \quad (4)$$

That is, we show

$$\epsilon > 0, \quad \eta > 0, \quad |D(h) - G| < \epsilon, \quad 0 < |h| < \eta. \quad (5)$$

Let us break up the sums 3), 4) which figure in 5), into three parts

$$\sum_1^\infty = \sum_1^r + \sum_{r+1}^s + \sum_{s+1}^\infty. \quad (6)$$

Thus

$$|D - G| \leq |D_r - G_r| + |D_{r,s} - G_{r,s}| + |\bar{D}_s - \bar{G}_s| \quad (7)$$

$$\leq A + B + C.$$

Since $g'(e_{mn})$ exists, the first term may be made as small as we please for an arbitrary but fixed r ; thus

$$A < \frac{\epsilon}{3}.$$

Let us now turn to B . We have

$$B \leq |D_{rs}| + |G_{rs}|,$$

$$\frac{g(e_{mn} + h) - g(e_{mn})}{h} = g'(e_{mn} + h'), \quad |h'| < |h|$$

provided $g'(x)$ exists in the interval $(e_{mn}, e_{mn} + h)$.

But by 2),

$$|e_{mn} + h'| \geq |e_{mn}| - |h| \geq \frac{1}{2mn} \geq \frac{1}{2ms}, \quad \text{for } r < n \leq s$$

$$\eta < \frac{1}{2ms}. \quad (8)$$

Thus by 1),

$$|g'(e_{mn} + h')| \leq 2^a M m^a n^a < M_1 n^a, \quad M_1 \text{ a constant.}$$

Hence *a fortiori*, $|g'(e_{mn})| < M_1 n^a$. (9)

Now the sum

$$H = \sum \frac{1}{n^{1+\mu}}$$

converges if $\mu > 0$. Hence $H_{p,q}$ and \bar{H}_p may be made as small as we choose, by taking p sufficiently large. Let us note that by 91,

$$\bar{H}_p < \frac{1}{\mu} \frac{1}{p^\mu}. \quad (10)$$

Thus if $\mu = \text{Min}(\alpha, \beta)$,

$$B \leq |D_{rs}| + |G_{rs}| \leq 2 \sum_{r=1}^s \frac{M_1 n^a}{n^{2+\alpha+\beta}} = 2 M_1 H_{r,s} < \frac{\epsilon}{3},$$

for a sufficiently large r .

We consider finally C . We have

$$\begin{aligned} C &\leq |\bar{D}_s| + |\bar{G}_s| \\ &\leq \frac{1}{|h|} \sum_{s+1}^{\infty} a_n |g(e_{mn} + h)| + \frac{1}{|h|} \sum_{s+1}^{\infty} a_n |g(e_{mn})| + |\bar{G}_s| \\ &\leq C_1 + C_2 + C_3. \end{aligned}$$

From 9) we see that

$$C_3 < M_1 \bar{H}_s < \frac{\epsilon}{6},$$

for s sufficiently large. Since $g(x)$ is continuous in \mathfrak{B} ,

$$|g(x)| < N.$$

Hence

$$\begin{aligned} C_1 \text{ and } C_2 &\leq \frac{1}{|h|} \sum_{s+1}^{\infty} \frac{N}{n^{2+\alpha+\beta}} \leq \frac{N}{|h|} \frac{1}{1+\alpha+\beta} \cdot \frac{1}{s^{1+\alpha+\beta}} \\ &\leq \frac{N}{1+\alpha+\beta} \cdot \frac{1}{s^{\alpha+\beta}}, \end{aligned}$$

if $s \geq \frac{1}{|h|}$, on using 10).

Taking s still larger if necessary, we can make

$$C_1, C_2 < \frac{\epsilon}{6}.$$

Thus

$$C < \frac{\epsilon}{3}.$$

The reader now sees why we broke the sum (6) into three parts. $\neq 0$, the middle term contains an increasing number of terms. Whatever given value h has, s has a finite value. As A, B, C are each $< \epsilon/3$, the relation (5) is established. Hence E has a differential coefficient at $x = e_m$, and as

$$\frac{\Delta F}{h} = a_m \frac{\Delta(0)}{h} + \frac{\Delta E}{h},$$

the theorem is established.

Example 1. Let

$$g(x) = \sqrt[3]{x^2}.$$

For $x \neq 0$, $g'(x) = \frac{2}{3} \frac{1}{\sqrt[3]{x}}$. Here $\alpha = \frac{1}{3}$.

$$x = 0, \quad Rg'(x) = +\infty, \quad Lg'(x) = -\infty.$$

$$F(x) = \sum \frac{\sqrt[3]{(x - e_n)^2}}{n^{\frac{1}{3} + \beta}}, \quad \beta > 0$$

is a continuous function, and at the rational points e_m in the interval \mathfrak{A} ,

$$RF'(x) = +\infty, \quad LF'(x) = -\infty.$$

Since the graph of F has a punctate set of cuspidal tangents, the curve is not monotone in any interval of \mathfrak{A} , however

Example 2. Let

$$g(x) = x \sin \frac{1}{x}, \quad x \neq 0$$

$$= 0, \quad x = 0.$$

$$g'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}, \quad x \neq 0.$$

$\alpha = 1$. For $x = 0$,

$$\overline{g}'(x) = +1, \quad \underline{g}'(x) = -1.$$

Then

$$F(x) = \sum \frac{1}{n^{3+\beta}} (x - e_n) \sin \frac{1}{x - e_n}, \quad \beta > 0$$

is a continuous function in \mathfrak{A} , and at the rational point e_m ,

$$\overline{F}'(x) = \frac{1}{n^{3+\beta}} + E'(e_m)$$

$$\underline{F}'(x) = -\frac{1}{n^{3+\beta}} + E'(e_m),$$

where E is the series obtained from F by deleting the m^{th} term.

538. Pompeiu Curves.* Let us now show the existence of curves which have a tangent at each point, and a pantactic set of vertical inflectional tangents.

We first prove the theorem (*Borel*):

$$\text{Let} \quad B(x) = \sum \frac{a_n}{x - e_n} = \sum \frac{a_n}{r_n}, \quad a_n > 0,$$

where $\mathfrak{E} = \{e_n\}$ is an enumerable set in the interval \mathfrak{A} , and

$$A = \sum \sqrt{a_n}$$

is convergent. Then B converges absolutely and uniformly in a set $\mathfrak{B} < \mathfrak{A}$, and \mathfrak{B} is as near \mathfrak{A} as we choose.

The points \mathfrak{D} where adjoint B is divergent form a null set.

For let us enclose each point e_n in an interval δ_n of length $\frac{2\sqrt{a_n}}{k}$, with e_n as center.

The sum of these intervals is

$$\leq \sum \frac{2\sqrt{a_n}}{k} = \frac{2A}{k} < \epsilon,$$

for $k > 0$ sufficiently large. Let now k be fixed. A point x of \mathfrak{A} will not lie in any δ_n if

$$r_n = |x - e_n| > \frac{\sqrt{a_n}}{k}.$$

Then at such a point,

$$\text{Adjoint } B < \sum a_n \frac{k}{\sqrt{a_n}} = k \sum \sqrt{a_n} = kA.$$

* *Math. Annalen*, v. 63 (1907), p. 326.

$\mathfrak{B} > \mathfrak{H} - \epsilon$, the points \mathfrak{D} where B does not converge ab-
solutely form a null set.

1. We now consider the function

$$F(x) = \sum_1^{\infty} a_n(x - e_n)^{\frac{1}{3}} = \sum f_n(x) \quad (1)$$

$\mathfrak{E} = \{e_n\}$ is an enumerable pantactic set in an interval \mathfrak{A} , and

$$A = \sum a_n \quad (2)$$

convergent positive term series.

Let F be a continuous function of x in \mathfrak{A} . For $|x - e_n|^{\frac{1}{3}}$ is $<$
 M in \mathfrak{A} .

Let us note that each $f_n(x)$ is an increasing function and the
corresponding to it has a vertical inflectional tangent at the
 $x = e_n$.

Next show that $F(x)$ is an increasing function in \mathfrak{A} . For let
 $x' < x''$. Then

$$f_n(x') < f_n(x'').$$

$$F_n(x') < F_n(x'').$$

$$F'_n(x') \leq F'_n(x'').$$

$$F(x') < F(x'').$$

Let us now consider the convergence of

$$D(x) = \frac{1}{3} \sum \frac{a_n}{(x - e_n)^{\frac{2}{3}}} \quad (3)$$

obtained by differentiating F termwise at the points of $\mathfrak{A} - \mathfrak{E}$.
Let \mathfrak{D} denote the points in \mathfrak{A} where

$$B = \sum \left| \frac{a_n}{x - e_n} \right| \quad (4)$$

converges. We have seen \mathfrak{D} is a null set if

$$\sum \sqrt{a_n} < \infty \quad (5)$$

is convergent. Let $\mathfrak{A} = \mathfrak{D} + \mathfrak{E}$. Let x be a point of \mathfrak{E} , i.e. a point where 4) is convergent. We break 3) into two parts

$$D = D_1 + D_2,$$

such that in D_1 , each $\xi_n < 1$. Then D_2 is obviously convergent, since each of its terms

$$\frac{a_n}{\xi_n^{\frac{2}{3}}} \leq a_n, \quad \text{where } \xi_n = |x - e_n|,$$

and the series 2) is convergent.

The series D_1 is also convergent. For as $\xi_n < 1$, the term

$$\frac{a_n}{\xi_n^{\frac{2}{3}}} < \frac{a_n}{\xi_n}$$

and the series 4) converges by hypothesis, at a point x in \mathfrak{E} . Hence $D(x)$ is convergent at any point in \mathfrak{E} , and $\mathfrak{E} = \mathfrak{A}$ when 5) is convergent.

3. Let C denote the points in \mathfrak{A} where 3) converges. Let $\mathfrak{A} = C + \Delta$.

We next show that $F'(x) = D(x)$, for x in C . For taking x at pleasure in C but fixed,

$$Q(h) = \frac{\Delta F}{\Delta x} = \sum a_n \frac{(x+h-e_n)^{\frac{1}{3}} - (x-e_n)^{\frac{1}{3}}}{h}, \quad \Delta x = h. \quad (6)$$

We now apply 156, 2, showing that Q is uniformly convergent in $(0^*, \eta)$. By direct multiplication we find that

$$\frac{(a+b)^{\frac{1}{3}} - a^{\frac{1}{3}}}{b} = \frac{1}{(a+b)^{\frac{2}{3}} + a^{\frac{1}{3}}(a+b)^{\frac{1}{3}} + a^{\frac{2}{3}}}.$$

Thus 6) gives

$$Q(h) = \sum \frac{a_n}{(x+h-e_n)^{\frac{2}{3}} + (x+h-e_n)^{\frac{1}{3}}(x-e_n)^{\frac{1}{3}} + (x-e_n)^{\frac{2}{3}}}.$$

Let us set

$$h_n = \left(\frac{x+h-e_n}{x-e_n} \right)^{\frac{1}{3}}.$$

Then

$$Q(h) = \sum \frac{1}{1+h_n+h_n^2} \cdot \frac{a_n}{(x-e_n)^{\frac{2}{3}}} \leq 2 \sum \frac{a_n}{\xi_n^{\frac{2}{3}}}, \quad (7)$$

for $0 < |h| \leq \eta$, η sufficiently small. As the series on the right is independent of h , Q converges uniformly in $(0^*, \eta)$. Thus by 156, ²

$$F' = D, \quad \text{for any } x \text{ in } C.$$

4. Let now x be a point of Δ , not in \mathfrak{E} . At such a point we show that

$$F'(x) = +\infty, \quad (8)$$

and thus the curve F has a vertical inflectional tangent. For as D is divergent at x , there exists for each M an m , such that

$$D_m > 2M.$$

But the middle term in 7) shows that for $|h| < \text{some } \eta'$ each term in Q_m is $> \frac{1}{2}$ the corresponding term in D_m . Thus

$$Q_m(h) > M, \quad 0 < |h| < \eta'.$$

Since each term of Q is > 0 , as 7) shows,

$$Q(h) > M.$$

Hence 8) is established.

5. Let us finally consider the points $x = e_m$. If Φ denotes the series obtained from F by deleting the m^{th} term, we have

$$\frac{\Delta F}{\Delta x} = \frac{\alpha_m}{h^{\frac{2}{3}}} + \frac{\Delta \Phi}{\Delta x}, \quad \text{for } x = e_m.$$

As F is increasing, the last term is ≥ 0 .

Hence

$$F'(x) = +\infty, \quad \text{in } \mathfrak{E}.$$

As a result we see the curve F has at each point a tangent. At an enumerable pantactic set V , it has points of inflection with vertical tangents.

7. Let us now consider the inverse of the function F , which we denote by

$$x = G(t). \quad (9)$$

As x in 1) ranges over the interval \mathfrak{A} , $t = F(x)$ will range over an interval \mathfrak{B} , and by I, 381, the inverse function 9) is a one-valued continuous function of t in \mathfrak{B} which has a tangent at each

point of \mathfrak{B} . If W are the points in \mathfrak{B} which correspond to the points V in \mathfrak{A} , then the tangent is parallel to the t -axis at the points W , or $G'(t) = 0$, at these points. The points W are pantactic in \mathfrak{B} .

Let Z denote the points of \mathfrak{B} at which $G'(t) = 0$. We show that Z is of the 2° category, and therefore

$$\text{Card } Z = c.$$

For $G'(t)$ being of class ≤ 1 in \mathfrak{B} , its points of discontinuity Z form a set of the 1° category, by 486, 2. On the other hand, the points of continuity of G' form precisely the set Z , since the points W are pantactic in \mathfrak{B} and $G' = 0$ in W . In passing let us note that the points Z in \mathfrak{B} correspond 1-1 to a set of points \mathfrak{B} at which the series 3) diverges. For at these points the tangent to F is vertical. But at any point of convergence of 3), we saw in 2 that the tangent is not vertical.

Finally we observe that 3) shows that

$$\text{Min } D(x) > \frac{1}{3} \cdot \frac{1}{\mathfrak{M}^{\frac{1}{3}}} \sum a_n, \quad \text{in } \mathfrak{A}.$$

Hence

$$\text{Max } G'(t) \leq \frac{3 \mathfrak{M}^{\frac{1}{3}}}{\sum a_n}.$$

Summing up, we have this result :

8. Let the positive term series $\sum \sqrt{a_n}$ converge. Let $\mathfrak{E} = \{e_n\}$ be an enumerable pantactic set in the interval \mathfrak{A} . The Pompeiu curve defined by

$$F(x) = \sum a_n (x - c_n)^{\frac{1}{3}}$$

have a tangent at each point in \mathfrak{A} , whose slope is given by

$$F'(x) = \frac{1}{3} \sum \frac{a_n}{(x - e_n)^{\frac{2}{3}}},$$

when this series is convergent, i.e. for all x in \mathfrak{A} except a null set. At a point set \mathfrak{B} of the 2° category which embraces \mathfrak{E} , the tangents are vertical. The ordinates of the curve F increase with x .

540. 1. *Faber Curves*.* Let $F(x)$ be continuous in the interval $\mathfrak{A} = (0, 1)$. Its graph we denote by \mathcal{F} . For simplicity let

* *Math. Annalen*, v. 66 (1908), p. 81.

$f(0) = 0$, $F(1) = l_0$. We proceed to construct a sequence of broken lines or polygons,

$$L_0, L_1, L_2 \dots \quad (1)$$

which converge to the curve F as follows:

As first line L_0 we take the segment joining the end points of

Let us now divide \mathfrak{A} into n_1 equal intervals

$$\delta_{11}, \delta_{12} \dots \delta_{1, n_1} \quad (2)$$

length

$$\delta_1 = \frac{1}{n_1},$$

and having

$$a_{11}, a_{12}, a_{13} \dots \quad (3)$$

end points. As second line L_1 we take the broken line or polygon joining the points on F whose abscissae are the points 3).

We now divide each of the intervals 2) into n_2 equal intervals, obtaining the $n_1 n_2$ intervals

$$\delta_{21}, \delta_{22}, \delta_{23} \dots \quad (4)$$

length

$$\delta_2 = \frac{1}{n_1 n_2},$$

and having

$$a_{21}, a_{22}, a_{23} \dots \quad (5)$$

end points. In this way we proceed on indefinitely. Let us call the points

$$A = \{a_{mn}\}$$

terminal points. The number of intervals in the r^{th} division is

$$\nu_r = n_1 \cdot n_2 \dots n_r.$$

If $L_m(x)$ denote the one-valued continuous function in \mathfrak{A} whose value is the ordinate of a point on L_m , we have

$$F(a_{mn}) = L_m(a_{mn}), \quad (6)$$

since the vertices of L_m lie on the curve F .

2. For each x in \mathfrak{A} ,

$$\lim_{m \rightarrow \infty} L_m(x) = F(x). \quad (7)$$

For if x is a terminal point, 7) is true by 6).

If x is not a terminal point, it lies in a sequence of intervals

$$\delta_1 > \delta_2 > \dots$$

belonging to the $1^\circ, 2^\circ \dots$ division of \mathfrak{A} .

Let

$$\delta_m = (a_{m,n}, a_{m,n+1}).$$

Since $F(x)$ is continuous, there exists an s , such that

$$|F(x) - F(a_{m,n})| \leq \frac{\epsilon}{2}, \quad m > s \quad (8)$$

for any x in δ_m . As $L_m(x)$ is monotone in δ_m ,

$$\begin{aligned} |L_m(x) - L_m(a_{mn})| &\leq |L_m(a_{mn}) - L_m(a_{m,n+1})| \\ &\leq |F_m(a_{mn}) - F_m(a_{m,n+1})| \\ &\leq \frac{\epsilon}{2}, \quad \text{by 8).} \end{aligned}$$

Thus

$$|L_m(x) - F_m(a_{mn})| \leq \frac{\epsilon}{2}. \quad (9)$$

Hence from 8), 9),

$$|F(x) - L_m(x)| < \epsilon, \quad m \geq s$$

which is 7).

3. We can write 7) as a telescopic series. For

$$L_1 = L_0 + (L_1 - L_0)$$

$$L_2 = L_1 + (L_2 - L_1) = L_0 + (L_1 - L_0) + (L_2 - L_1)$$

etc. Hence

$$F(x) = \lim L_n(x) = L_0(x) + \sum_1^{\infty} \{L_n(x) - L_{n-1}(x)\}.$$

If we set

$$f_0(x) = L_0(x), \quad f_n(x) = L_n(x) - L_{n-1}(x), \quad (10)$$

we have

$$F(x) = \sum_0^{\infty} f_n(x), \quad (11)$$

and

$$F_n(x) = \sum_0^n f_s(x) = L_n(x). \quad (12)$$

The function $f_n(x)$, as 10) shows, is the difference between the ordinates of two successive polygons L_{n-1} , L_n at the point x . It may be positive or negative. In any case its graph is a polygon

which has a vertex on the x -axis at the end point of each interval δ_{n-1} . Let l_{ns} be the value of $f_n(x)$ at the point $x = a_{ns}$, that is, at a point corresponding to one of the vertices of f_n . We call l_{ns} the *vertex differences* of the polygon L_n .

Let

$$p_n = \min_s |l_{ns}|, \quad q_n = \max_s |l_{ns}|.$$

Then

$$|f_n(x)| \leq q_n, \quad \text{in } \mathfrak{A}. \quad (13)$$

In the foregoing we have supposed $F(x)$ given. Obviously if the vertex differences were given, the polygons 1) could be constructed successively.

We now show:

If

$$\sum q_n \quad (14)$$

is convergent,

$$F(x) = \sum f_n(x)$$

is uniformly convergent in \mathfrak{A} , and is a continuous function in \mathfrak{A} .

For by 13), 14), F converges uniformly in \mathfrak{A} . As each $f_n(x)$ is continuous, F is continuous in \mathfrak{A} .

The functions so defined may be called *Fubler functions*.

541. 1. We now investigate the *derivatives of Fubler's functions*, and begin by proving the theorem:

If

$$\sum_s n_1 \cdots n_s q_s = \sum \nu_s q_s \quad (1)$$

converge, the unilateral derivatives of $F(x)$ exist in $\mathfrak{A} = (0, 1)$. Moreover they are equal, except possibly at the terminal points $A = \{a_{mn}\}$.

For let x be a point not in A . Let x', x'' lie in $V = V_\eta^*(x)$; let $x' - x = h'$, $x'' - x = h''$.

Let

$$Q = \frac{F(x') - F(x)}{h'} - \frac{F(x'') - F(x)}{h''}.$$

Then $F'(x)$ exists at x , if

$$\epsilon > 0, \quad \eta > 0, \quad |Q| < \epsilon, \quad \text{for any } x', x'' \text{ in } V. \quad (2)$$

Now

$$|Q| \leq \left| \frac{F_m(x') - F_m(x)}{h'} - \frac{F_m(x'') - F_m(x)}{h''} \right| + \left| \frac{\bar{F}_m(x') - \bar{F}_m(x)}{h'} \right| + \left| \frac{\bar{F}_m(x'') - \bar{F}_m(x)}{h''} \right|$$

$$\leq Q_1 + Q_2 + Q_3.$$

But

$$\left| \frac{f_s(x') - f_s(x)}{x' - x} \right| \leq \frac{2q_s}{\delta_s}.$$

Hence

$$Q_2 \leq 2 \sum_{s=m+1}^{\infty} q_s < \frac{\epsilon}{2}, \quad m \text{ sufficiently large.}$$

Similarly

$$Q_3 < \frac{\epsilon}{2}.$$

Finally, if η is taken sufficiently small, x, x', x'' will correspond to the side of the polygon L_m . Hence using 540, 12), we see that $Q_1 = 0$. Thus 2) holds, and $F'(x)$ exists at x .

If x is a terminal point a_{mn} , and the two points x', x'' are taken on the same side of a_{mn} , the same reasoning shows that the unilateral derivatives exist at a_{mn} . They may, however, be different.

2. Let $n_1 = n_2 = \dots = 2$. For the differential coefficient $F'(x)$ to exist at the terminal point x , it is necessary that

$$\overline{\lim} 2^n q_n < \infty. \quad (3)$$

If

$$\overline{\lim} 2^n p_n = \infty, \quad (4)$$

the points where the differential coefficient does not exist form a pantactic set in \mathfrak{A} .

Let us first prove 3). Let $b < a < c$ be terminal points. Then they belong to every division after a certain stage. We will therefore suppose that b, c are consecutive points in the n^{th} division, and a is a point of the $n+1^{\text{st}}$ division falling in the interval $\delta_n = (b, c)$. If a differential coefficient is to exist at a ,

$$\frac{F(a) - F(b)}{a - b} \text{ and } \frac{F(a) - F(c)}{a - c} \quad (5)$$

must be numerically less than some M , as $n \rightarrow \infty$, and hence their sum Q remains numerically $< 2M$.

Now

$$F(a) = L_{n+1}(a) \quad , \quad F(b) = L_n(b) \quad , \quad F(c) = L_n(c),$$

$$|a - b| = |a - c| = \delta_n = \frac{1}{2^{n+1}}.$$

Thus
$$Q = 2^{n+1} \{ 2 L_{n+1}(a) - [L_n(b) + L_n(c)] \}$$

$$= 2 \cdot 2^{n+1} \left\{ L_{n+1}(a) - \frac{L_n(b) + L_n(c)}{2} \right\},$$

$$|Q| = 4 \cdot 2^n l_{n,s} \quad , \quad \text{supposing } a = a_{ns}.$$

Hence
$$2^n q_n < M,$$

which establishes 3).

Let us now consider 4). By hypothesis there exists a sequence $1 < n_2 < \dots \doteq \infty$, such that

$$2^{n_m} p_{n_m} > G \quad , \quad m = 1, 2, \dots,$$

n_m being large at pleasure. Hence at least one of the difference quotients 5) belonging to this sequence of divisions is numerically large at pleasure.

3. If
$$\lambda = \sum l_{ms} \tag{1}$$

absolutely convergent, the functions $F(x)$ have limited variation in

For $f_m(x)$ is monotone in each interval δ_{ms} . Hence in δ_{ms} ,

$$\text{Var } f_m = |l_{ms} - l_{m, s+1}| \leq |l_{ms}| + |l_{m, s+1}|.$$

Hence in \mathfrak{A} ,
$$\text{Var } f_m(x) \leq 2 \sum_s l_{ms}.$$

Hence
$$\text{Var } F_n(x) \leq 2 \sum_{m=1}^n \sum_s l_{ms} = 2 \lambda \quad , \quad \text{in } \mathfrak{A}.$$

We apply now 531.

542. Faber Functions without Finite or Infinite Derivatives.

To simplify matters let us consider the following example. The method employed admits easy generalization and gives a class of functions of this type. We use the notation of the preceding sections.

Let $f_0(x)$ have as graph Fig. 1. We next divide $\mathcal{H} = (0, 1)$ into 2^{11} equal parts δ_{11}, δ_{12} and take $f_1(x)$ as in Fig. 2. We now divide \mathcal{H} into 2^{21} equal parts $\delta_{21}, \delta_{22}, \delta_{23}, \delta_{24}$ and take $f_2(x)$ as in Fig. 3. The height of the peaks is $l_2 = \frac{l}{10^2}$.

In the m^{th} division \mathcal{H} falls into 2^{m1} equal parts

$$\delta_{m1}, \delta_{m2} \dots$$

one of which may be denoted by

$$\delta_m = (\alpha_m, \alpha_{m, n+1}).$$

Its length may be denoted by the same letter, thus

$$\delta_m = \frac{1}{2^{m1}}.$$

In Fig. 4, δ_m is an interval of the $m-1^{\text{st}}$ division.

The maximum ordinate of $f_m(x)$ is $l_m = \frac{l}{10^m} = \frac{1}{2} \cdot \frac{1}{10^m}$. The part of the curve whose points have an ordinate $\leq \frac{1}{2} l_m$ have been marked more heavily. The x of such points, form class 1. The other x 's make up class 2. With each x in class 1, we associate the points $\alpha_m < \beta_m$ corresponding to the peaks of f_m adjacent to x . Thus $\alpha_m < x < \beta_m$. If x is in class 2, the points α_m, β_m are the adjacent valley points, where $f_m = 0$.

Let now x be a point of class 1. The numerators in

$$\frac{f_m(\beta_m) - f(x)}{\beta_m - x} \quad \frac{f_m(\alpha_m) - f(x)}{\alpha_m - x} \quad (1)$$

have like signs, while their denominators are of opposite sign. Thus the signs of the quotients 1) are different. Similarly if x belongs to class 2, the signs of 1) are opposite. Hence for any x ,

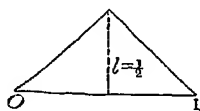


FIG. 1

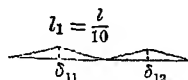


FIG. 2

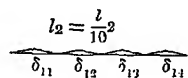


FIG. 3

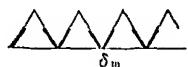


FIG. 4

the signs of 1) are opposite. It will be convenient to let e_m denote either α_m or β_m . We have

$$|x - e_m| < \delta_m,$$

$$|f_m(x) - f_m(e_m)| \geq \frac{1}{2} l_m. \quad (2)$$

Hence

$$\left| \frac{f_m(x) - f_m(e_m)}{x - e_m} \right| > \frac{1}{4} \frac{2^{m1}}{10^m}. \quad (3)$$

On the other hand, for any $x \neq x'$ in δ_m ,

$$\left| \frac{f_m(x') - f_m(x)}{x' - x} \right| < \frac{2 l_m}{\delta_m}.$$

Hence setting $x' = e_n$, and letting $n > m$,

$$\begin{aligned} |f_m(e_n) - f_m(x)| &\leq \frac{l_m}{\delta_m} |e_n - x| < l_m \cdot \frac{\delta_n}{\delta_m} \\ &\leq \frac{1}{10^m} \cdot \frac{2^{m1}}{2^{n1}} < \frac{1}{10^m} \cdot \frac{2^{n-11}}{2^{n1}} \\ &< \frac{1}{10^n} \cdot \frac{1}{10^m}. \end{aligned} \quad (4)$$

For if $\log_2 a$ be the logarithm of a with the base 2,

$$n - 1! > \frac{n}{n-1} \log_2 10, \quad \text{for } n \text{ sufficiently large.}$$

Hence
$$(n-1)!(n-1) > \log_2 10^n.$$

Thus
$$\frac{2^{n1}}{2^{(n-1)1}} > 10^n, \quad \text{or} \quad \frac{2^{n-11}}{2^{n1}} < \frac{1}{10^n},$$

and this establishes 4).

Let us now extend the definition of the functions $f_n(x)$ by giving them the period 1. The corresponding Faber function $F(x)$ (defined by 540, 12) will admit 1 as period. We have now

$$\begin{aligned} F(e_n) - F(x) &= \{f_n(e_n) - f_n(x)\} + \{F_{n-1}(e_n) - F_{n-1}(x)\} \\ &\quad + \{\bar{F}_{n+1}(e_n) - \bar{F}_{n+1}(x)\} = T_1 + T_2 + T_3. \end{aligned}$$

From 2) we have

$$T_1 \geq \frac{1}{2} l_n.$$

As to \dot{T}_2 , we have, using 4) and taking n sufficiently large,

$$|T_2| < \frac{1}{10^n} \sum_{m=1}^{n-1} \frac{1}{10^m} < \frac{1}{10^n} \sum_1^{\infty} \frac{1}{10^m} = \frac{1}{9} \cdot \frac{1}{10^n} \\ < \frac{2}{9} l_n.$$

Similarly

$$|T_3| \leq \sum_{m=n+1}^{\infty} |f_m(e_n) - f_m(x)| \leq \sum_{m=n+1}^{\infty} \{f_m(e_n) + f_m(x)\} \\ \leq \sum_{n+1}^{\infty} 2 l_m = \frac{1}{9} \cdot \frac{1}{10^n} \\ \leq \frac{2}{9} l_n.$$

Thus finally

$$|F(e_n) - F(x)| > l_n \left(\frac{1}{2} - \frac{2}{9} - \frac{2}{9} \right) \\ > \frac{1}{18} l_n.$$

As

$$|T_1| > |T_2| + |T_3|$$

$$\operatorname{sgn} \frac{F(e_n) - F(x)}{e_n - x} = \operatorname{sgn} \frac{f_n(e_n) - f_n(x)}{e_n - x}.$$

Thus

$$\left| \frac{F(e_n) - F(x)}{e_n - x} \right| > \frac{1}{18} \frac{l_n}{\delta_n} \\ > \frac{1}{36} \frac{2^{n!}}{10^n} = \infty.$$

As e_n may be at pleasure α_n or β_n , and as the signs of 1) are opposite, we see that

$$\overline{F}'(x) = +\infty, \quad \underline{F}'(x) = -\infty;$$

and $F(x)$ has neither a finite nor an infinite differential coefficient at any point.

CHAPTER XVI

SUB- AND INFRA-UNIFORM CONVERGENCE

Continuity

543. In many places in the preceding pages we have seen how important the notion of uniform convergence is when dealing with iterated limits. We wish in this chapter to treat a kind of uniform convergence first introduced by *Arzelà*, and which we will call *subuniform*. By its aid we shall be able to give conditions for integrating and differentiating series termwise much more general than those in Chapter V.

We refer the reader to *Arzelà's* two papers, "*Sulle Serie di funzioni*," *R. Accad. di Bologna*, ser. V, vol. 8 (1899). Also to a fundamental paper by *Osgood*, *Am. Journ. of Math.*, vol. 19 (1897), and to another by *Hobson*, *Proc. Lond. Math. Soc.*, ser. 2, vol. 1 (1904).

544. 1. Let $f(x_1 \dots x_m, t_1 \dots t_n) = f(x, t)$ be a function of two sets of variables. Let $x = (x_1 \dots x_m)$ range over \mathfrak{X} in an m -way space, and $t = (t_1 \dots t_n)$ range over \mathfrak{T} in an n -way space. As x ranges over \mathfrak{X} and t over \mathfrak{T} , the point $(x_1 \dots t_1 \dots) = (x, t)$ will range over a set \mathfrak{U} lying in a space \mathfrak{N}_p , $p = m + n$.

Let τ , finite or infinite, be a limiting point of \mathfrak{T} .

Let
$$\lim_{t=\tau} f(x_1 \dots x_m, t_1 \dots t_n) = \phi(x_1 \dots x_m) \quad \text{in } \mathfrak{X}.$$

Let the point x range over $\mathfrak{B} \leq \mathfrak{X}$, while t remains fixed, then the point (x, t) will range over a *layer of ordinate t* , which we will denote by \mathfrak{L}_t . We say x belongs to or is associated with this layer.

We say now that $f \doteq \phi$, *subuniformly* in \mathfrak{X} when for each $\epsilon > 0$, > 0 :

1° There exists a finite number of layers \mathfrak{L}_i whose ordinates t lie in $V_\eta^*(\tau)$.

2° Each point x of \mathfrak{X} is associated with one or more of these layers. Moreover if $x = a$ belongs to the layer \mathfrak{L}_i , all the points x in some $V_\delta(a)$ also belong to \mathfrak{L}_i .

$$3^\circ \quad |f(x, t) - \phi(x)| < \epsilon$$

while (x, t) ranges over *any* one of the layers \mathfrak{L}_i . When $m=1$, that is when there is but a single variable x which ranges over an interval, the layers reduce to segments. For this reason Arzelà calls the convergence *uniform in segments*.

2. In case that subuniform convergence is applied to the series

$$F(x_1 \dots x_m) = \sum f_n(x_1 \dots x_m)$$

convergent in \mathfrak{X} , we may state the definition as follows:

F converges subuniformly in \mathfrak{X} when

1° For each $\epsilon > 0$, and for each ν there exists a finite set of layers of ordinates $\geq \nu$, call them

$$\mathfrak{L}_1, \mathfrak{L}_2 \dots \quad (2)$$

such that each point x of \mathfrak{X} belongs to one or more of them, and if $x = a$ belongs to \mathfrak{L}_m , then all the points of \mathfrak{X} near a also belong to \mathfrak{L}_m .

$$2^\circ \quad |\bar{F}_n(x_1 \dots x_m)| < \epsilon$$

as the point (x, n) ranges over any one of the layers 2).

545. *Example.* Let

$$F(x) = \sum_1^\infty \left\{ \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2} \right\} \quad \text{in } \mathfrak{X} = (-1, 1).$$

Here

$$F_n(x) = \frac{nx}{1+n^2x^2}, \quad F(x) = 0.$$

The series converges uniformly in \mathfrak{X} , except at $x=0$. The convergence is therefore not uniform in \mathfrak{X} ; it is, however, subuniform. For

$$|\bar{F}_n(x)| = \frac{n|x|}{1+n^2x^2}.$$

Hence taking m at pleasure and fixed,

$$|\bar{F}_m| < \epsilon, \quad x \text{ in } s_1 = (-\delta, \delta),$$

sufficiently small. On the other hand,

$$|\bar{F}_n| \leq \frac{n}{1 + \frac{1}{4}n^2\delta^2} \quad \text{in } (-1, -\frac{1}{2}\delta) + (\frac{1}{2}\delta, 1) = s_2 + s_3.$$

Thus for n sufficiently large,

$$|\bar{F}_n| < \epsilon.$$

Hence we need only three segments s_1, s_2, s_3 to get subuniform convergence.

546. 1. Let $f(x_1 \cdots x_m, t_1 \cdots t_n) \doteq \phi(x_1 \cdots x_m)$ in \mathfrak{X} , as $t \doteq \tau$, finite or infinite. Let $f(x, t)$ be continuous in \mathfrak{X} for each t near τ . For ϕ to be continuous at the point $x = a$ in \mathfrak{X} , it is necessary that for each $\epsilon > 0$, there exists an $\eta > 0$, and a d_t for each t in $V_\eta^*(\tau)$ such that

$$|f(x, t) - \phi(x)| < \epsilon \quad (1)$$

for each t in V_η and for any x in $V_{d_t}(a)$.

It is sufficient if there exists a single $t = \beta$ in $V_\eta^*(\tau)$ for which the inequality 1) holds for any x in some $V_\delta(a)$.

It is necessary. For since ϕ is continuous at $x = a$,

$$|\phi(x) - \phi(a)| < \frac{\epsilon}{3}, \quad \text{for any } x \text{ in some } V_\delta(a).$$

Also since $f \doteq \phi$,

$$|f(a, t) - \phi(a)| < \frac{\epsilon}{3}, \quad \text{for any } t \text{ in some } V_\eta^*(\tau).$$

Finally, since f is continuous in x for any t near τ ,

$$|f(x, t) - f(a, t)| < \frac{\epsilon}{3}, \quad \text{for any } x \text{ in some } V_{d_t}(a).$$

Adding these three inequalities we get 1), on taking

$$d_t < \delta, d_t.$$

It is sufficient. For by hypothesis

$$|f(x, \beta) - \phi(x)| > \frac{\epsilon}{3}, \quad \text{for any } x \text{ in some } V_{\delta'}(a);$$

and hence in particular.

$$|f(a, \beta) - \phi(a)| < \frac{\epsilon}{3}.$$

Also since $f(x, \beta)$ is continuous in x ,

$$|f(x, \beta) - f(a, \beta)| < \frac{\epsilon}{3}, \quad \text{for any } x \text{ in some } V_{\delta''}(a).$$

Thus if $\delta < \delta', \delta''$, these inequalities hold simultaneously. Adding them we get

$$|\phi(x) - \phi(a)| < \epsilon, \quad \text{for any } x \text{ in } V_{\delta}(a),$$

and thus ϕ is continuous at $x = a$.

2. As a corollary we get:

$$\text{Let} \quad F(x) = \sum f_{i_1 \dots i_n}(x_{i_1} \dots x_{i_n})$$

converge in \mathfrak{A} , each term being continuous in \mathfrak{A} . For $F(x)$ to be continuous at the point $x = a$ in \mathfrak{A} , it is necessary that for each $\epsilon > 0$, and for any cell $R_{\mu} > \text{some } R_{\lambda}$, there exists a δ_{μ} such that

$$|\bar{F}_{\mu}(x)| < \epsilon, \quad \text{for any } x \text{ in } V_{\delta_{\mu}}(a).$$

It is sufficient if there exists an R_{λ} and a $\delta > 0$ such that

$$|\bar{F}_{\lambda}(x)| < \epsilon, \quad \text{for any } x \text{ in } V_{\delta}(a).$$

547. 1. Let $\lim_{x \rightarrow \tau} f(x_1 \dots x_m, t_1 \dots t_n) = \phi(x_1 \dots x_m)$ in \mathfrak{X} , τ finite or infinite. Let $f(x, t)$ be continuous in \mathfrak{X} for each t near τ .

1° If $f \doteq \phi$ subuniformly in \mathfrak{X} , ϕ is continuous in \mathfrak{X} .

2° If \mathfrak{X} is complete, and ϕ is continuous in \mathfrak{X} , $f \doteq \phi$ subuniformly in \mathfrak{X} .

To prove 1°. Let $x = a$ be a point of \mathfrak{X} . Let $\epsilon > 0$ be taken at pleasure and fixed. Then there is a layer \mathfrak{E}_{β} to which the point a belongs and such that

$$|f(x, t) - \phi(x)| < \epsilon, \tag{1}$$

then (x, t) ranges over the points of \mathfrak{R}_β . But then 1) holds for $x = \beta$ and x in some $V_\delta(a)$. Thus the condition of 546, 1 is satisfied.

To prove 2°. Since ϕ is continuous at $x = a$, the relation 1) holds by 546, 1, for each t in $V_\eta^*(\tau)$ and for any x in $V_{\delta_t}(a)$. With the point a let us associate a cube $C_{a,t}$ lying in $D_{\delta_t}(a)$ and having a as center. Then each point of \mathfrak{X} lies within a cube. Hence by Borel's theorem there exists a finite number of these cubes C_i such that each point of \mathfrak{X} lies within one of them, say

$$C_{a,t_1}, \quad C_{a,t_2} \dots \quad (2)$$

but the cubes 2) determine a set of layers

$$\mathfrak{X}_{t_1}, \quad \mathfrak{X}_{t_2} \dots \quad (3)$$

such that 1) holds as (x, t) ranges over the points of \mathfrak{X} in each layer of 3). Thus the convergence of f to ϕ is subuniform in \mathfrak{X} .

2. As a corollary we have the theorem:

Let
$$F(x_1 \dots x_m) = \Sigma f_{i_1, \dots, i_n}(x_1 \dots x_m)$$

converge in \mathfrak{X} , each f_i being continuous in \mathfrak{X} . If F converges subuniformly in \mathfrak{X} , F is continuous in \mathfrak{X} . If \mathfrak{X} is complete and F is continuous in \mathfrak{X} , F converges subuniformly in \mathfrak{X} .

548. 1. Let
$$F(x) = \Sigma f_{i_1, \dots, i_n}(x_1 \dots x_m)$$

converge in \mathfrak{X} .

Let the convergence be uniform in \mathfrak{X} except possibly for the points of a complete discrete set $\mathfrak{B} = \{b\}$. For each b , let there exist a λ_0 such that for any $\lambda \geq \lambda_0$,

$$\lim_{x \rightarrow b} \bar{F}_\lambda(x) = 0.$$

Then F converges subuniformly in \mathfrak{X} .

For let D be a cubical division of norm d of the space \mathfrak{R}_m in which \mathfrak{X} lies. We may take d so small that $\bar{\mathfrak{B}}_D$ is small at measure. Let B_D denote the cells of D containing points of \mathfrak{X} at none of \mathfrak{B} . Then by hypothesis F converges uniformly in B_D . Thus there exists a μ_0 such that for any $\mu \geq \mu_0$,

$$|\bar{F}_\mu(x)| < \epsilon, \quad \text{for any } x \text{ of } \mathfrak{X} \text{ in } B_D.$$

At a point b of \mathfrak{B} , there exists by hypothesis a $V_\delta(b)$ and a λ_0 such that for each $\lambda \geq \lambda_0$

$$|\bar{F}_\lambda(x)| < \epsilon \quad , \quad \text{for any } x \text{ in } V_\delta(b).$$

Let $C_{b,\lambda}$ be a cube lying in $D_\delta(b)$, having b as center. Since \mathfrak{B} is complete there exists a finite number of these cubes

$$C_{b_1\lambda_1} \quad , \quad C_{b_2\lambda_2} \dots \tag{1}$$

such that each point of \mathfrak{B} lies within one of them.

Moreover

$$|\bar{F}_{\lambda_\kappa}(x)| < \epsilon,$$

for any x of \mathfrak{X} lying in the κ^{th} cube of 1).

As B_D embraces but a finite number of cubes, and as the same is true of 1), there is a finite set of layers \mathfrak{L} such that

$$|\bar{F}_\nu(x)| < \epsilon \quad , \quad \text{in each } \mathfrak{L}.$$

The convergence is thus subuniform, as λ, μ are arbitrarily large.

2. The reasoning of the preceding section gives us also the theorem:

Let

$$\lim_{t=\tau} f(x_1 \dots x_m, t_1 \dots t_n) = \phi(x_1 \dots x_m)$$

in \mathfrak{X} , τ finite or infinite. Let the convergence be uniform in \mathfrak{X} except possibly for the points of a complete discrete set $\mathfrak{E} = \{e\}$. For each point e , let there exist an η such that setting $\epsilon(x, t) = f(x, t) - \phi(x)$,

$$\lim_{x=e} \epsilon(x, t) = 0 \quad , \quad \text{for any } t \text{ in } V_\eta^*(\tau).$$

Then $f \doteq \phi$ subuniformly in \mathfrak{X} .

3. As a special case of 1 we have the theorem:

Let

$$F(x) = f_1(x) + f_2(x) + \dots$$

converge in \mathfrak{X} , and converge uniformly in \mathfrak{X} , except at $x = \alpha_1, \dots x = \alpha_s$. At $x = \alpha_i$ let there exist a ν_i such that

$$\lim_{x=\alpha_i} \bar{F}_{n_i}(x) = 0 \quad , \quad n_i \geq \nu_i \quad , \quad i = 1, 2 \dots s.$$

Then F converges subuniformly in \mathfrak{X} .

4. When

$$\lim_{t=\tau} f(x, t) = \phi(x)$$

we will often set

$$f(x, t) = \phi(x) + \epsilon(x, t),$$

and call ϵ the *residual function*.

549. *Example 1.*

$$f(x, n) = \frac{n^\lambda x^\alpha}{e^{n^\mu x^\beta}} \doteq \phi(x) = 0 \quad , \quad \text{for } n \doteq \infty \text{ in } \mathfrak{A} = (0 < x),$$

$$\alpha, \beta, \lambda \geq 0 \quad , \quad \mu > 0.$$

The convergence is subuniform in \mathfrak{A} . For $x = 0$ is the only possible point of non-uniform convergence, and for any n ,

$$|\epsilon(x, n)| = \frac{n^\lambda x^\alpha}{e^{n^\mu x^\beta}} \doteq 0 \quad , \quad \text{as } x \doteq 0.$$

Example 2. $f(x, n) = \frac{n^\lambda x^\alpha}{c + n^\mu x^\beta} \doteq \phi(x) = 0 \quad , \quad \text{as } n \doteq \infty,$

$$x \text{ in } \mathfrak{A} = (0 < x) \quad , \quad \alpha, \beta, \lambda, \mu > 0 \quad , \quad \mu > \lambda \quad , \quad c > 0.$$

The convergence is uniform in $\mathfrak{B} = (e < x)$, where $e > 0$. For

$$|\epsilon(x, n)| \leq \frac{n^\lambda c^\alpha}{c + n^\mu c^\beta} \quad , \quad \text{in } \mathfrak{B}$$

$$< \frac{c^\alpha}{c^\beta} \cdot \frac{n^\lambda}{n^\mu}$$

$$< \epsilon \quad , \quad \text{for } n > \text{some } m.$$

Thus the convergence is uniform in \mathfrak{A} , except possibly at $x = 0$. The convergence is subuniform in \mathfrak{A} . For obviously for a given n

$$\lim_{x=0} f(x, n) = 0.$$

550. 1. Let $\lim_{t=\tau} f(x_1 \cdots x_m t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$ in \mathfrak{X} , τ finite infinite.

Let the convergence be uniform in \mathfrak{X} except at the points

$$\mathfrak{B} = (b_1, b_2, \cdots b_n).$$

For the convergence to be sub-uniform in \mathfrak{X} , it is necessary that for each b in \mathfrak{B} , and for each $\epsilon > 0$, there exists a $t = \beta$ near τ , such that

$$\overline{\lim}_{x=b} |\epsilon(x, t)| \geq \epsilon. \quad (1)$$

For if the convergence is subuniform, there exists for each ϵ and $\eta > 0$ a finite set of layers \mathfrak{X}_t , t in $V_\eta^*(\tau)$ such that

$$|\epsilon(x, t)| < \epsilon, \quad x \text{ in } \mathfrak{X}_t.$$

Now the point $x = b$ lies in one of these layers, say in \mathfrak{X}_β . Then

$$|\epsilon(x, \beta)| < \epsilon, \quad \text{for all } x \text{ in some } V^*(b).$$

But then 1) holds.

2. *Example.* Let
$$F(x) = \sum_0^\infty x^n (1 - x).$$

This is the series considered in 140, Ex. 2.

F converges uniformly in $\mathfrak{A} = (-1, 1)$, except at $x = 1$.

As
$$\bar{F}_m(x) = -x^m,$$

we see that
$$\lim_{x=1} \bar{F}_m(x) = -1.$$

Hence F is not subuniformly convergent in \mathfrak{A} .

Integrability

551. 1. *Infra-uniform Convergence.* It often happens that

$$f(x_1 \cdots x_m t_1 \cdots t_n) \doteq \phi(x_1 \cdots x_m)$$

subuniformly in \mathfrak{X} except possibly at certain points $\mathfrak{E} = \{e\}$ forming a discrete set. To be more specific, let Δ be a cubical division of \mathfrak{X}_m in which \mathfrak{X} lies, of norm δ . Let X_Δ denote those cells containing points of \mathfrak{X} , but none of \mathfrak{E} . Since \mathfrak{E} is discrete, $\bar{X}_\Delta \doteq \bar{\mathfrak{X}}$. Suppose now $f \doteq \phi$ subuniformly in any X_Δ ; we shall say the convergence is *infra-uniform* in \mathfrak{X} . When there are no exceptional points, infra-uniform convergence goes over into subuniform convergence.

This kind of convergence Arzelà calls uniform convergence by segments, in general.

2. We can make the above definition independent of the set \mathfrak{E} , and this is desirable at times.

Let $\mathfrak{X} = (X, \mathfrak{x})$ be an unmixed division of \mathfrak{X} such that $\bar{\mathfrak{x}}$ may be taken small at pleasure. If $f \doteq \phi$ subuniformly in each X , we say the convergence is infra-uniform in \mathfrak{X} .

3. Then to each $\epsilon, \eta > 0$, and a given X , there exists a set of layers $\mathfrak{f}_1, \mathfrak{f}_2, \dots, \mathfrak{f}_t$ in $V_\eta^*(\tau)$, such that the residual function $\epsilon(x, t)$ numerically $< \epsilon$ for each of these layers. As the projections of these layers \mathfrak{f} do not in general embrace all the points of \mathfrak{X} , we call them *deleted layers*.

4. The points \mathfrak{x} we shall call the *residual points*.

5. *Example 1.*
$$F = \sum_0^\infty \frac{x^2}{(1 + nx^2)(1 + (n+1)x^2)}.$$

This series was studied in 150. We saw that it converges uniformly in $\mathfrak{A} = (0, 1)$, except at $x = 0$.

As

$$\bar{F}_n(x) = \frac{1}{1 + nx^2}, \quad x \neq 0$$

and as this $\doteq 1$ as $x \doteq 0$ for an arbitrary but fixed n , F does not converge subuniformly in \mathfrak{A} , by 550. The series converges infra-uniformly in \mathfrak{A} , obviously.

6. *Example 2.*
$$F = \sum_0^\infty x^n(1 - x).$$

This series was considered in 550, 2. Although it does not converge subuniformly in an interval containing the point $x = 1$, the convergence is obviously infra-uniform.

552. 1. Let $\lim_{x \rightarrow \tau} f(x_1 \dots x_m t_1 \dots t_n) = \phi(x_1 \dots x_m)$ be limited in \mathfrak{X} , finite or infinite. For each t near τ , let f be limited and R -integrable in \mathfrak{X} . For ϕ to be R -integrable in \mathfrak{X} , it is sufficient that $f \doteq \phi$ infra-uniformly in \mathfrak{X} . If \mathfrak{X} is complete, this condition is necessary.

It is sufficient. We show that for each $\epsilon, \omega > 0$ there exists a division D of \mathfrak{R}_m such that the cells in which

$$\text{Osc } \phi \geq \omega \quad (1)$$

have a volume $< \sigma$. For setting as usual

$$f = \phi + \epsilon,$$

we have in any point set,

$$\text{Osc } \phi \leq \text{Osc } f + \text{Osc } \epsilon.$$

Using the notation of 551,

$$|\epsilon(x, t)| < \frac{\omega}{4}$$

in the finite set of deleted layers $t_1, t_2 \dots$ corresponding to $t = t_1, t_2 \dots$. For each of these ordinates t_i , $f(x, t_i)$ is integrable in \mathfrak{X} . There exists, therefore, a rectangular division D of \mathfrak{R}_m , such that those cells in which

$$\text{Osc } f(x, t_i) \geq \frac{\omega}{2}$$

have a content $< \frac{\sigma}{2}$, whichever ordinate t_i is used. Let E be a division of \mathfrak{R}_m such that the cells containing points of the residual set \mathfrak{r} have a content $< \sigma/2$. Let $F = D + E$. Then those cells of F in which

$$\text{Osc } f(x, t_i) \geq \frac{\omega}{2}, \quad \text{or} \quad \text{Osc } |\epsilon(x, t_i)| \geq \frac{\omega}{2}$$

$i = 1, 2 \dots$ have a content $< \sigma$. Hence those cells in which 1) holds have a content $< \sigma$.

It is necessary, if \mathfrak{X} is complete. For let

$$t_1, t_2 \dots \doteq \tau.$$

Since ϕ and $f(x, t_n)$ are integrable, the points of discontinuity of $\phi(x)$ and of $f(x, t_n)$ are null sets by 462, a. Hence if $\mathfrak{C}, \mathfrak{C}_i$ denote the points of continuity of $\phi(x)$ and $f(x, t)$ in \mathfrak{X} ,

$$\widehat{\mathfrak{C}} = \widehat{\mathfrak{C}}_i = \widehat{\mathfrak{X}},$$

since \mathfrak{X} is measurable, as it is complete.

$$\begin{array}{ll}
\text{Let} & \mathfrak{G} = \text{Qdv } \{\mathfrak{G}_i\}, \\
\text{then} & \widehat{\mathfrak{G}} = \widehat{\mathfrak{K}} \\
\text{by 410, 6.} & \\
\text{Let} & \mathfrak{D} = \text{Dv}(\mathfrak{G}, \mathfrak{G}), \\
\text{then} & \widehat{\mathfrak{D}} = \widehat{\mathfrak{K}}, \tag{1}
\end{array}$$

as we proceed to show. For if $G = \mathfrak{K} - \mathfrak{G}$,

$$\mathfrak{G} = \text{Dv}(\mathfrak{G}, \mathfrak{G}) + \text{Dv}(\mathfrak{G}, G) = \mathfrak{D} + \text{Dv}(\mathfrak{G}, G).$$

But G is a null set. Hence $\text{Meas } \text{Dv}(\mathfrak{G}, G) = 0$, and thus $\widehat{\mathfrak{G}} = \widehat{\mathfrak{K}} = \widehat{\mathfrak{D}}$, which is 1).

Let now ξ be a point of \mathfrak{D} , let it lie in $\mathfrak{G}_{t_1}, \mathfrak{G}_{t_2} \dots$ where $t_1, t_2 \dots$ form a monotone sequence $\doteq \tau$. Then since

$$f(\xi, t_n) \doteq \phi(\xi),$$

there is an m such that

$$|\epsilon(\xi, t_n)| < \frac{\epsilon}{3}, \quad \text{for any } n > m. \tag{2}$$

But ξ lying in \mathfrak{D} , it lies in \mathfrak{G} and \mathfrak{G}_{t_n} .

$$\text{Thus} \quad |\phi(x) - \phi(\xi)| < \frac{\epsilon}{3},$$

$$|f(x, t_n) - f(\xi, t_n)| < \frac{\epsilon}{3},$$

for any x in $V_\delta(\xi)$. Hence

$$|\epsilon(x, t_n) - \epsilon(\xi, t_n)| < \frac{2\epsilon}{3}, \quad x \text{ in } V_\delta(\xi). \tag{3}$$

Now

$$\epsilon(x, t_n) = \epsilon(x, t_n) - \epsilon(\xi, t_n) + \epsilon(\xi, t_n).$$

Hence from 2), 3),

$$|\epsilon(x, t_n)| < \epsilon, \quad \text{for any } x \text{ in } V_\delta(\xi).$$

Thus associated with the point ξ , there is a cube Γ lying in $D_\delta(\xi)$, having ξ as center. As $D = \mathfrak{K} - \mathfrak{D}$ is a null set, each of its points can be enclosed within cubes C , such that the resulting enclosure

\mathfrak{E} has a measure $< \sigma$, small at pleasure. Thus each point of \mathfrak{X} lies within a cube. By Borel's theorem there exists a finite set of these cubes

$$\Gamma_1, \Gamma_2 \dots \Gamma_r \quad ; \quad \mathcal{C}_1, \mathcal{C}_2 \dots \mathcal{C}_s,$$

such that each point of \mathfrak{X} lies within one of them. But corresponding to the Γ 's, are layers

$$\mathfrak{L}_1, \mathfrak{L}_2, \dots \mathfrak{L}_r$$

such that in each of them

$$|\epsilon(x, t)| < \epsilon.$$

Thus $f \doteq \phi$ subuniformly in $X = (\Gamma_1, \Gamma_2 \dots \Gamma_r)$. Let \mathfrak{z} be the residual set. Obviously $\mathfrak{z} < \sigma$. Thus the convergence is infra-uniform.

2. As a corollary we have :

Let
$$F(x) = \Sigma f_{i_1, \dots, i_n}(x_1 \dots x_m)$$

converge in \mathfrak{A} . Let F be limited, and each f_i be limited and R -integrable in \mathfrak{A} . For F to be R -integrable in \mathfrak{A} , it is sufficient that F converge infra-uniformly in \mathfrak{A} .

If \mathfrak{A} is complete, this condition is necessary.

553. *Infinite Peaks.* 1. Let $\lim_{t \rightarrow \tau} f(x_1 \dots x_m t_1 \dots t_n) = \phi(x)$ in \mathfrak{X} , τ finite or infinite. Although $f(x, t)$ is limited in \mathfrak{X} for each t near τ , and although $\phi(x)$ is also limited in \mathfrak{X} , we cannot say that

$$|f(x, t)| < \text{some } M \quad (1)$$

for any x in \mathfrak{X} and any t near τ , as is shown by the following

Example. Let $f(x, t) = \frac{tx}{e^{tx^2}} \doteq \phi(x) = 0$, as $t \doteq \infty$ for x in $\mathfrak{X} = (-\infty, \infty)$.

It is easy to see that the peak of f becomes infinitely high as $n \doteq \infty$.

In fact, for $x = \frac{1}{\sqrt{t}}$, $f = \frac{\sqrt{t}}{e}$. Thus the peak is at least as high as $\frac{\sqrt{t}}{e}$, which $\doteq \infty$.

The origin is thus a point in whose vicinity the peaks of the family of curves $f(x, t)$ are infinitely high. In general, if the peaks of

$$f(x_1 \cdots x_m t_1 \cdots t_n)$$

in the vicinity V_δ of $x = \xi$ become infinitely high as $t \doteq \tau$, however small δ is taken, we say ξ is a point with infinite peaks.

On the other hand, if the relation 1) holds for all x and t involved, we shall say $f(x, t)$ is uniformly limited.

2. If $\lim_{t \rightarrow \tau} f(x_1 \cdots x_m t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$, and if $f(x, t)$ is uniformly limited in \mathfrak{X} , then ϕ is limited in \mathfrak{X} .

For x being taken at pleasure in \mathfrak{X} and fixed, $\phi(x)$ is a limit point of the points $f(x, t)$ as $t \doteq \tau$. But all these points lie in some interval $(-G, G)$ independent of x . Hence ϕ lies in this interval.

3. If \mathfrak{X} is complete, the points \mathfrak{R} in \mathfrak{X} with infinite peaks also form complete set. If these points \mathfrak{R} are enumerable, they are discrete.

That \mathfrak{R} is complete is obvious. But then $\bar{\mathfrak{R}} = \widehat{\mathfrak{R}} = 0$, as \mathfrak{R} is enumerable.

554. 1. Let $\lim_{t \rightarrow \tau} f(x_1 \cdots x_m t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$ in \mathfrak{X} , metric or complete. Let $f(x, t)$ be uniformly limited in \mathfrak{X} , and R -integrable for each t near τ . For the relation

$$\lim_{t \rightarrow \tau} \int_{\mathfrak{X}} f(x, t) = \int_{\mathfrak{X}} \phi(x)$$

to hold, it is sufficient that $f \doteq \phi$ infra-uniformly in \mathfrak{X} . If \mathfrak{X} is for each t complete, this condition is necessary.

For by 552, ϕ is R -integrable if $f \doteq \phi$ infra-uniformly, and when \mathfrak{X} is complete, this condition is necessary. By 424, 4, each $f(x, t)$ is measurable. Thus we may apply 381, 2 and 413, 2.

2. As a corollary we have the theorem:

Let
$$F(x) = \sum f_{i_1, \dots, i_n}(x_1 \cdots x_m)$$

converge in the complete or metric field \mathfrak{A} . Let the partial sums F_λ be uniformly limited in \mathfrak{A} . Let each term f_i be limited and R -integrable in \mathfrak{A} . Then for the relation

$$\int_{\mathfrak{A}} F = \sum \int_{\mathfrak{A}} f_i$$

to hold it is sufficient that F is infra-uniformly convergent in \mathfrak{A} . If \mathfrak{A} is complete, this condition is necessary.

555. *Example 1.* Let us reconsider the example of 150,

$$F(x) = \sum_0^{\infty} \frac{x^2}{(1 + nx^2)(1 + (n+1)x^2)}.$$

We saw that we may integrate termwise in $\mathfrak{A} = (0, 1)$, although F does not converge uniformly in \mathfrak{A} . The only point of non-uniform convergence is $x = 0$. In 551, 5, we saw that it converges, however, infra-uniformly in \mathfrak{A} . As

$$|F_n(x)| \leq 1, \quad \text{for any } x \text{ in } \mathfrak{A}, \text{ and for every } n,$$

all the conditions of 554 are satisfied and we can integrate the series termwise, in accordance with the result already obtained in 150.

Example 2. Let
$$F(x) = \sum_1^{\infty} \left\{ \frac{nx}{e^{nx^2}} - \frac{(n-1)x}{e^{(n-1)x^2}} \right\} = 0.$$

Then

$$F_n(x) = \frac{nx}{e^{nx^2}}.$$

We considered this series in 152, 1. We saw there that this series cannot be integrated termwise in $\mathfrak{A} = (0 < a)$. It is, however, subuniformly convergent in \mathfrak{A} as we saw in 549, Ex. 1. We cannot apply 554, however, as F_n is not uniformly limited. In fact we saw in 152, 1, that $x = 0$ is a point with an infinite peak.

Example 3.
$$F(x) = \sum_0^{\infty} x^n(1-x).$$

We saw in 551, 6, that F converges infra-uniformly in $\mathfrak{A} = (0, 1)$. Here

$$|F_n(x)| = |1 - x^n| < \text{some } M,$$

for any x in $\mathfrak{A} = (0 \leq u)$, $u \leq 1$, and any n . Thus the F_n are uniformly limited in \mathfrak{A} .

We may therefore integrate termwise by 554, 2. We may verify this at once. For

$$\begin{aligned} F(x) &= 1, & 0 \leq x < 1 \\ &= 0, & x = 0. \end{aligned}$$

Hence
$$\int_0^u F(x) dx = u. \quad (1)$$

On the other hand,

$$\int_0^u F_n dx = u - \frac{u^{n+1}}{n+1} \doteq u, \quad \text{as } n \doteq \infty. \quad (2)$$

From 1), 2) we have

$$G(u) = u = \sum_0^\infty \left\{ \frac{u^{n+1}}{n+1} - \frac{u^{n+2}}{n+2} \right\}.$$

556. 1. If $1^\circ f(x_1 \dots x_m t_1 \dots t_n) \doteq \phi(x_1 \dots x_m)$ *infra-uniformly* the metric or complete field \mathfrak{X} , as $t \doteq \tau$, τ finite or infinite;

$2^\circ f(x, t)$ is uniformly limited in \mathfrak{X} and R -integrable for each t at τ ;

Then

$$\lim_{t \rightarrow \tau} \int_{\mathfrak{X}} f(x, t) = \int_{\mathfrak{X}} \phi,$$

uniformly with respect to the set of measurable fields \mathfrak{A} in \mathfrak{X} .

If \mathfrak{X} is complete, condition 1° may be replaced by $3^\circ \phi(x)$ is R -integrable in \mathfrak{X} .

For by 552, 1, when 3° holds, 1° holds; and when 1° holds, ϕ R -integrable in \mathfrak{X} .

Now the points \mathfrak{E}_t where

$$|\epsilon(x, t_n)| > \epsilon$$

are such that

$$\lim_{t \rightarrow \tau} \widehat{\mathfrak{E}}_t = 0, \quad \text{by 412.}$$

Let $\mathfrak{X} = \mathfrak{E}_t + \mathfrak{X}_t$. Then

$$\int_{\mathfrak{X}} \epsilon(x, t) = \int_{\mathfrak{E}_t} \epsilon(x, t) + \int_{\mathfrak{X}_t} \epsilon(x, t),$$

$$\left| \int_{\mathfrak{A}} \epsilon \right| \leq \int_{\mathfrak{X}} |\epsilon| \leq 2 M \widehat{\mathfrak{E}}_t + \epsilon \widehat{\mathfrak{X}}.$$

But

$$\lim_{t \rightarrow \tau} \widehat{\mathfrak{E}}_t = 0,$$

which establishes the theorem.

2. As a corollary we have:

If $1^\circ F(x) = \sum_{f_1, \dots, f_n} (x_1 \dots x_m)$ converges *infra-uniformly*, and each of its terms f_i is R -integrable in the metric or complete field \mathfrak{A} ;

2° $F_\lambda(x)$ is uniformly limited in \mathfrak{A} ;

Then

$$\int_{\mathfrak{A}} F(x) = \sum \int_{\mathfrak{A}} f_i,$$

and the series on the right converges uniformly with respect to all measurable $\mathfrak{B} \leq \mathfrak{A}$.

3. If 1° $\lim_{t \rightarrow \tau} f(x, t_1 \dots t_n) = \phi(x)$ is R -integrable in the interval $\mathfrak{A} = (a < b)$, τ finite or infinite;

2° $f(x, t)$ is uniformly limited, and R -integrable for each t near τ ;

Then

$$\lim_{t \rightarrow \tau} \int_a^x f(x, t) dx = \int_a^x \phi(x) dx = \Phi(x),$$

uniformly in \mathfrak{A} , and $\Phi(x)$ is continuous in \mathfrak{A} .

4. If 1°

$$F(x) = \sum f_{i_1 \dots i_n}(x)$$

and also each term f_i are R -integrable in the interval $\mathfrak{A} = (a < b)$;

2° $F_\lambda(x)$ is uniformly limited in \mathfrak{A} ;

Then

$$G(x) = \sum \int_a^x f_i(x) dx, \quad x \text{ in } \mathfrak{A}$$

is continuous.

For G is a uniformly convergent series in \mathfrak{A} , each of whose terms

$$\int_a^x f_i dx$$

is a continuous function of x .

Differentiability

557. 1. If 1° $\lim_{t \rightarrow \tau} f(x, t_1 \dots t_n) = \phi(x)$ in $\mathfrak{A} = (a < b)$, τ finite or infinite;

2° $f'_x(x, t)$ is R -integrable for each t near τ , and uniformly limited in \mathfrak{A} ;

3° $f'_x(x, t) \doteq \psi(x)$ infra-uniformly in \mathfrak{A} , as $t \doteq \tau$;

Then at a point x of continuity of ψ in \mathfrak{A}

$$\phi'(x) = \psi(x), \tag{1}$$

or what is the same

$$\frac{d}{dx} \lim_{t \rightarrow \tau} f(x, t) = \lim_{t \rightarrow \tau} \frac{d}{dx} f(x, t). \tag{2}$$

For by 554,

$$\begin{aligned}\lim_{t \rightarrow \tau} \int_a^x f'_x(x, t) dx &= \int_a^x \psi(x) dx \\ &= \lim_{t \rightarrow \tau} [f(x, t) - f(a, t)] \quad , \quad \text{by I, 538} \\ &= \phi(x) - \phi(a) \quad , \quad \text{by 1}^\circ.\end{aligned}\tag{3}$$

Now by I, 537, at a point of continuity of ψ ,

$$\frac{d}{dx} \int_a^x \psi(x) dx = \psi(x).\tag{4}$$

From 3), 4), we have 1), or what is the same 2).

2. In the interval \mathfrak{A} , if

$$1^\circ \quad F(x) = \sum f_{i_1 \dots i_n}(x) \text{ converges;} \tag{1}$$

2° Each $f'_i(x)$ is limited and R -integrable;

3° $F'_\lambda(x)$ is uniformly limited;

4° $G(x) = \sum f'_i$ is infra-uniformly convergent;

Then at a point of continuity of $G(x)$ in \mathfrak{A} , we may differentiate the series 1) termwise, or $F'(x) = G(x)$.

3. In the interval \mathfrak{A} , if

1° $f(x, t_1 \dots t_n) \doteq \phi(x)$ as $t \doteq \tau$, τ finite or infinite;

2° $f(x, t)$ is uniformly limited, and a continuous function of x ;

3° $\psi(x) = \lim_{t \rightarrow \tau} f'_x(x, t)$ is continuous;

$$\text{Then} \quad \phi'(x) = \psi(x), \tag{1}$$

or what is the same

$$\frac{d}{dx} \lim_{t \rightarrow \tau} f(x, t) = \lim_{t \rightarrow \tau} \frac{d}{dx} f(x, t). \tag{2}$$

For by 547, 1, condition 3° requires that $f' \doteq \psi$ subuniformly in \mathfrak{A} . But then the conditions of 1 are satisfied and 1) and 2) hold.

4. In the interval \mathfrak{A} let us suppose that .

$$1^\circ \quad F(x) = \sum f_{i_1 \dots i_n}(x) \text{ converges;} \tag{1}$$

2° Each term f_n is continuous;

3° $F'_n(x)$ is uniformly limited;

4° $G(x) = \sum f'_n(x)$ is continuous;

Then we may differentiate 1) termwise, or $F'(x) = G(x)$.

558. Example 1. We saw in 555, Ex. 3 that

$$F(x) = x = \sum_0^n \left\{ \frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2} \right\}, \quad \text{in } \mathfrak{A} = (0, 1). \quad (1)$$

The series got by differentiating termwise is

$$\begin{aligned} G(x) &= \sum_0^\infty x^n (1-x) = 1, & 0 \leq x < 1 \\ &= 0, & x = 0. \end{aligned} \quad (2)$$

Thus by 557, 4, $G(x) = F'(x)$ in $(0^*, 1) = \mathfrak{A}^*$. (3)

The relation 3) does not hold for $x = 0$.

Example 2.

$$F(x) = \sum_1^\infty \left\{ \frac{\operatorname{arctg} x\sqrt{n}}{\sqrt{n}} - \frac{\operatorname{arctg} x\sqrt{n+1}}{\sqrt{n+1}} \right\} = \sum f_n(x)$$

$$G(x) = \sum_1^\infty \left\{ \frac{1}{1+nx^2} - \frac{1}{1+(n+1)x^2} \right\} = \sum f'_n(x).$$

Here $F(x) = \operatorname{arctg} x$, for any x . (1)

$$G(x) = \frac{1}{1+x^2}, \quad x \neq 0 \quad (2)$$

$$= 0, \quad x = 0.$$

Hence $G(x)$ is continuous in any interval \mathfrak{A} , not containing $x = 0$. Thus we should have by 557, 4,

$$F'(x) = G(x), \quad x \text{ in } \mathfrak{A}. \quad (3)$$

This relation is verified by 1), 2). The relation 3) does not hold for $x = 0$, since

$$F'(0) = 1, \quad G(0) = 0.$$

Example 3.

$$\begin{aligned} F(x) &= \sum_1^n \left\{ \frac{1}{2n} \log(1 + n^2 x^2) - \frac{1}{2(n+1)} \log(1 + (n+1)^2 x^2) \right\} \\ &= \Sigma f_n(x) \\ &= \frac{1}{2} \log(1 + x^2) \quad , \quad \text{for any } x. \end{aligned} \quad (1)$$

$$\begin{aligned} F'(x) &= \sum_1^n f'_n(x) = \sum \left\{ \frac{nx}{1 + n^2 x^2} - \frac{(n+1)x}{1 + (n+1)^2 x^2} \right\} \\ &= \frac{x}{1 + x^2} \quad , \quad \text{for any } x. \end{aligned} \quad (2)$$

In any interval \mathfrak{A} , all the conditions of 557, 4, hold.

Hence $F'(x) = G(x)$, for any x in \mathfrak{A} .
 In case we did not know the value of the sums 1), 2) we could assert that 3) holds. For by 545, G is subuniformly convergent in \mathfrak{A} , and hence is continuous.

Example 4.

$$F(x) = \sum_1^\infty \left\{ \frac{1 + nx}{ne^{nx}} - \frac{1 + (n+1)x}{(n+1)e^{(n+1)x}} \right\} = \frac{1 + x}{e^x}. \quad (1)$$

Here

$$F'(x) = -\frac{x}{e^x}. \quad (2)$$

The series obtained by differentiating F termwise is

$$G(x) = \sum_1^\infty \left\{ \frac{(n+1)x}{e^{(n+1)x}} - \frac{nx}{e^{nx}} \right\} = -\frac{x}{e^x}, \quad (3)$$

hence

$$G_{n-1}(x) = -\frac{x}{e^x} + \frac{nx}{e^{nx}}.$$

The peaks of the residual function

$$\epsilon(x, n) = \frac{nx}{e^{nx}},$$

of height $= 1/e$. The convergence of G is not uniform at 0. The conditions of 557, 4, are satisfied and we can differentiate 1) termwise. This is verified by 2), 3).

559. 1. If $1^\circ \lim_{t \rightarrow \tau} f(x, t_1 \dots t_n) = \phi(x)$ is limited and R -integrable in the interval $\mathfrak{A} = (a < b)$;

$2^\circ f(x, t)$ is limited, and R -integrable in \mathfrak{A} , for each t near τ ;

$$3^\circ \quad \psi(x) = \lim_{t \rightarrow \tau} \int_a^x f(x, t) = \lim_{t \rightarrow \tau} g(x, t)$$

is a continuous function in \mathfrak{A} ;

4° The points \mathfrak{E} in \mathfrak{A} in whose vicinity the peaks of $f(x, t)$ as $t \rightarrow \tau$ are infinitely high form an enumerable set;

Then

$$\theta(x) = \int_a^x \phi(x) = \lim_{t \rightarrow \tau} \int_a^x f(x, t) dx = \psi(x), \quad (1)$$

or

$$\lim_{t \rightarrow \tau} \int_a^x f(x, t) dx = \int_a^x \lim_{t \rightarrow \tau} f(x, t) dx,$$

and the set \mathfrak{E} is complete and discrete.

For \mathfrak{E} is discrete by 553, 3.

Let α be a point of $A = \mathfrak{A} - \mathfrak{E}$. Then in an interval α about α ,

$$|f(x, t)| < \text{some } M, \quad x \text{ in } \alpha, \text{ any } t \text{ near } \tau. \quad (2)$$

Now by 556, 3, taking $\epsilon > 0$ small at pleasure, there exists an $\eta > 0$ such that

$$\psi(x) - \psi(\alpha) = \int_\alpha^x f(x, t) + \epsilon', \quad |\epsilon'| < \epsilon$$

for any x in α , and t in $V_\eta^*(\tau)$. If we set $x = \alpha + h$, we have

$$\frac{\Delta\psi}{\Delta x} = \frac{\psi(x) - \psi(\alpha)}{h} = \frac{1}{h} \int_\alpha^x f(x, t) dx + \frac{\epsilon'}{h}. \quad (3)$$

Also by 556, 3, we have

$$\int_\alpha^x f(x, t) dx = \int_\alpha^x \phi(x) dx + \epsilon'', \quad |\epsilon''| < \epsilon$$

for any x in α , and t in $V_\eta^*(\tau)$. Thus

$$\frac{1}{h} \int_\alpha^x f(x, t) dx = \frac{\theta(x) - \theta(\alpha)}{h} + \frac{\epsilon''}{h} = \frac{\Delta\theta}{\Delta x} + \frac{\epsilon''}{h}. \quad (4)$$

From 3), 4) we have

$$\frac{\Delta\psi}{\Delta x} - \frac{\epsilon'}{h} = \frac{\Delta\theta}{\Delta x} + \frac{\epsilon''}{h}, \quad |\epsilon'|, |\epsilon''| < \epsilon.$$

Now ϵ may be made small at pleasure, and that independent of h . Thus the last relation gives

$$\frac{\Delta\psi}{\Delta x} = \frac{\Delta\theta}{\Delta x}, \quad \text{for } x \text{ in } A.$$

As this holds however small $h = \Delta x$ is taken, we have

$$\frac{d\psi}{dx} = \frac{d\theta}{dx}, \quad \text{for } x \text{ in } A.$$

Hence by 515, 3,

$$\psi(x) = \theta(x) + \text{const}, \quad \text{in } \mathfrak{A}.$$

$$\text{For } x = a, \quad \psi(a) = \theta(a) = 0;$$

and thus

$$\psi(x) = \theta(x), \quad \text{in } \mathfrak{A}.$$

2. As a corollary we have :

If 1° $F(x) = \sum f_i(x)$ is limited and R integrable in the interval $\mathfrak{A} = (a < b)$;

2° $F'_\lambda(x)$ is limited and each term f_i is R -integrable;

3° $G(x) = \sum \int_a^x f_i$ is continuous;

4° The points \mathfrak{E} in \mathfrak{A} in whose vicinity the peaks of $F'_\lambda(x)$ are infinitely high form an enumerable set;

Then

$$\int_a^x F(x) = \sum \int_a^x f_i,$$

or we may integrate the F series termwise.

560. 1. If 1° $\lim_{t \rightarrow \tau} f(x, t_1 \cdots t_n) = \phi(x)$ in $\mathfrak{A} = (a < b)$, τ finite or infinite;

2° $f'_x(x, t)$ is limited and R -integrable for each t near τ ;

3° The points \mathfrak{E} of \mathfrak{A} in whose vicinity $f'_x(x, t)$ has infinite peaks as $t \rightarrow \tau$ form an enumerable set;

4° $\phi(x)$ is continuous at the points \mathfrak{E} ;

5° $\psi(x) = \lim_{t \rightarrow \tau} f'_x(x, t)$ is limited and R -integrable in \mathfrak{A} ;

Then at a point of continuity of $\psi(x)$ in \mathfrak{A}

$$\phi'(x) = \psi(x), \quad (1)$$

or what is the same

$$\frac{d}{dx} \lim_{t \rightarrow \tau} f(x, t) = \lim_{t \rightarrow \tau} \frac{d}{dx} f(x, t).$$

For let $\delta = (\alpha < \beta)$ be an interval in \mathfrak{A} containing no point of \mathfrak{E} . Then for any x in δ

$$\int_{\alpha}^x f'_x(x, t) dx = f(x, t) - f(\alpha, t) \quad , \quad \text{by } 2^{\circ}.$$

Hence

$$\begin{aligned} \lim_{t \rightarrow \tau} \int_{\alpha}^x f'_x(x, t) dx &= \lim_{t \rightarrow \tau} \{f(x, t) - f(\alpha, t)\} \\ &= \phi(x) - \phi(\alpha) \quad , \quad \text{by } 1^{\circ}. \end{aligned} \quad (2)$$

By 556, 3, $\phi(x)$ is continuous in δ . Thus $\phi(x)$ is continuous at any point not in \mathfrak{E} . Hence by 4° it is continuous in \mathfrak{A} .

We may thus apply 559, 1, replacing therein $f(x, t)$ by $f'_x(x, t)$. We get

$$\lim_{t \rightarrow \tau} \int_{\alpha}^x f'_x(x, t) dx = \int_{\alpha}^x \lim_{t \rightarrow \tau} f'_x(x, t) dx = \int_{\alpha}^x \psi(x) dx. \quad (3)$$

Since 2) obviously holds when we replace α by a , this relation with 3) gives

$$\int_a^x \psi(x) dx = \phi(x) - \phi(a).$$

At a point of continuity, this gives 1) on differentiating.

2. If 1° $F(x) = \sum f_1, \dots, f_n(x)$ converges in the interval \mathfrak{A} ;

2° $G(x) = \sum f'_i(x)$ and each of its terms are limited and R -integrable in \mathfrak{A} ;

3° The points of \mathfrak{A} in whose vicinity $G_{\lambda}(x)$ has infinite peaks as $\lambda \rightarrow \infty$, form an enumerable set at which $F(x)$ is continuous;

Then at a point of continuity of $G(x)$ we have

$$F'(x) = G(x),$$

or what is the same

$$\frac{d}{dx} \sum f_i(x) = \sum \frac{df_i(x)}{dx}.$$

561. *Example.*

$$F(x) = \sum_1^{\infty} \left\{ \frac{nx^2}{e^{nx^2}} - \frac{(n+1)x^2}{e^{(n+1)x^2}} \right\} = \frac{x^2}{e^{x^2}}.$$

Hence

$$F'(x) = \frac{2x}{e^{x^2}} - \frac{2x^3}{e^{x^2}}. \quad (1)$$

The series obtained by differentiating F termwise is

$$G(x) = \sum_1^{\infty} \left\{ \frac{2nx}{e^{nx^2}} - \frac{2n^2x^3}{e^{nx^2}} - \frac{2(n+1)}{e^{(n+1)x^2}} + \frac{2(n+1)^2x^3}{e^{(n+1)x^2}} \right\}.$$

Here

$$G_{n-1}(x) = \frac{2x}{e^{x^2}} - \frac{2x^3}{e^{x^2}} - \left\{ \frac{2nx}{e^{nx^2}} - \frac{2n^2x^3}{e^{nx^2}} \right\}.$$

Hence

$$G(x) = \frac{2x}{e^{x^2}} - \frac{2x^3}{e^{x^2}} \quad (2)$$

a continuous function of x .

The convergence of the G series is not uniform at $x=0$. For $a_n = 1/n$. Then

$$G_{n-1}(a_n) = G(a_n) - \left\{ \frac{2}{e^{\frac{1}{n}}} - \frac{\frac{2}{n}}{e^{\frac{1}{n}}} \right\} \doteq -2.$$

To get the peaks of the residual function we consider the points of extreme of

$$y = \frac{nx(1-nx^2)}{e^{nx^2}}. \quad (3)$$

We find

$$y' = \frac{n(1-5nx^2+2n^2x^4)}{e^{nx^2}}.$$

Thus $y' = 0$ when

$$2n^2x^4 - 5nx^2 + 1 = 0,$$

when $x = \frac{a}{\sqrt{n}}$ or $\frac{\alpha}{\sqrt{n}}$, a, α constants.

Putting these values in 3), we find that y has the form

$$y = c\sqrt{n}.$$

Hence $x=0$ is the only point where the residual function has infinite peak. Thus the conditions of 560, 2, are satisfied, and should have $F'(x) = G(x)$ for any x . This is indeed so, as 1), show.

CHAPTER XVII

GEOMETRIC NOTIONS

Plane Curves

562. In this chapter we propose to examine the notions of curve and surface together with other allied geometric concepts. Like most of our notions, we shall see that they are vague and uncertain as soon as we pass the confines of our daily experience. In studying some of their complexities and even paradoxical properties, the reader will see how impossible it is to rely on his unschooled intuition. He will also learn that the demonstration of a theorem in analysis which rests on the evidence of our geometric intuition cannot be regarded as binding until the geometric notions employed have been clarified and placed on a sound basis.

Let us begin by investigating our ideas of a plane curve.

563. Without attempting to define a curve we would say on looking over those curves most familiar to us that a plane curve has the following properties :

- 1° It can be generated by the motion of a point.
- 2° It is formed by the intersection of two surfaces.
- 3° It is continuous.
- 4° It has a tangent at each point.
- 5° The arc between any two of its points has a length.
- 6° A curve is not superficial.
- 7° Its equations can be written in any one of the forms

$$y = f(x), \tag{1}$$

$$x = \phi(t) \quad , \quad y = \psi(t), \tag{2}$$

$$F(x, y) = 0; \tag{3}$$

and conversely such equations define curves.

8° When closed it forms the complete boundary of a region.

9° This region has an area.

Of all these properties the first is the most conspicuous and characteristic to the naïve intuition. Indeed many employ this as the definition of a curve. Let us therefore look at our ideas of motion.

564. *Motion.* In this notion, two properties seem to be essential. 1° motion is continuous, 2° it takes place at each instant in a definite direction and with a definite speed. The direction of motion, we agree, shall be given by dy/dx , its speed by ds/dt . We see that the notion of motion involves properties 4°, 5°, and 7°. Waiving this point, let us notice a few peculiarities which may arise.

Suppose the curve along which the motion takes place has an angle point or a cusp as in I, 366. What is the direction of motion at such a point? Evidently we must say that motion is impossible along such a curve, or admit that the ordinary idea of motion is imperfect and must be extended in accordance with the notion of right-hand and left-hand derivatives.

Similarly ds/dt may also give two speeds, a posterior and an anterior speed, at a point where the two derivatives of $s = \phi(t)$ are different.

Again we will admit that at any point of the path of motion, motion may begin and take place in either direction. Consider what happens for a path defined by the continuous function in I, 367. This curve has no tangent at the origin. We ask how does the point move as it passes this point, or to make the question still more embarrassing, suppose the point at the origin. In what direction does it start to move? We will admit that no such motion is possible, or at least it is not the motion given us by our intuition. Still more complicated paths of this nature are given in I, 369, 371, and in Chapter XV of the present volume.

It thus appears that to define a curve as the path of a moving point, is to define an unknown term by another unknown term, equally if not more obscure.

565. 2° *Property. Intersection of Two Surfaces.* This property has also been used as the definition of a curve. As the notion

of a surface is vastly more complicated than that of a curve, it hardly seems advisable to define a complicated notion by one still more complicated and vague.

566. 3° Property. Continuity. Over this knotty concept philosophers have quarreled since the days of Democritus and Aristotle. As far as our senses go, we say a magnitude is continuous when it can pass from one state to another by imperceptible gradations. The minute hand of a clock appears to move continuously, although in reality it moves by little jerks corresponding to the beats of the pendulum. Its velocity to our senses appears to be continuous.

We not only say that the magnitude shall pass from one state to another by gradations imperceptible to our senses, but we also demand that between any two states another state exists and so without end. Is such a magnitude continuous? No less a mathematician than Bolzano admitted this in his philosophical tract *Paradoxien des Unendlichen*. No one admits it, however, to-day. The different states of such a magnitude are pantactic, but their ensemble is not a continuum.

But we are not so much interested in what constitutes a continuum in the abstract, as in what constitutes a continuous curve or even a continuous straight line or segment. The answer we have adopted to these questions is given in the theory of irrational numbers created by Cantor and Weierstrass [see Vol. I, Chap. II], and in the notion of a continuous function due to Cauchy and Weierstrass [see Vol. I, Chap. III].

These definitions of continuity are not very intuitive, but with them we can reason with the utmost precision and deduce consequences which our intuition would not suggest. We can show by purely analytic methods that a continuous function $f(x)$ has in its extreme values [I, 354], that if such a function takes the value a at the point P , and the value b at the point Q , it takes in all intermediary values between a , b , as x runs from P to Q [I, 357]. We can also show that a closed curve has a point which does not form the boundary of a complete region.

567. 4° Property. Tangents. To be a tangent is a tangent? Euclid defines a tangent to a circle as a straight line which meets

the circle and being produced does not cut it again. In commenting on this definition Casey says, "In modern geometry a curve is made up of an infinite number of points which are placed in order along the curve, and then the secant through two consecutive points is a tangent." If the points on a curve were like beads on a string, we might speak of consecutive points. As, however, there are always an infinite number of points between any two points on a continuous curve, this definition is quite illusory. The definition we have chosen is given in I, 365. That property does not hold at each point of a continuous curve was brought out in the discussion of property 1°. Not only is it not necessary that a curve has a tangent at each of its points, but a curve does not need to have a tangent at a pantactic set of points, as we saw Chapter XV.

For a long time it was supposed that every curve has a tangent at each point, or if not at each point, at least in general. Analytically, this property would go over into the following: *every continuous function has a derivative*. A celebrated attempt to prove this was made by Ampère.

Mathematicians were greatly surprised when Weierstrass exhibited the function we have studied in 502 and which has no derivative.

Weierstrass* himself remarks: "Bis auf die neueste Zeit hat man allgemein angenommen, dass eine eindeutige und continuirliche Function einer reellen Veränderlichen auch stets eine erste Ableitung habe, deren Werth nur an einzelnen Stellen unbestimmt oder unendlich gross werden könne. Selbst in den Schriften von Gauss, Cauchy, Dirichlet findet sich meines Wissens keine Aeusserung, aus der unzweifelhaft hervorginge, dass diese Mathematiker, welche in ihrer Wissenschaft die strengste Kritik überall zu üben gewohnt waren, anderer Ansicht gewesen seien."

568. Property 5°. Length. We think of a curve as having length. Indeed we read as the definition of a curve in Euclid's elements: a line is length without breadth. When we see two simple curves we can often compare one with the other in regard to length without consciously having established a way to measure

* *Werke*, vol. 2, p. 71.

them. Perhaps we unconsciously suppose them described at a uniform rate and estimate the time it takes. It may be that we regard them as inextensible strings whose length is got by straightening them out. A less obvious way to measure their lengths would be to roll a straightedge over them and measure the distance on the edge between the initial and final points of contact.

We ask how shall we formulate arithmetically our intuitional ideas regarding the length of a curve? The intuitionist says, a curve or the arc of a curve *has* length. This length is expressed by a number L which is obtained by taking a number of points $P_1, P_2, P_3 \dots$ on the curve between the end points P, P' , and forming the sum

$$\sum P_i P_{i+1}. \quad (1)$$

The limit of this sum as the points became pantactic is the length L of the arc PP' .

Our point of view is different. We would say: Whatever arithmetic formulation we choose we have no *a priori* assurance that it adequately represents our intuitional ideas of length. With the intuitionist we will, however, form the sum 1) and see if it has a limit, however the points P_i are chosen. If it has, we will investigate this number used as a definition of length and see if it leads to consequences which are in harmony with our intuition.

This we now proceed to do.

$$569. \quad 1. \quad \text{Let} \quad x = \phi(t) \quad , \quad y = \psi(t) \quad (1)$$

be one-valued continuous functions of t in the interval $\mathfrak{A} = (a < b)$. As t ranges over \mathfrak{A} the point x, y will describe a *curve* or an *arc* of a curve C . We might agree to call such curves *analytic*, in distinction to those given by our intuition. The interval \mathfrak{A} is the *interval corresponding to C*.

Let D be a finite division of \mathfrak{A} of norm d , defined by

$$a < t_1 < t_2 < \dots < b.$$

To these values of t will correspond points

$$P, P_1, P_2 \dots Q \quad (2)$$

on C , which may be used to define a polygon P_D whose vertices are 2).

Let $(m, m+1)$ denote the side $P_m P_{m+1}$, as well as its length. If we denote the length of P_D by the same letter, we have

$$P_D = \Sigma(m, m+1) = \Sigma \sqrt{\Delta x_m^2 + \Delta y_m^2}.$$

If

$$\lim_{d=0} P_D \quad (3)$$

exists, it is called the *length of the arc C* , and C is *rectifiable*.

2. (Jordan.) For the arc PQ to be rectifiable, it is necessary and sufficient that the functions ϕ, ψ in 1) have limited variation in \mathfrak{A} .

For

$$\sqrt{\Delta x^2 + \Delta y^2} \geq |\Delta x|.$$

Hence

$$P_D \geq \Sigma |\Delta x|.$$

But the sum on the right is the variation of ϕ for the division D . If now ϕ does not have limited variation in \mathfrak{A} , the limit 3) does not exist. The same holds for ψ . Hence limited variation is a necessary condition.

The condition is *sufficient*. For

$$P_D \leq \Sigma |\Delta x| + \Sigma |\Delta y| = \text{Var}_D \phi + \text{Var}_D \psi.$$

As ϕ, ψ have limited variation, this shows that

$$P_0 = \text{Max}_D P_D$$

is finite. We show now that

$$\lim_{d=0} P_D = P_0. \quad (4)$$

For there exists a division Δ such that

$$P_0 - \frac{\epsilon}{2} < P_\Delta \leq P_0. \quad (5)$$

Let Δ cause \mathfrak{A} to fall into ν intervals, the smallest of which has the length λ . Let D be a division of \mathfrak{A} of norm $d \leq d_0 < \lambda$. Then no interval of D contains more than one point of Δ . Let $E = D + \Delta$.

Obviously

$$P_E \geq P_D \quad \text{or} \quad P_\Delta.$$

Suppose that the point t_κ of Δ falls in the interval $(t_\iota, t_{\iota+1})$ of D . Then the chord $(\iota, \iota+1)$ in P_D is replaced by the two chords (ι, κ) , $(\kappa, \iota+1)$ in P_E . Hence

$$P_E - P_D = \Sigma G_\kappa, \quad \kappa = 1, 2 \dots \mu \leq \nu$$

where

$$G_\kappa = (\iota, \kappa) + (\kappa, \iota+1) - (\iota, \iota+1).$$

Obviously as ϕ, ψ are continuous we may take d_0 so small that each

$$G_\kappa < \frac{\epsilon}{2\nu}, \quad \text{for any } d \leq d_0.$$

Hence

$$P_E - P_D < \frac{\epsilon}{2}. \quad (6)$$

From 5), 6) we have

$$P_0 - P_D < \epsilon, \quad \text{for any } d \leq d_0,$$

which gives 4).

3. *If the arc PQ is rectifiable, any arc contained in PQ is also rectifiable.*

For ϕ, ψ having limited variation in interval \mathfrak{A} , have *a fortiori* limited variation in any segment of \mathfrak{A} .

4. *Let the rectifiable arc C fall into two arcs C_1, C_2 . If s, s_1, s_2 are the lengths of C, C_1, C_2 , then*

$$s = s_1 + s_2. \quad (7)$$

For we saw that C_1, C_2 are rectifiable since C is. Let $\mathfrak{A}_1, \mathfrak{A}_2$ be the intervals in \mathfrak{A} corresponding to C_1, C_2 . Let D_1, D_2 be divisions of $\mathfrak{A}_1, \mathfrak{A}_2$ of norm d . Then

$$s_1 = \lim_{d \rightarrow 0} P_{D_1}, \quad s_2 = \lim_{d \rightarrow 0} P_{D_2}.$$

But D_1, D_2 effect a division of \mathfrak{A} , and since

$$s = \lim_{\epsilon \rightarrow 0} P_E \quad (8)$$

with respect to the class of all divisions of \mathfrak{A} , the limit 8) is the same when E is restricted to range over divisions of the type of D . Now

$$P_D = P_{D_1} + P_{D_2}.$$

Passing to the limit, we get 7).

The preceding reasoning also shows that if C_1, C_2 are rectifiable curves, then C is, and 7) holds again.

5. If 1) define a rectifiable curve, its length s is a continuous function $s(t)$ of t .

For ϕ, ψ having limited variation,

$$\phi = \phi_1 - \phi_2, \quad \psi = \psi_1 - \psi_2,$$

where the functions on the right are continuous monotone increasing functions of t in the interval $\mathfrak{A} = (a < b)$.

For a division D of norm d of the interval $\Delta\mathfrak{A} = (t, t+h)$ we have

$$\begin{aligned} P_D &= \sum \sqrt{\Delta x^2 + \Delta y^2} \\ &\leq \sum |\Delta x| + \sum |\Delta y| \\ &\leq \sum \Delta \phi_1 + \sum \Delta \phi_2 + \sum \Delta \psi_1 + \sum \Delta \psi_2 \\ &\leq \delta \phi_1 + \delta \phi_2 + \delta \psi_1 + \delta \psi_2, \end{aligned}$$

where $\delta \phi_1 = \phi_1(t+h) - \phi_1(t)$, and similarly for the other functions. As ϕ_1 is continuous, $\delta \phi_1 \rightarrow 0$, etc., as $h \rightarrow 0$. We may therefore take $\eta > 0$ so small that $\delta \phi_1, \delta \phi_2, \delta \psi_1, \delta \psi_2 < \epsilon/4$, if $h < \eta$. Hence $\Delta s = s(t+h) - s(t) \leq \text{Max } P_D < \epsilon$, if $0 < h < \eta$.

Thus s is continuous.

6. The length s of the rectifiable arc C corresponding to the interval $\mathfrak{A} (a < t)$ is a monotone increasing function of t .

This follows from 4.

7. If x, y do not have simultaneous intervals of invariability, $s(t)$ is an increasing function of t . The inverse function is one-valued and increasing and the coördinates x, y are one-valued functions of s .

That the inverse function $t(s)$ is one-valued follows from I, 214. We can thus express t in terms of s , and so eliminate t in 1).

570. 1. If ϕ', ψ' are continuous in the interval \mathfrak{A} ,

$$s = \int_{\mathfrak{A}} dt \sqrt{\phi'^2 + \psi'^2}. \quad (1)$$

For

$$s = \lim_{d \rightarrow 0} \sum \sqrt{\Delta \phi_k^2 + \Delta \psi_k^2}. \quad (2)$$

Now

$$\Delta\phi_\kappa = \phi'(t'_\kappa)\Delta t_\kappa, \quad \Delta\psi_\kappa = \psi'(t''_\kappa)\Delta t_\kappa \quad (3)$$

where t'_κ, t''_κ lie in the interval Δt_κ .

As ϕ', ψ' are continuous they are uniformly continuous. Hence for any division D of norm \leq some d_0 ,

$$\phi'(t'_\kappa) = \phi'(t_\kappa) + \alpha_\kappa, \quad \psi'(t''_\kappa) = \psi'(t_\kappa) + \beta_\kappa$$

where $|\alpha_\kappa|, |\beta_\kappa| <$ some η , small at pleasure, for any κ . Thus

$$\sqrt{\Delta\phi_\kappa^2 + \Delta\psi_\kappa^2} = \Delta t_\kappa \sqrt{\phi'(t_\kappa)^2 + \psi'(t_\kappa)^2} + \epsilon_\kappa \Delta t_\kappa,$$

and we may take

$$|\epsilon_\kappa| < \epsilon/\mathfrak{N}, \quad \kappa = 1, 2, \dots$$

Thus

$$s = \lim_{d \rightarrow 0} \Sigma \Delta t_\kappa \sqrt{\phi'(t_\kappa)^2 + \psi'(t_\kappa)^2} + \lim \Sigma \epsilon_\kappa \Delta t_\kappa.$$

Hence

$$\left| s - \int_{\mathfrak{N}} \right| < \epsilon,$$

which establishes 1).

For simplicity we have assumed ϕ', ψ' to be continuous in \mathfrak{N} . This is not necessary, as the following shows.

2. Let $a_1, \dots, a_n, b_1, \dots, b_n \geq 0$ but not all $= 0$.

Then

$$|\sqrt{a_1^2 + \dots + a_n^2} - \sqrt{b_1^2 + \dots + b_n^2}| \leq \sum_m |a_m - b_m|,$$

$$m = 1, 2, \dots, n. \quad (4)$$

For

$$\begin{aligned} & (\sqrt{a_1^2 + \dots} - \sqrt{b_1^2 + \dots})(\sqrt{a_1^2 + \dots} + \sqrt{b_1^2 + \dots}) \\ &= (a_1^2 + \dots + a_n^2) - (b_1^2 + \dots + b_n^2) \\ &= (a_1^2 - b_1^2) + \dots + (a_n^2 - b_n^2) \\ &= (a_1 - b_1)(a_1 + b_1) + \dots + (a_n - b_n)(a_n + b_n). \end{aligned}$$

Hence

$$\sqrt{a_1^2 + \dots} - \sqrt{b_1^2 + \dots} = \sum_{m=1}^n (a_m - b_m) \frac{a_m + b_m}{\sqrt{a_1^2 + \dots} + \sqrt{b_1^2 + \dots}} \quad (5)$$

But

$$\left| \frac{a_m + b_m}{\sqrt{a_1^2 + \dots} + \sqrt{b_1^2 + \dots}} \right| \leq 1.$$

This in 5) gives 4).

Let us apply 4) to prove the following theorem, more general than 1.

3. (Baire.) If ϕ' , ψ' are limited and \mathcal{R} -integrable, then

$$s = \int_{\mathfrak{A}} \sqrt{\phi'^2 + \psi'^2} dt. \quad (1)$$

For by 4),

$$\begin{aligned} |\sqrt{\phi'(t'_\kappa)^2 + \psi'(t''_\kappa)^2} - \sqrt{\phi'(t_\kappa)^2 + \psi'(t_\kappa)^2}| &\leq |\phi'(t'_\kappa) - \phi'(t_\kappa)| \\ &\quad + |\psi'(t''_\kappa) - \psi'(t_\kappa)|; \end{aligned}$$

$$\Phi_\kappa - \Psi_\kappa = \eta'_\kappa \text{Osc } \phi'(t) + \eta''_\kappa \text{Osc } \psi'(t) \quad , \quad \text{in } \delta_\kappa = \Delta t_\kappa,$$

where η'_κ , η''_κ are numerically ≤ 1 . Thus

$$|\Sigma \delta_\kappa \Phi_\kappa - \Sigma \delta_\kappa \Psi_\kappa| = \Sigma \delta_\kappa \eta'_\kappa \text{Osc } \phi' + \Sigma \delta_\kappa \eta''_\kappa \text{Osc } \psi'. \quad (6)$$

As ϕ' , ψ' are integrable, the right side $\doteq 0$, as $d \doteq 0$. Now

$$\lim_{d \doteq 0} \Sigma \delta_\kappa \Psi_\kappa = \int_{\mathfrak{A}} \sqrt{\phi'^2 + \psi'^2} dt.$$

Thus passing to the limit in 6), we have

$$\lim \Sigma \Delta t_\kappa \sqrt{\phi'(t'_\kappa)^2 + \psi'(t''_\kappa)^2} = \int_{\mathfrak{A}}.$$

This with 2), 3) gives 1) at once.

571. Volterra's Curve. It is interesting to note that *there are rectifiable curves for which $\phi'(t)$, $\psi'(t)$ are not both \mathcal{R} -integrable.* Such a curve is Volterra's curve, discussed in 503. Let its equation be $y = f(x)$. Then $f'(x)$ behaves as

$$2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

in the vicinity of a non null set in $\mathfrak{A} = (0, 1)$. Hence $f'(x)$ is not \mathcal{R} -integrable in \mathfrak{A} . But then it is easy to show that

$$\int_0^1 \sqrt{1 + f'(x)^2} dx$$

does not exist. For suppose that

$$g = \sqrt{1 + f'(x)^2}$$

were R -integrable. Then $g^2 = 1 + f'(x)^2$ is R -integrable, and hence $f'(x)^2$ also. But the points of discontinuity of f'^2 in \mathfrak{A} do not form a null set. Hence f'^2 is not R -integrable.

On the other hand, Volterra's curve is rectifiable by 569, 2, and 528, 1.

572. Taking the definition of length given in 569, 1, we saw that the coördinates

$$x = \phi(t) \quad , \quad y = \psi(t)$$

must have limited variation for the curve to be rectifiable. But we have had many examples of functions not having limited variation in an interval \mathfrak{A} . Thus the curve defined by

$$\begin{aligned} y &= x \sin \frac{1}{x} \quad , \quad x \neq 0 \\ &= 0 \quad , \quad x = 0 \end{aligned} \tag{4}$$

does not have a length in $\mathfrak{A} = (-1, 1)$; while

$$\begin{aligned} y &= x^2 \sin \frac{1}{x} \quad , \quad x \neq 0 \\ &= 0 \quad , \quad x = 0 \end{aligned} \tag{5}$$

does.

It certainly astonishes the naïve intuition to learn that the curve 4) has no length in any interval δ about the origin however small, or if we like, that this length is infinite, however small δ is taken. For the same reason we see that

No arc of Weierstrass' curve has a length (or its length is infinite) however near the end points are taken to each other, when $ab > 1$.

573. 1. 6° *Property. Space-filling Curves.* We wish now to exhibit a curve which passes through every point of a square, *i.e.* which completely fills a square. Having seen how to define one such curve, it is easy to construct such curves in great variety, not only for the plane but for space. The first to show how this may be done was *Peano* in 1890. The curve we wish now to define is due to *Hilbert*.

We start with a unit interval $\mathfrak{A} = (0, 1)$ over which t ranges, and a unit square \mathfrak{B} over which the point x, y ranges. We define

$$x = \phi(t) \quad , \quad y = \psi(t) \tag{1}$$

one-valued continuous functions of t in \mathfrak{A} so that xy ranges over \mathfrak{B} as t ranges over \mathfrak{A} . The analytic curve C defined by 1) thus completely fills the square \mathfrak{B} .

We do this as follows. We effect a division of \mathfrak{A} into four equal segments $\delta'_1, \delta'_2, \delta'_3, \delta'_4$, and of \mathfrak{B} into equal squares $\eta'_1, \eta'_2, \eta'_3, \eta'_4$, as in Fig. 1.

We call this the first division or D_1 . The correspondence between \mathfrak{A} and \mathfrak{B} is given in *first approximation* by saying that to each point P in \mathfrak{A} shall correspond some point Q in η'_i .

We now effect a second division D_2 by dividing each interval and square of D_1 into four equal parts.

We number them as in Fig. 2,

$$\begin{array}{cccc} \delta''_1 & , & \delta''_2 & \cdots & \delta''_{16} \\ \eta''_1 & , & \eta''_2 & \cdots & \eta''_{16} \end{array}$$

As to the numbering of the η 's we observe the following two principles: 1° we may pass over the squares 1 to 16 continuously without passing the same square twice, and 2° in doing this we pass over the squares of D_1 in the same order as in Fig. 1. The correspondence between \mathfrak{A} and \mathfrak{B} is given in *second approximation* by saying that to each point P in \mathfrak{A} shall correspond some point Q in η''_i . In this way we continue indefinitely.

To find the point Q in \mathfrak{B} corresponding to P in \mathfrak{A} we observe that P lies in a sequence of intervals

$$\delta' > \delta'' > \delta''' > \cdots \doteq 0, \quad (2)$$

to which correspond uniquely a sequence of squares

$$\eta' > \eta'' > \eta''' > \cdots \doteq 0. \quad (3)$$

The sequence 3) determines *uniquely* a point whose coördinates are one-valued functions of t , viz. the functions given in 1).

The functions 1) are continuous in \mathfrak{A} .

For let t' be a point near t ; it either lies in the same interval as t in D_n or in the adjacent interval. Thus the point Q' corre-

| | |
|---|---|
| 2 | 3 |
| 1 | 4 |

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
|---|---|---|---|

Fig. 1.

| | | | |
|---|---|----|----|
| 6 | 7 | 10 | 11 |
| 5 | 2 | 9 | 12 |
| 4 | 3 | 14 | 13 |
| 1 | 8 | 15 | 16 |

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
|---|---|---|---|

Fig. 2.

sponding to t' either lies in the same square of D_n as the point Q corresponding to t , or in an adjacent square. But the diagonal of the squares $\doteq 0$, as $n \doteq \infty$. Thus

$$\text{Dist } (Q'Q) \doteq 0, \quad \text{as } n \doteq \infty.$$

Thus

$$\phi(t') - \phi(t), \quad \text{and } \psi(t') - \psi(t)$$

both $\doteq 0$, as $t' \doteq t$.

As t ranges over \mathfrak{A} , the point x, y ranges over every point in the square \mathfrak{B} .

For let Q be a given point of \mathfrak{B} . It lies in a sequence of squares as \mathfrak{B}). If Q lies on a side or at a vertex of one of the η squares, there is more than one such sequence. But having taken such a sequence, the corresponding sequence 2) is uniquely determined. Thus to each Q corresponds at least one P . A more careful analysis shows that to a given Q never more than four points P can correspond.

2. The method we have used here may obviously be extended to space. By passing median planes through a unit cube we divide it into 2^3 equal cubes. Thus to get our correspondence each division D_n should divide each interval and cube of the preceding division D_{n-1} into 2^3 equal parts. The cubes of each division should be numbered according to the 1° and 2° principles of enumeration mentioned in 1.

By this process we define

$$x = \phi_1(t), \quad y = \phi_2(t), \quad z = \phi_3(t)$$

as one-valued continuous functions of t such that as t ranges over the unit interval $(0, 1)$, the point x, y, z ranges over the unit cube.

574. 1. *Hilbert's Curve.* We wish now to study in detail the correspondence between the unit interval \mathfrak{A} and the unit square \mathfrak{B} afforded by Hilbert's curve defined in 573. A number of interesting facts will reward our labor. We begin by seeking the points P in \mathfrak{A} which correspond to a given Q in \mathfrak{B} .

To this end let us note how P enters and leaves an η square. Let B be a square of D_n . In the next division B falls into four

squares $B_1 \dots B_4$ and in the $n + 2^d$ division in 16 squares B_{ij} . These last, four lie at the vertices of B ; we call them *vertex squares*. The other 12 are *median squares*. A simple consideration shows that the η squares of D_{n+2} are so numbered that we always enter a square B belonging to D_n , and also leave it by a vertex square.

Since this is true of every division, we see on passing to the next that the point Q enters and leaves any η square at the vertices of η . We call this the *vertex law*.

Let us now *classify the points* P, Q .

P is an end point of some division D_n ; we call it a *terminal point*, otherwise an *inner point*, because it lies within a sequence of intervals $\delta' > \delta'' > \dots \doteq 0$.

The points Q we divide into four classes:

1° *vertex points*, when Q is a vertex of some division.

2° *inner points*, when Q lies within a sequence of squares

$$\eta' > \eta'' > \dots \doteq 0.$$

3° *lateral points*, when Q lies on a side of some η square but not at a vertex.

4° *points lying on the edge of the original square* \mathfrak{B} . Points of this class also lie in 1°, 3°.

We now seek the points P corresponding to a Q lying in one of the four classes.

Class 1°. Q a Vertex Point. Let D_n be the first division such that Q is at a vertex. Then Q lies in four squares $\eta_i, \eta_j, \eta_k, \eta_l$ of

there are 5 cases:

- a) i, j, k, l are consecutive.
- b) i, j, k are consecutive, but not l .
- c) i, j are consecutive, but not k, l .
- d) i, j , also k, l , are consecutive.
- e) no two are consecutive.

A simple analysis shows that a), b) are not permanent in the following divisions; c), d) may or may not be permanent; e) is permanent.

Now, whenever a case is permanent, we can enclose Q in a sequence of η squares whose sides $\doteq 0$. To this sequence corresponds uniquely a sequence of δ intervals of lengths $\doteq 0$. Thus to two consecutive squares will correspond two consecutive intervals which converge to a single point P in \mathfrak{A} . If the squares are not consecutive, the corresponding intervals converge to two distinct points in \mathfrak{A} . Thus we see that when γ) is permanent, to Q correspond three points P . When δ) is permanent, to Q correspond two points P . While when Q belongs to ϵ), four points P correspond to it.

Class 2°. Q an Inner Point. Obviously to each Q corresponds one point P and only one.

Class 3°. Q a Lateral Point. To fix the ideas let Q lie on a vertical side of one of the η 's. Let it lie between η_i, η_j of D_n . There are two cases:

$$\alpha) \quad j = i + 1.$$

$$\beta) \quad j > i + 1.$$

We see easily that α) is not permanent, while of course β) is. Thus to each Q in class 3°, there correspond two points P .

Class 4°. Q lies on the edge of \mathfrak{B} . If Q is a vertex point, to it may correspond one or two points P . If Q is not a vertex point, only one point P corresponds to it.

To sum up we may say:

To each inner point Q corresponds one inner point P .

To each lateral point Q correspond two points P .

To each edge point Q correspond one or two points P .

To each vertex point Q , correspond two, three, or four points P .

2. As a result of the preceding investigation we show easily that:

To the points on a line parallel to one of the sides of \mathfrak{B} correspond in \mathfrak{A} an apantactic perfect set.

3. Let us now consider the tangents to Hilbert's curve which we denote by H .

Let Q be a vertex point. We saw there were three permanent cases γ), δ), ϵ).

In cases γ), δ) we saw that to two consecutive δ intervals correspond permanently two contiguous vertical or horizontal squares.

Thus as t ranges over $\underbrace{\hspace{1.5cm}}_{\delta_i \quad \delta_{i+1}}^P$ Q

| |
|--------------|
| η_i |
| η_{i+1} |

| | |
|----------|--------------|
| η_i | η_{i+1} |
|----------|--------------|

 δ_i, δ_{i+1} , the point x, y ranges over these squares, and the secant line joining Q and this variable point x, y oscillates through 180° . There is thus no tangent at Q . In case ϵ) we see similarly that the secant line ranges through 90° . Again there is no tangent at Q .

In the same way we may treat the three other classes. We find that the secant line never converges to a fixed position, and may oscillate through 360° , viz. when Q is an inner point. As a result we see that *Hilbert's curve has at no point a tangent, nor even a unilateral tangent.*

4. Associated with Hilbert's curve H are two other curves,

$$x = \phi(t) \quad , \quad \text{and} \quad y = \psi(t).$$

The functions ϕ, ψ being one-valued and continuous in \mathfrak{A} , these curves are continuous and they do not have a multiple point. A very simple consideration shows that *they do not have even a unilateral tangent at a pantactic set of points in \mathfrak{A} .*

575. *Property 7°. Equations of a Curve.* As already remarked, it is commonly thought that the equation of a curve may be written in any one of the three forms

$$y = f(x), \tag{1}$$

$$\Phi(x, y) = 0, \tag{2}$$

$$x = \phi(t) \quad , \quad y = \psi(t), \tag{3}$$

and if these functions are continuous, these equations define continuous curves.

Let us look at the Hilbert curve H . We saw its equation could be expressed in the form 3). H cuts an ordinate at every point of it for which $0 \leq y \leq 1$. Thus if we tried to define H by

an equation of the type 1), $f(x)$ would have to take on every value between 0 and 1 for each value of x in $\mathfrak{A} = (0, 1)$. No such functions are considered in analysis.

Again, we saw that to any value $x = a$ in \mathfrak{A} corresponds a perfect apantactic set of values $\{t_a\}$ having the cardinal number c . Thus the inverse function of $x = \phi(t)$ is a many-valued function of x whose different values form a set whose cardinal number is c . Such functions have not yet been studied in analysis.

How is it possible in the light of such facts to say that we may pass from 3) to 1) or 2) by *eliminating* t from 3). And if we cannot, how can we say a curve can be represented equally well by any of the above three equations, or if the curve is given by one of these three equations, we may suppose it replaced by one of the other two whenever convenient. Yet this is often done.

In this connection we may call attention to the loose way *elimination* is treated. Suppose we have a set of equations

$$f_1(x_1 \cdots x_m t_1 \cdots t_n) = 0,$$

$$f_{n+1}(x_1 \cdots x_m t_1 \cdots t_n) = 0.$$

We often see it stated that one can eliminate $t_1 \cdots t_n$ and obtain a relation involving the x 's alone. Any reasoning based on such a procedure must be regarded as highly unsatisfactory, in view of what we have just seen, until this elimination process has been established.

576. Property 8°. Closed Curves. A circle, a rectangle, an ellipse are examples of closed curves. Our intuition tells us that it is impossible to pass from the inside to the outside without crossing the curve itself. If we adopt the definition of a closed curve without multiple point given in I, 362, we find it no easy matter to establish this property which is so obvious for the simple closed curves of our daily experience. The first to effect the demonstration was *Jordan* in 1892. We give here * a proof due to *de la Vallée-Poussin*.†

Let us call for brevity a continuous curve without double point

* The reader is referred to a second proof due to Brouwer and given in 508 seq.

† *Cours d'Analyse*, Paris, 1903, Vol. 1, p. 307.

Jordan curve. A continuous closed curve without double point will then be a *closed Jordan curve*. Cf. I, 362.

577. Let C be a closed Jordan curve. However small $\sigma > 0$ is chosen, there exists a polygonal ring R containing C and such that

1° Each point of R is at a distance $< \sigma$ from C .

2° Each point of C is at a distance $< \sigma$ from the edges of R .

For let $x = \phi(t)$, $y = \psi(t)$ (1

be continuous one-valued functions of t in $T = (a < b)$ defining C . Let $D = (a, a_1, a_2 \dots b)$ be a division of T of norm d . Let $\alpha_1, \alpha_2 \dots$ be points of C corresponding to $a, a_1 \dots$. If d is sufficiently small, the distance between two points on the arc $C_i = (\alpha_{i-1}, \alpha_i)$ is $< \epsilon'$, small at pleasure. Let Δ be a quadrangle division of the x, y plane of norm δ . Let us shade all cells containing a point of C_i . These form a connected domain since C_i is continuous. We can thus go around its outer edge without a break.* If this shaded domain contains unshaded cells, let us shade these too. We call the result a link A_i . It has only one edge E_i , and the distance between any two points of E_i is obviously $< \epsilon' + 2\sqrt{2}\delta$. We can choose d, δ so small that

$$\epsilon' + 2\sqrt{2}\delta < \sigma, \quad \text{arbitrarily small.} \quad (1)$$

Then the distance between any two points of A_i is $< \sigma$. Let ϵ'' be the least distance between non-consecutive arcs C_i . We take δ so small that we also have

$$\sqrt{2}\delta < \frac{\epsilon''}{2}. \quad (2)$$

Then two non-consecutive links A_i, A_j have no point in common. For then their edges would have a common point P . As P lies on E_i its distance from C_i is $\leq \sqrt{2}\delta$. Its distance from C_j is also $\leq \sqrt{2}\delta$. Thus there is a point P_i on C_i , and a point P_j on C_j such that

$$\eta = \overline{P_i P_j} \leq 2\sqrt{2}\delta.$$

* Here and in the following, intuitional properties of polygons are assumed as known.

But by hypothesis $\epsilon'' < \eta$. Hence

$$\epsilon'' \leq 2\sqrt{2}\delta,$$

which contradicts 2).

Thus the union of these links form a ring R whose edges are polygons without double point. One of the edges, say G_i , lies within the other, which we call G_e . The curve C lies within R . The inner polygon G_i must exist, since non-consecutive links have no point in common.

578. 1. *Interior and Exterior Points.* Let $\sigma_1 > \sigma_2 > \dots \doteq 0$. Let $R_1, R_2 \dots$ be the corresponding rings, and let

$$G_i', \quad G_i'' \dots$$

$$G_e', \quad G_e'' \dots$$

be their inner and outer edges. A point P of the plane not on C which lies inside some G_i we call an *interior or inner point of C* . If P lies outside some G_e , we call it an *exterior or outer point of C* .

Each point P not on C must belong to one of these two classes. For let $\rho = \text{Dist}(P, C)$; then ρ is $>$ some σ_n . It therefore lies within $G_i^{(n)}$ or without $G_e^{(n)}$, and is thus an inner or an outer point. Obviously this definition is independent of the sequence of rings $\{R_n\}$ employed. The points of the curve C are interior to each $G_e^{(n)}$ and exterior to each $G_i^{(n)}$.

Inner points must exist, since the inner polygons exist as already observed. Let us denote the inner points by \mathfrak{I} and the outer points by \mathfrak{D} . Then the frontiers of \mathfrak{I} and \mathfrak{D} are the curve C .

2. We show now that

1° *Two inner points can be joined by a broken line L_i lying in \mathfrak{I} .*

2° *Two outer points can be joined by a broken line L_e lying in \mathfrak{D} .*

3° *Any continuous curve \mathfrak{R} joining an inner point i and an outer point e has a point in common with C .*

To prove 3°, let

$$x=f(t) \quad , \quad y=g(t)$$

be the equations of \mathfrak{R} , the variable t ranging over an interval $\tau = (\alpha < \beta)$, $t = \alpha$ corresponding to i and $t = \beta$ to e . Let t' be

such that $\alpha \leq t < t'$ gives inner points, while $t = t'$ does not give an inner point. Thus the point corresponding to $t = t'$ is a frontier point of \mathfrak{S} and hence a point of \mathcal{C} .

To prove 1°. If A, B are inner points, they lie within some G_i . We may join A, B, G_i by broken lines L_a, L_b meeting G_i at the points A', B' , say. Let G_{ab} be the part of G_i lying between A', B' . Then

$$L_a + G_{ab} + L_b$$

is a broken line joining A to B .

The proof of 2° is similar.

579. 1. Let P', P'' correspond to $t = t', t = t''$, on the curve \mathcal{C} defined by 577, 1). If $t' < t''$, we say P' precedes P'' and write $P' < P''$.

Any set of points on \mathcal{C} corresponding to an increasing set of values of t is called an *increasing set*.

As t ranges from a to b , the point P ranges over \mathcal{C} in a *direct sense*.

We may thus consider a Jordan curve as an ordered set, in the sense of 265.

2. (De la Vallée-Poussin.) On each arc \mathcal{C}_i of the curve \mathcal{C} , there exists at least one point P_i such that

$$P_1 < P_2 < P_3 < \dots \quad (1)$$

may be regarded as the vertices of a closed polygon without double point and whose sides are all $< \epsilon$.

For in the first place we may take $\delta > 0$ so small that no square of Δ contains a point lying on non-consecutive arcs \mathcal{C}_i of \mathcal{C} . Let us also take Δ so that the point α corresponding to $t = a$ lies within a square, call it S_1 , of Δ . As t increases from $t = a$, there is a last point P_1 on \mathcal{C} where the curve leaves S_1 . The point P_1 lies in another square of Δ , call it S_2 , containing other points of \mathcal{C} . Let P_2 be the last point of \mathcal{C} in S_2 . In this way we may continue, getting a sequence 1).

There exists at least one point of 1) on each arc \mathcal{C}_i . For otherwise a square of Δ would contain points lying on non-consecutive arcs \mathcal{C}_k . The polygon determined by 1) cannot have a double

point, since each side of it lies in one square. The sides are $< \epsilon$, provided we take $\delta\sqrt{2} < \epsilon$, since the diagonal is the longest line we can draw in a square of side δ .

580. Existence of Inner Points. To show that the links form a ring with inner points, *Schönflies** has given a proof which may be rendered as follows:

Let us take the number of links to be even, and call them L_1, L_2, \dots, L_{2n} . Then $L_1, L_3, L_5 \dots$ lie entirely outside each other. Since L_1, L_2 overlap, let P be an inner common point. Similarly let Q be an inner common point of L_2, L_3 . Then P, Q lying within L_2 may be joined by a finite broken line b lying within L_2 . Let b_2 be that part of it lying between the last point of leaving L_1 and the following point of meeting L_3 . In this way the pairs of links

$$L_1 L_3 \quad ; \quad L_3 L_5 \quad ; \dots$$

define finite broken lines

$$b_2 \quad , \quad b_4 \quad , \quad \dots \quad b_{2n}.$$

No two of these can have a common point, since they lie in non-consecutive links. The union of the points in the sets

$$L_1 \quad , \quad b_2 \quad , \quad L_3 \quad , \quad b_4 \dots L_{2n-1} \quad , \quad b_{2n}$$

we call a *ring*, and denote it by \mathfrak{R} . The points of the plane not in \mathfrak{R} fall into two parts, *separated* by \mathfrak{R} . Let \mathfrak{T} denote the part which is limited, together with its frontier. We call \mathfrak{T} the *interior* of \mathfrak{R} . That \mathfrak{T} has *inner points* is regarded as obvious since it is defined by the links

$$L_1 \quad , \quad L_3 \quad , \quad L_5 \dots$$

which pairwise have no point in common, and by the broken lines

$$b_2 \quad , \quad b_4 \quad , \quad b_6 \dots$$

each of which latter lies entirely *within* a link.

$$\text{Let} \quad \mathfrak{R}_{2m} = Dv(L_{2m}, \mathfrak{T}) \quad , \quad m = 1, 2, \dots$$

* *Die Entwicklung der Lehre von den Punktmannigfaltigkeiten.* Leipzig, 1908, Part 2, p. 170.

Then these \mathfrak{L} have pairwise no point in common since the L_{2m} have not.

Let $\mathfrak{T} = \mathfrak{L}_2 + \mathfrak{L}_4 + \dots + \mathfrak{L}_{2n} + \mathfrak{L}$.

Then $\mathfrak{R} > 0$. For let us adjoin L_2 to \mathfrak{R} , getting a ring \mathfrak{R}_2 whose interior call \mathfrak{T}_2 . That \mathfrak{T}_2 has inner points follows from the fact that it contains $\mathfrak{L}_4, \mathfrak{L}_6 \dots$. Let us continue adjoining the links $L_4, L_6 \dots$. Finally we reach L_{2n} , to which corresponds the ring \mathfrak{R}_{2n} , whose interior, if it exists, is \mathfrak{T}_{2n} . If \mathfrak{T}_{2n} does not exist, \mathfrak{T}_{2n-2} contains only \mathfrak{L}_{2n} . This is not so, for on the edge of L_1 bounding \mathfrak{T} , is a point P , such that some $D_\rho(P)$ contains points of no L except L_1 . In fact there is a point P on the edge of L_1 not in either L_2 or L_{2n} , as otherwise these would have a point in common. Now, if however small $\rho > 0$ is taken, $D_\rho(P)$ contains points of some L other than L_1 , the point P must lie in L_k which is absurd, since L_1 has only points in common with L_2, L_{2n} , and P is not in either of these. Thus the adjunction of L_2, L_4, \dots, L_{2n} produces a ring \mathfrak{R}_{2n} whose interior \mathfrak{T}_{2n} does not reduce to 0; it has inner points.

581. Property 9°. Area. That a figure defined by a closed curve without double point, i.e. the interior of a Jordan curve, has an area, has long been an accepted fact in intuitional geometry. Thus Lindemann, *Vorlesungen über Geometrie*, vol. 2, p. 557, says "einer allseitig umgrenzten Figur kommt ein bestimmter Flächeninhalt zu." The truth of such a statement rests of course on the definition of the term area. In I, 487, 702 we have given a definition of area for any limited plane point set \mathfrak{A} which reduces to the ordinary definition when \mathfrak{A} becomes an ordinary plane figure. In our language \mathfrak{A} has an area when its frontier points form a discrete set. Let

$$x = \phi(t) \quad , \quad y = \psi(t)$$

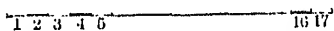
define a Jordan curve \mathfrak{C} , as t ranges over $T = (a < b)$. The figure \mathfrak{A} defined by this curve has the curve as frontier. In I, 708, 710, we gave various cases in which \mathfrak{C} is discrete. The reasoning of I, 710, gives us also this important case:

If one of the continuous functions ϕ, ψ defining the Jordan curve \mathfrak{C} , has limited variation in T , then \mathfrak{C} is discrete.

It was not known whether \mathfrak{C} would remain discrete if the condition of limited variation was removed from both coördinates, until Osgood* exhibited a Jordan curve which is not discrete. This we will now discuss.

582. 1. Osgood's Curve. We start with a unit segment $T = (0, 1)$ on the t axis, and a unit square S in the xy plane.

We divide T into 17 equal parts



$$T_1, T_2, \dots T_{17}, \quad (1)$$

and the square S into 9 equal squares

$$S_1, S_2, S_3 \dots S_{17}, \quad (2)$$

by drawing 4 bands B_1 which are shaded in the figure. On these bands we take 8 segments,

$$s_2, s_4, s_6 \dots s_{16}, \quad (3)$$

marked heavy in the figure.

Then as t is ranging from left to right over the even or black intervals $T_2, T_4, \dots T_{16}$ marked heavy in the figure, the point x, y on Osgood's curve, call it \mathfrak{D} , shall range univariantly over the segments 3).

While t is ranging over the odd or white intervals $T_1, T_3 \dots T_{17}$ the point xy on \mathfrak{D} shall range over the squares 2) as determined below.

Each of the odd intervals 1) we will now divide into 17 equal intervals T_{ij} and in each of the squares 2) we will construct horizontal and vertical bands B_2 as we did in the original square S . Thus each square 2) gives rise to 8 new segments on \mathfrak{D} corresponding to the new black intervals in T , and 9 new squares S_{ij} corresponding to the white intervals. In this way we may continue indefinitely.

The points which finally get in a black interval call β , the others are limit points of the β 's and we call them λ . The point

* *Trans. Am. Math. Soc.*, vol. 4 (1903), p. 107.

on \mathfrak{D} corresponding to a β point has been defined. The point of \mathfrak{D} corresponding to a point λ is defined to be the point lying in the sequence of squares, one inside the other, corresponding to the sequence of white intervals, one inside the other, in which λ falls, in the successive divisions of T .

Thus to each t in T corresponds a single point x, y in S . The aggregate of these points constitutes Osgood's curve. Obviously the x, y of one of its points are one-valued functions of t in T , say

$$x = \phi(t) \quad , \quad y = \psi(t). \quad (4)$$

The curve \mathfrak{D} has no double point. This is obvious for points of \mathfrak{D} lying in black segments. Any other point falls in a sequence of squares

$$S_i > S_{ij} > S_{ijk} \dots$$

to which correspond intervals

$$T_i > T_{ij} > T_{ijk} \dots$$

in which the corresponding t 's lie. But only one point t is thus determined.

The functions 4) are continuous. This is obvious for points β lying within the black intervals of T . It is true for the points λ . For λ lies within a sequence of white intervals, and while t ranges over one of these, the point on \mathfrak{D} ranges in a square. But these squares shut down to a point as the intervals do. Thus ϕ, ψ are continuous at $t = \lambda$. In a similar manner we show they are continuous at the end points of the black intervals.

We note that to $t = 0$ corresponds the upper left-hand corner of S , and to $t = 1$, the diagonally opposite point.

2. Up to the present we have said nothing as to the width of the shaded bands

$$B_1 \quad , \quad B_2 \dots$$

introduced in the successive steps. Let

$$A = a_1 + a_2 + \dots$$

be a convergent positive term series whose sum $A \leq 1$. We choose B_1 so that its area is a_1 , B_2 so that its area is a_2 , etc. Then

$$\underline{\mathfrak{D}} = 0 \quad , \quad \bar{\mathfrak{D}} = 1 - A, \quad (5)$$

as we now show. For \mathfrak{D} has obviously only frontier points; hence $\underline{\mathfrak{D}} = 0$. Since \mathfrak{D} is complete, it is measurable and

$$\bar{\mathfrak{D}} = \overline{\mathfrak{D}}.$$

Let $O = S - \mathfrak{D}$, and $B = \{B_n\}$. Then $O < B$. For any point which does not lie in some B_n lies in a sequence of convergent squares $S_i > S_{i,j} > \dots$ which converge to a point of \mathfrak{D} . Now

$$\bar{B} = \bar{B}_1 + \bar{B}_2 + \dots = A.$$

On the other hand, B contains a null set of points of \mathfrak{D} , viz. the black segments. Thus

$$\bar{O} = \bar{B} = A, \quad \text{and hence } \bar{\mathfrak{D}} = 1 - A$$

and 5) is established.

Thus *Osgood's curve is continuous, has no double point, and its upper content is $1 - A$.*

3. To get a continuous closed curve C without double point we have merely to join the two end points α, β of Osgood's curve by a broken line which does not cut itself or have a point in common with the square S except of course the end points α, β . Then C bounds a figure \mathfrak{F} whose frontier is not discrete, and \mathfrak{F} does not have an area. Let us call such curves *closed Osgood curves*.

Thus, we see that there exist regions bounded by Jordan curves which do not have area in the sense current since the Greek geometers down to the present day.

Suppose, however, we discard this traditional definition, and employ as definition of area its measure. Then we can say:

A figure \mathfrak{F} formed of a closed Jordan curve J and its interior \mathfrak{F} has an area, viz. $\text{Meas } \mathfrak{F}$.

For $\text{Front } \mathfrak{F} = J$. Hence \mathfrak{F} is complete, and is therefore measurable.

We note that

$$\mathfrak{F} = \mathcal{F} + \mathfrak{F}.$$

We have seen there are Jordan curves such that

$$\mathcal{F} > 0.$$

We now have a definition of area which is in accordance with the promptings of our geometric intuition. It must be remembered, however, that this definition has been only recently discovered, and that the definition which for centuries has been accepted leads to results which flatly contradict our intuition, which leads us to say that a figure bounded by a continuous closed curve has an area.

583. At this point we will break off our discussion of the relation between our intuitional notion of a curve, and the configuration determined by the equations

$$x = \phi(t) \quad , \quad y = \psi(t) \quad (1)$$

where ϕ, ψ are one-valued continuous functions of t in an interval T . Let us look back at the list of properties of an intuitional curve drawn up in 563. We have seen that the analytic curve 1) does not need to have tangents at a pantactic set of points on it; no arc on it needs have a finite length; it may completely fill the interior of a square; its equations cannot always be brought in the forms $y = f(x)$ or $F(xy) = 0$, if we restrict ourselves to functions f or F employed in analysis up to the present; it does not need to have an area as that term is ordinarily understood.

On the other hand, it is continuous, and when closed and without double point it forms the complete boundary of a region.

Enough in any case has been said to justify the thesis that geometric reasoning in analysis must be used with the greatest circumspection.

Detached and Connected Sets

584. In the foregoing sections we have studied in detail some of the properties of curves defined by the equations

$$x = \phi(t) \quad , \quad y = \psi(t).$$

Now the notion of a curve, like many other geometric notions, is independent of an analytic representation. We wish in the following sections to consider some of these notions from this point of view.

585. 1. Let \mathfrak{A} , \mathfrak{B} be point sets in m -way space \mathfrak{R}_m . If

$$\text{Dist}(\mathfrak{A}, \mathfrak{B}) > 0,$$

we say \mathfrak{A} , \mathfrak{B} are *detached*. If \mathfrak{A} cannot be split up into two parts \mathfrak{B} , \mathfrak{C} such that they are detached, we say \mathfrak{A} has *no detached parts*. If $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$ and $\text{Dist}(\mathfrak{B}, \mathfrak{C}) > 0$, we say \mathfrak{B} , \mathfrak{C} are *detached parts* of \mathfrak{A} .

Let the set of points, finite or infinite,

$$a, a_1, a_2, \dots b \tag{1}$$

be such that the distance between two successive ones is $\leq \epsilon$. We call 1) an ϵ -sequence between a , b ; or a sequence with segments (a_i, a_{i+1}) of length $\leq \epsilon$. We suppose the segments ordered so that we can pass continuously from a to b over the segments without retracing. If 1) is a finite set, the sequence is *finite*, otherwise *infinite*.

2. Let \mathfrak{A} have no detached parts. Let a , b be two of its points. For each $\epsilon > 0$, there exists a finite ϵ -sequence between a , b , and lying in \mathfrak{A} .

For about a describe a sphere of radius ϵ . About each point of \mathfrak{A} in this sphere describe a sphere of radius ϵ . About each point of \mathfrak{A} in each of these spheres describe a sphere of radius ϵ . Let this process be repeated indefinitely. Let \mathfrak{B} denote the points of \mathfrak{A} made use of in this procedure. If $\mathfrak{B} < \mathfrak{A}$, let $\mathfrak{C} = \mathfrak{A} - \mathfrak{B}$. Then $\text{Dist}(\mathfrak{B}, \mathfrak{C}) > \epsilon$, and \mathfrak{A} has detached parts, which is contrary to hypothesis. Thus there are sets of ϵ -spheres in \mathfrak{A} joining a and b .

Among these sets there are finite ones. For let \mathfrak{F} denote the set of points in \mathfrak{A} which may be joined to a by finite sequences; let $\mathfrak{G} = \mathfrak{A} - \mathfrak{F}$. Then $\text{Dist}(\mathfrak{F}, \mathfrak{G}) \geq \epsilon$. For if $< \epsilon$, there is a point f in \mathfrak{F} , and a point g in \mathfrak{G} whose distance is $< \epsilon$. Then a and g can be joined by a finite ϵ -sequence, which is contrary to hypothesis.

3. If \mathfrak{A} has no detached parts, it is dense.

For if not dense, it must have at least one isolated point a . But then a , and $\mathfrak{A} - a$ are detached parts of \mathfrak{A} , which contradicts the hypothesis.

4. Let \mathfrak{A} , \mathfrak{B} , \mathfrak{C} be complete and $\mathfrak{A} = (\mathfrak{B}, \mathfrak{C})$. If \mathfrak{A} has no detached parts, \mathfrak{B} , \mathfrak{C} have at least one common point.

or if \mathfrak{B} , \mathfrak{C} have no common point, $\delta = \text{Dist}(\mathfrak{B}, \mathfrak{C})$ is > 0 . δ cannot > 0 , since \mathfrak{B} , \mathfrak{C} would then be detached parts of \mathfrak{A} . If $\delta = 0$ and since \mathfrak{B} , \mathfrak{C} are complete, they have a point in common.

If \mathfrak{A} is such that any two of its points may be joined by an infinite sequence lying in \mathfrak{A} , where $\epsilon > 0$ is small at pleasure, \mathfrak{A} has no detached parts.

or if \mathfrak{A} had \mathfrak{B} , \mathfrak{C} as detached parts, let $\text{Dist}(\mathfrak{B}, \mathfrak{C}) = \delta$. Then $\delta > 0$. Hence there is no sequence joining a point of \mathfrak{B} with a point of \mathfrak{C} with segments $< \delta$.

If \mathfrak{A} is complete and has no detached parts, it is said to be connected. We also call \mathfrak{A} a connex.

As a special case, a point may be regarded as a connex.

If \mathfrak{A} is connected, it is perfect.

or by 3 it is dense, and by definition it is complete.

If \mathfrak{A} is a rectilinear connex, it has a first point α and a last point β , and contains every point in the interval (α, β) .

or being limited and complete its minimum and maximum exist in \mathfrak{A} and these are respectively α and β . Let now

$$\epsilon_1 > \epsilon_2 > \dots \doteq 0.$$

there exists an ϵ_1 -sequence C_1 between α , β . Each segment has an ϵ_2 -sequence C_2 . Each segment of C_2 has an ϵ_3 -sequence C_3 .

Let C be the union of all these sequences. It is pantactic between α , β . As \mathfrak{A} is complete,

$$\mathfrak{A} = (\alpha, \beta).$$

Images

16. Let $x_1 = f_1(t_1 \dots t_m) \dots x_n = f_n(t_1 \dots t_m)$ (1) be n -valued functions of t in the point set \mathfrak{T} . As t ranges over \mathfrak{T} , the point $x = (x_1 \dots x_n)$ will range over a set \mathfrak{A} in an n -way space \mathfrak{R}_n . We have called \mathfrak{A} the image of \mathfrak{T} . Cf. I, 238, 3. If the functions f are not one-valued, to a point t may correspond several images x' , $x'' \dots$ finite or infinite in number. Conversely

to the point x may correspond several values of t . If to each point t correspond in general r values of x , and to each x in general s values of t , we say the correspondence between \mathfrak{T} , \mathfrak{N} is r to s . If $r = s = 1$ the correspondence is 1 to 1 or *unifold*; if $r > 1$, it is *manifold*. If $r = 1$, \mathfrak{N} is a *simple* image of \mathfrak{T} , otherwise it is a *multiple* image. If the functions 1) are one-valued and continuous in \mathfrak{T} , we say \mathfrak{N} is a *continuous image* of \mathfrak{T} .

587. Transformations of the Plane. Example 1. Let

$$u = x \sin y, \quad v = x \cos y. \quad (1)$$

We have in the first place

$$u^2 + v^2 = x^2.$$

This shows that the image of a line $x = a$, $a \neq 0$, parallel to the y -axis is a circle whose center is the origin in the u, v plane, and whose radius is a . To the y -axis in the x, y plane corresponds the origin in the u, v plane.

From 1) we have, secondly,

$$\frac{u}{v} = \tan y.$$

This shows that the image of a line $y = b$, is a line through the origin in the u, v plane.

From 1) we have finally that u, v are periodic in y , having the period 2π . Thus as x, y ranges in the band B , formed by the two parallels $y = \pm \pi$, or $-\pi < y \leq \pi$, the point u, v ranges over the entire u, v plane once and once only.

The correspondence between B and the u, v plane is unifold, except, as is obvious, to the origin in the u, v plane corresponds the points on the y -axis.

Let us apply the theorem of I, 4-41, on implicit functions. The determinant Δ is here

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \sin y, & \cos y \\ x \cos y, & -x \sin y \end{vmatrix} = -x.$$

As this is $\neq 0$ when x, y is not on the y -axis, we see that the correspondence between the *domain* of any such point and its image is 1 to 1. This accords with what we have found above.

is, however, a much more restricted result than we have found; we have seen that the correspondence between any limited point set \mathcal{H} in B which does not contain a point of the y -axis and its image is unifold.

588. *Example 2.* Let

$$u = \frac{y}{\sqrt{x^2 + y^2}}, \quad v = \sqrt{x^2 + y^2}, \quad (1)$$

the radical having the positive sign. Let us find the image of the first quadrant Q in the x, y plane.

From 1) we have at once

$$0 \leq u \leq 1, \quad v \geq 0.$$

Hence the image of Q is a band B parallel to the v -axis.

From 1) we get secondly

$$y = uv, \quad x = v\sqrt{1 - u^2}. \quad (2)$$

Hence

$$x^2 + y^2 = v^2.$$

Thus the image of a circle in Q whose center is the origin and whose radius is a is a segment of a right line $v = a$.

When $x = y = 0$, the equations 1) do not define the corresponding point in the u, v plane. If we use 2) to define the correspondence, we may say that to the line $v = 0$ in B corresponds the origin in the x, y plane. With this exception the correspondence between Q and B is uniform, as 1), 2) show.

The determinant Δ of 1) is, setting

$$r = \sqrt{x^2 + y^2},$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} -\frac{xy}{r^3} & \frac{x^2}{r^3} \\ \frac{x}{r} & \frac{y}{r} \end{vmatrix} = \frac{-x}{x^2 + y^2}$$

for any point x, y different from the origin.

589. *Example 3. Reciprocal Radii.* Let O be the origin in the x, y plane and Ω the origin in the u, v plane. To any point $P = (x, y)$ in the x, y plane different from the origin shall correspond a point $Q = (u, v)$ in the u, v plane such that ΩQ has

the same direction as OP and such that $OP \cdot \Omega Q = 1$. Analytically we have

$$x = \lambda y, \quad u = \lambda v, \quad \lambda > 0,$$

and

$$(u^2 + v^2)(x^2 + y^2) = 1.$$

From these equations we get

$$x = \frac{u}{u^2 + v^2}, \quad y = \frac{v}{u^2 + v^2} \quad (1)$$

and also

$$u = \frac{x}{x^2 + y^2}, \quad v = \frac{y}{x^2 + y^2}.$$

The correspondence between the two planes is obviously unifold except that no point in either plane corresponds to the origin in the other plane. We find for any point x, y different from the origin that

$$\Delta = \frac{\partial(u, v)}{\partial(x, y)} = -\frac{1}{(x^2 + y^2)^2}.$$

Obviously from the definition, to a line through the origin in the x, y plane corresponds a similar line in the u, v plane. As xy moves toward the origin, u, v moves toward infinity.

Let x, y move on the line $x = a \neq 0$. Then 1) shows that u, v moves along the circle

$$a(u^2 + v^2) - u = 0$$

which passes through the origin. A similar remark holds when x, y moves along the line $y = b \neq 0$.

590. Such relations between two point sets $\mathfrak{A}, \mathfrak{B}$ as defined in 586 may be formulated independently of the functions f . In fact with each point a of \mathfrak{A} we may associate one or more points $b_1, b_2 \dots$ of \mathfrak{B} according to some law. Then \mathfrak{B} may be regarded as the image of \mathfrak{A} . We may now define the terms simple, manifold, etc., as in 586. When b corresponds to a we may write $b \sim a$.

We shall call \mathfrak{B} a *continuous image* of \mathfrak{A} when the following conditions are satisfied. 1° To each a in \mathfrak{A} shall correspond but one b in \mathfrak{B} , that is, \mathfrak{B} is a simple image of \mathfrak{A} . 2° Let $b \sim a$, let $a_1, a_2 \dots$ be any sequence of points in \mathfrak{A} which $\doteq a$. Let $b_n \sim a_n$. Then b_n must $\doteq b$ however the sequence $\{a_n\}$ is chosen.

When \mathfrak{B} is a simple image of \mathfrak{A} , the law which determines which b of \mathfrak{B} is associated with a point a of \mathfrak{A} determines obviously n one-valued functions as in 586, 1), where $t_1 \dots t_m$ are the m coördinates of a , and $x_1 \dots x_n$ are the n coördinates of b . We call these functions 1) the *associated functions*. Obviously when \mathfrak{B} is a continuous image, the associated functions are continuous in \mathfrak{A} .

591. 1. *Let \mathfrak{B} be a simple continuous image of the limited complete set \mathfrak{A} . Then 1° \mathfrak{B} is limited and complete. If 2° \mathfrak{A} is perfect and only a finite number of points of \mathfrak{A} correspond to any point of \mathfrak{B} , then \mathfrak{B} is perfect. If 3° \mathfrak{A} is a connex, so is \mathfrak{B} .*

To prove 1°. The case that \mathfrak{B} is finite requires no proof. Let $b_1, b_2 \dots$ be points of \mathfrak{B} which $\doteq \beta$. We wish to show that β lies in \mathfrak{B} . To each b_n will correspond one or more points in \mathfrak{A} ; call the union of all these points α . Since \mathfrak{B} is a simple image, α is an infinite set. Let $a_1, a_2 \dots$ be a set of points in α which $\doteq \alpha$, a limiting point of \mathfrak{A} . As \mathfrak{A} is complete, α lies in \mathfrak{A} . Let $b \sim \alpha$. Let $b_n \sim a_n$. As $a_n \doteq \alpha$, $b_n \doteq \beta$. But \mathfrak{B} being continuous, b_n must $\doteq b$. Thus β lies in \mathfrak{B} . That \mathfrak{B} is limited follows from the fact that the associated functions are continuous in the limited complete set \mathfrak{A} . To prove 2°. Suppose that \mathfrak{B} had an isolated point b . Let $b \sim \alpha$. Since \mathfrak{A} is perfect, let $a_1, a_2 \dots \doteq \alpha$. Let $b_n \sim a_n$. Then as \mathfrak{B} is continuous, $b_n \doteq b$, and b is not an isolated point. To prove 3°. We have only to show that there exists an ϵ -sequence between any two points α, β of \mathfrak{B} , ϵ small at pleasure. Let $\alpha \sim a, \beta \sim b$. Since \mathfrak{A} is connected there exists an η -sequence between a, b . Also the associated functions are uniformly continuous in \mathfrak{A} , and hence η may be taken so small that each segment of the corresponding sequence in \mathfrak{B} is $\geq \epsilon$.

2. *Let $f(t_1 \dots t_m)$ be one-valued and continuous in the connex \mathfrak{A} , then the image of \mathfrak{A} is an interval including its end points.*

This follows from the above and from 585, 8.

3. *Let the correspondence between $\mathfrak{A}, \mathfrak{B}$ be unifold. If \mathfrak{B} is a continuous image of \mathfrak{A} , then \mathfrak{A} is a continuous image of \mathfrak{B} .*

For let $\{b_n\}$ be a set of points in \mathfrak{B} which $\doteq b$. Let $a_n \sim b_n, a \sim b$. We have only to show that $a_n \doteq a$. For suppose that it does not, suppose in fact that there is a sequence $a_1, a_2 \dots$ which

$\doteq a \neq a$. Let $\beta \sim a$. Then $b_1, b_2, \dots \doteq \beta$. But any partial sequence of $\{b_n\}$ must $\doteq b$. Thus $b = \beta$, hence $a = a$, hence $a_n \doteq a$.

4. A Jordan curve J is a unifold continuous image of an interval T . Conversely if J is a unifold continuous image of an interval T , there exist two one-valued continuous functions

$$x = \phi(t) \quad , \quad y = \psi(t)$$

such that as t ranges over T , the point x, y ranges over J . In case J is closed it may be regarded as the image of a circle Γ .

All but the last part of the theorem has been already established. To prove the last sentence we have only to remark that if we set

$$x = r \cos t \quad , \quad y = r \sin t$$

we have a unifold continuous correspondence between the points of the interval $(0, 2\pi^*)$ and the points of a circle.

5. The first part of 4 may be regarded as a *geometrical definition* of a Jordan curve. The image of a segment of the interval T or of the circle Γ , will be called an *arc* of J .

592. *Side Lights on Jordan Curves.* These curves have been defined by means of the equations

$$x = \phi(t), \quad y = \psi(t). \quad (1)$$

As t ranges over the interval $T = (a < b)$, the point $P = (x, y)$ ranges over the curve J . This curve is a certain point set in the x, y plane. We may now propose this problem: We have given a point set \mathfrak{E} in the x, y plane; may it be regarded as a Jordan curve? That is, do there exist two continuous one-valued functions 1) such that as t ranges over some interval T , the point P ranges over the given set \mathfrak{E} without returning on itself, except possibly for $t = a, t = b$, when the curve would be closed?

Let us look at a number of point sets from this point of view.

593. Example 1.

1. Let $y = \sin \frac{1}{x}$, x in the interval $\mathfrak{A} = (-1, 1)$, but $\neq 0$
 $= 0$, for $x = 0$.

Is this point set \mathfrak{C} a Jordan curve? The answer is, No. For a Jordan curve is a continuous image of an interval \mathfrak{A} . By 591, 1, \mathfrak{C} is complete. But \mathfrak{C} is not complete, as all the points on the y -axis, $-1 \leq y \leq 1$ are limiting points of \mathfrak{C} , and only one of them belongs to \mathfrak{C} , viz. the origin.

2. Let us modify \mathfrak{C} by adjoining to it all these missing limiting points, and call the resulting point set \mathcal{C} . Is \mathcal{C} a Jordan curve? The answer is again, No. For if it were, we can divide the interval \mathcal{T} into intervals δ so small that the oscillation of ϕ, ψ in any one of them is $< \omega$. To the intervals δ_i will correspond arcs \mathcal{C}_i on the curve, and two non-consecutive arcs \mathcal{C}_i are distant from each other by an amount $>$ some ϵ , small at pleasure. This shows that one of these arcs, say \mathcal{C}_* , must contain the segment on the y -axis $-1 \leq y \leq 1$. But then $\text{Osc } \psi = 2$ as t ranges over the corresponding δ_* interval. Thus the oscillation of ψ cannot be made $< \epsilon$, however small δ_* is taken.

3. Let us return to the set \mathfrak{C} defined in 1. Let A, B be the two end points corresponding to $x = -1, x = 1$. Let us join them by an ordinary curve, a polygon if we please, which does not cut itself or \mathfrak{C} . The resulting point set \mathfrak{R} divides all the other points of the plane into two parts which cannot be joined by a continuous curve without crossing \mathfrak{R} . For this point of view \mathfrak{R} must be regarded as a *closed* configuration. Yet \mathfrak{R} is obviously not complete.

On the other hand, let us look at the curve formed by removing the points on a circle between two given points a, b on it. The remaining arc \mathfrak{L} including the end points a, b is a *complete* set, but as it does not divide the other points of the plane into two separated parts, we cannot say \mathfrak{L} is a closed configuration.

We mention this circumstance because many English writers use the term closed set where we have used the term complete. Cantor, who first introduced this notion, called such sets *abgeschlossen*, which is quite different from *geschlossen* = closed.

594. *Example 2.* Let $\rho = e^{-\frac{1}{\theta}}$, for θ in the interval $\mathfrak{A} = (0, 1)$ except $\theta = 0$, where $\rho = 0$. These polar coördinates may easily be replaced by Cartesian coördinates

$$x = \phi(\theta) = e^{-\frac{1}{\theta}} \cos \theta, \quad y = e^{-\frac{1}{\theta}} \sin \theta, \quad \text{in } \mathfrak{A},$$

except $\theta = 0$, when x, y both $= 0$. The curve thus defined is a Jordan curve.

Let us take a second Jordan curve

$$\rho = e^{-(\pi + \frac{1}{\theta})},$$

with $\rho = 0$ for $\theta = 0$. If we join the two end points on these curves corresponding to $\theta = 1$ by a straight line, we get a closed Jordan curve J , which has an interior \mathfrak{J} , and an exterior \mathfrak{D} .

The peculiarity of this curve J is the fact that one point of it, viz. the origin $x = y = 0$, cannot be joined to an arbitrary point of \mathfrak{J} by a finite broken line lying entirely in \mathfrak{J} ; nor can it be joined to an arbitrary point in \mathfrak{D} by such a line lying in \mathfrak{D} .

595. 1. It will be convenient to introduce the following terms.

Let \mathfrak{A} be a limited or unlimited point set in the plane. A set of distinct points in \mathfrak{A}

$$a_1, a_2, a_3 \dots \quad (1)$$

determine a broken line. In case 1) is an infinite sequence, let a_n converge to a fixed point. If this line has no double point, we call it a *chain*, and the segments of the line *links*. In case not only the points 1) but also the links lie in \mathfrak{A} , we call the chain a *path*. If the chain or path has but a finite number of links, it is called *finite*.

Let us call a *precinct* a region, i.e. a set all of whose points are inner points, limited or unlimited, such that any two of its points may be joined by a finite path.

2. Using the results of 578, we may say that, —

A closed Jordan curve J divides the other points of the plane into two precincts, an inner \mathfrak{J} and an outer \mathfrak{D} . Moreover, they have a common frontier which is J .

3. The closed Jordan curve considered in 594 shows that not every point of such a closed Jordan curve can always be joined to an arbitrary point of \mathfrak{J} or \mathfrak{D} by a finite path.

Obviously it can be joined by an infinite path. For about this point, call it P , we can describe a sequence of circles of radii $r \neq 0$. Between any two of these circles there lie points of \mathfrak{J} and of \mathfrak{D} , if r is suf-

sufficiently small. In this way we may get a sequence of points in \mathfrak{J} , viz. $I_1, I_2 \dots \doteq P$. Any two of these I_m, I_{m+1} may be joined by a path which does not cut the path joining I_1 to I_m . For if a loop were formed, it could be omitted.

4. Any arc \mathfrak{R} of a closed Jordan curve J can be joined by a path to an arbitrary point of the interior or exterior, which call \mathfrak{A} .

For let $J = \mathfrak{R} + \mathfrak{L}$. Let k be a point of \mathfrak{R} not an end point. Let $\delta = \text{Dist}(k, \mathfrak{L})$, let a be a point of \mathfrak{A} such that $\text{Dist}(a, k) < \frac{1}{2} \delta$. Then

$$\eta = \text{Dist}(a, \mathfrak{L}) > \frac{1}{2} \delta.$$

Hence the link $l = (a, k)$ has no point in common with \mathfrak{L} . Let b be the first point of l in common with \mathfrak{R} . Then the link $m = (a, b)$ lies in \mathfrak{A} . If now α is any point of \mathfrak{A} , it may be joined to a by a path p . Then $p + m$ is a path in \mathfrak{A} joining the arbitrary point α to a point b on the arc \mathfrak{R} .

596. *Example 3.* For θ in $\mathfrak{A} = (0^*, 1)$ let

$$\rho = a(1 + e^{-\frac{1}{\theta}}),$$

and

$$\rho = a(1 + e^{-(\pi + \frac{1}{\theta})}).$$

These equations in polar coördinates define two non-intersecting spirals S_1, S_2 which coil about $\rho = a$ as an asymptotic circle Γ . Let us join the end points of the spirals corresponding to $\theta = 1$ by a straight line L . Let \mathfrak{C} denote the figure formed by the spirals S_1, S_2 , the segment L and the asymptotic circle Γ . Is \mathfrak{C} a closed Jordan curve? The answer is, No. This may be seen in many ways. For example, \mathfrak{C} does not divide the other points into two precincts, but into three, one of which is formed of points within Γ .

Another way is to employ the reasoning of 593, 2. Here the circle Γ takes the place of the segment on the y -axis which figures here.

Still another way is to observe that no point on Γ can be joined to a point within \mathfrak{C} by a path.

597. *Example 4.* Let \mathfrak{C} be formed of the edge \mathfrak{C} of a unit square, together with the ordinates erected at the points

$x = \frac{m}{2^n}$, of length $\frac{1}{2^n}$, $n = 1, 2 \dots$ Although \mathfrak{C} divides the other points of the plane into two precincts \mathfrak{J} and \mathfrak{D} , we can say that \mathfrak{C} is not a closed Jordan curve.

For if it were, \mathfrak{J} and \mathfrak{D} would have to have \mathfrak{C} as a common frontier. But the frontier of \mathfrak{D} is \mathfrak{C} , while that of \mathfrak{J} is \mathfrak{C} .

That \mathfrak{C} is not a Jordan curve is seen in other ways. For example, let γ be an inner segment of one of the ordinates α . Obviously it cannot be reached by a path in \mathfrak{D} .

Brouwer's Proof of Jordan's Theorem

598. We have already given one proof of this theorem in 577 seq., based on the fact that the coördinates of the closed curve are expressed as one-valued continuous functions

$$x = \phi(t) \quad , \quad y = \psi(t).$$

Brouwer's proof* is entirely geometrical in nature and rests on the definition of a closed Jordan curve as the unfold continuous image of a circle, cf. 591, 6.

If $\mathfrak{A}, \mathfrak{B}, \dots$ are point sets in the plane, it will be convenient to denote their frontiers by $\mathfrak{F}_{\mathfrak{A}}, \mathfrak{F}_{\mathfrak{B}} \dots$ so that

$$\mathfrak{F}_{\mathfrak{A}} = \text{Front } \mathfrak{A} \quad , \quad \text{etc.}$$

We admit that any closed polygon \mathfrak{P} having a finite number of sides, without double point, divides the other points of the plane into an inner and an outer precinct $\mathfrak{P}_i, \mathfrak{P}_o$ respectively. In the following sections we shall call such a polygon *simple*, and usually denote it by \mathfrak{P} .

We shall denote the whole plane by \mathfrak{C} .

Then

$$\mathfrak{C} = \mathfrak{P} + \mathfrak{P}_i + \mathfrak{P}_o.$$

Let \mathfrak{A} be complete. The complementary set \mathcal{A} is formed, as we saw in 328, of an enumerable set of precincts, say $\mathcal{A} = \{A_n\}$.

* *Math. Annalen*, vol. 60 (1910), p. 109.

599. 1. *If a precinct \mathfrak{A} and its complement* A each contain a point of the connex \mathfrak{C} , then $\mathfrak{F}_{\mathfrak{A}}$ contains a point of \mathfrak{C} .*

For in the contrary case $c = Dv(\mathfrak{A}, \mathfrak{C})$ is complete. In fact $\mathfrak{B} = \mathfrak{A} + \mathfrak{F}_{\mathfrak{A}}$ is complete. As \mathfrak{C} is complete, $Dv(\mathfrak{B}, \mathfrak{C})$ is complete. But if $\mathfrak{F}_{\mathfrak{A}}$ does not contain a point of \mathfrak{C} , $c = Dv(\mathfrak{B}, \mathfrak{C})$. Thus on this hypothesis, c is complete. Now $c = Dv(A, \mathfrak{C})$ is complete in any case. Thus $\mathfrak{C} = c + c$, which contradicts 585, 4.

2. *If \mathfrak{P}_i , \mathfrak{P}_e , the interior and exterior of a simple polygon \mathfrak{P} each contain a point of a connex \mathfrak{C} , then \mathfrak{P} contains a point of \mathfrak{C} .*

3. *Let \mathfrak{R} be complete and not connected. There exists a simple polygon \mathfrak{P} such that no point of \mathfrak{R} lies on \mathfrak{P} , while a part of \mathfrak{R} lies in \mathfrak{P}_i and another part in \mathfrak{P}_e .*

For let $\mathfrak{R}_1, \mathfrak{R}_2$ be two non-connected parts of \mathfrak{R} whose distance from each other is $\rho > 0$. Let Δ be a quadrate division of the plane of norm δ , so small that no cell contains a point of \mathfrak{R}_1 and \mathfrak{R}_2 . Let Δ_1 denote the cells of Δ containing points of \mathfrak{R}_1 . It is complete, and the complementary set $\Delta_2 = \mathfrak{C} - \Delta_1$ is formed of one or more precincts. No point of \mathfrak{R}_1 lies in Δ_2 or on its frontier.

Let P_1, P_2 be points in $\mathfrak{R}_1, \mathfrak{R}_2$ respectively. Let D be that precinct containing P_2 . Then \mathfrak{F}_D embraces a simple polygon \mathfrak{P} which separates P_1 and P_2 .

4. *Let $\mathfrak{R}_1, \mathfrak{R}_2$ be two detached connexes. There exists a simple polygon \mathfrak{P} which separates them. One of them is in \mathfrak{P}_i , the other in \mathfrak{P}_e , and no point of either connex lies on \mathfrak{P} .*

For the previous theorem shows that there is a simple polygon \mathfrak{P} which separates a point P_1 in \mathfrak{R}_1 from a point P_2 in \mathfrak{R}_2 and no point of \mathfrak{R}_1 or \mathfrak{R}_2 lies on \mathfrak{P} . Call this fact F .

Let now P_1 lie in \mathfrak{P}_i . Then every point of \mathfrak{R}_1 lies in \mathfrak{P}_i . For otherwise \mathfrak{P}_i and \mathfrak{P}_e each contain a point of the connex \mathfrak{R}_1 . Then 2 shows that a point of \mathfrak{R}_1 lies on \mathfrak{P} , which contradicts F .

5. *Let \mathfrak{B} be a precinct determined by the connex \mathfrak{C} . Then $\text{Front } \mathfrak{B}$ is a connex.*

* Since the initial sets are all limited, their complements may be taken with reference to a sufficiently large square \mathfrak{Q} ; and when dealing with frontier points, points on the edge of \mathfrak{Q} may be neglected.

For suppose b is not a connex. Then by 3, there exists a simple polygon \mathfrak{P} which contains a part of b in \mathfrak{P}_i and another in \mathfrak{P}_r , while no point of b lies on \mathfrak{P} . Hence a point β' of b lies in \mathfrak{P}_i , and another point β'' in \mathfrak{P}_r . As \mathfrak{B} is a precinct, let us join β' , β'' by a path v in \mathfrak{B} . Thus \mathfrak{P} contains at least one point of v , that is, a point of \mathfrak{B} lies on \mathfrak{P} . As b and \mathfrak{P} have no point in common, and as one point of \mathfrak{P} lies in \mathfrak{B} , all the points of \mathfrak{P} lie in \mathfrak{B} . Hence

$$Dv(\mathfrak{P}, \mathfrak{C}) = 0. \quad (1)$$

As b is a part of \mathfrak{C} and hence some of the points of \mathfrak{C} are in \mathfrak{P}_i and some in \mathfrak{P}_r , it follows from 2 that a part of \mathfrak{P} lies in \mathfrak{C} . This contradicts 1).

6. Let $\mathfrak{R}_1, \mathfrak{R}_2$ be two connexes without double point. By 3 there exists a simple polygon \mathfrak{P} which separates them and has one connex inside, the other outside \mathfrak{P} .

Now $\mathfrak{R} = \mathfrak{R}_1 + \mathfrak{R}_2$ is complete and defines one or more precincts. *One of these precincts contains \mathfrak{P} .*

For say \mathfrak{P} lay in two of these precincts as \mathfrak{A} and \mathfrak{B} . Then the precinct \mathfrak{A} and its complement (in which \mathfrak{B} lies) each contain a point of the connex \mathfrak{P} . Thus $\mathfrak{F}_{\mathfrak{A}}$ contains a point of \mathfrak{P} . But $\mathfrak{F}_{\mathfrak{A}}$ is a part of \mathfrak{R} , and no point of \mathfrak{R} lies on \mathfrak{P} .

That precinct in Comp \mathfrak{R} which contains \mathfrak{P} we call the *intermediate precinct* determined by $\mathfrak{R}_1, \mathfrak{R}_2$, or more shortly the precinct between $\mathfrak{R}_1, \mathfrak{R}_2$ and denote it by Inter $(\mathfrak{R}_1, \mathfrak{R}_2)$.

7. *Let $\mathfrak{R}_1, \mathfrak{R}_2$ be two detached connexes, and let $\mathfrak{I} = \text{Inter}(\mathfrak{R}_1, \mathfrak{R}_2)$. Then $\mathfrak{R}_1, \mathfrak{R}_2$ can be joined by a path lying in \mathfrak{I} , except its end points which lie on the frontiers of $\mathfrak{R}_1, \mathfrak{R}_2$ respectively.*

For by hypothesis $\rho = \text{Dist}(\mathfrak{R}_1, \mathfrak{R}_2) > 0$. Let P_1 be a point of $\mathfrak{F}_{\mathfrak{R}_1}$ such that some domain δ of P_1 contains only points of \mathfrak{R}_1 and of \mathfrak{I} . Let Q_1 be a point of \mathfrak{I} in δ . Join P_1, Q_1 by a right line, let it cut $\mathfrak{F}_{\mathfrak{R}_1}$ first at the point P' . In a similar way we may reason on \mathfrak{R}_2 , obtaining the points P'', Q_2 . Then $P'Q_1Q_2P''$ is the path in question. If we denote it by v , we may let v^* denote this path after removing its two end points.

8. *Let $\mathfrak{R}_1, \mathfrak{R}_2$ be two detached connexes. A path v joining $\mathfrak{R}_1, \mathfrak{R}_2$ and lying in $\mathfrak{I} = \text{Inter}(\mathfrak{R}_1, \mathfrak{R}_2)$, end points excepted, determines one and only one precinct in \mathfrak{I} .*

For from an arbitrary point P in \mathfrak{k} , let us draw all possible paths to v . Those paths ending on the same side (left or right) of v certainly lie in one and the same precinct \mathfrak{k}_r or \mathfrak{k}_l in \mathfrak{k} . Then since one end point of v is inside, the other end point outside \mathfrak{B} , there must be a part of \mathfrak{B} which is not met by v and which joins the right and left sides of v . We take this as an evident property of finite broken lines and polygons without double points.

Thus \mathfrak{k}_l and \mathfrak{k}_r are not detached; they are parts of one precinct.

9. *Two paths v_1, v_2 without common point, lying in \mathfrak{k} and joining $\mathfrak{R}_1, \mathfrak{R}_2$, split \mathfrak{k} into two precincts.*

Let $i = \mathfrak{k} - v_1$; this we have just seen is a precinct. From any point of it let us draw paths to v_2 . Those paths ending on the same side of v_2 determine precincts i_l, i_r which may be identical. Suppose they are. Then the two sides of v_2 can be joined by a path lying in \mathfrak{k} , which does not touch v_2 (end points excepted), has no point in common with v_1 , and together with a segment of v_1 forms a simple polygon \mathfrak{B} which has one end point of v_1 in \mathfrak{B}_i , the other end point in \mathfrak{B}_o . Thus by 2, \mathfrak{B} contains a point of the connexion v_1 . This is contrary to hypothesis.

Similar reasoning shows that

10. *The n paths $v_1 \dots v_n$ pairwise without common point, lying in and joining the connexes $\mathfrak{R}_1, \mathfrak{R}_2$ split \mathfrak{k} into n precincts.*

Let us finally note that the reasoning of 595, 4, being independent of an analytic representation of a Jordan curve, enables us to use the geometric definition of 591, 5, and we have therefore the theorem

11. *Let \mathfrak{A} be a precinct whose frontier \mathfrak{F} is a Jordan curve. Then there exists a path in \mathfrak{A} joining an arbitrary point of \mathfrak{A} with any arc \mathfrak{F} .*

Having established these preliminary theorems, we may now take up the body of the proof.

600. 1. *Let \mathfrak{A} be a precinct determined by a closed Jordan curve. Then $\mathfrak{F} = \text{Front } \mathfrak{A}$ is identical with J .*

If J determines but one precinct \mathfrak{A} which is pantactic in \mathfrak{E} , we have obviously $\mathfrak{F} = J$.

Suppose then that \mathfrak{A} is a precinct, not pantactic in \mathfrak{E} . Let \mathfrak{B} be a precinct $\neq \mathfrak{A}$ determined by \mathfrak{F} . Let $b = \text{Front } \mathfrak{B}$. Then $b \leq \mathfrak{F} \leq J$. Suppose now $b < J$. As J is a connex by 591, 1, \mathfrak{F} is a connex by 599, 5. Similarly since \mathfrak{F} is a connex, b is a connex. Since $b < J$, let $b \sim b$ on the circle Γ whose image is J . We divide b into three arcs b_1, b_2, b_3 to which $\sim b_1, b_2, b_3$ in b .

Let $\beta = \text{Inter}(b_1, b_3)$.

Then by 599, 11, we can join b_1, b_3 by a path v_1 in \mathfrak{A} , and by a path v_2 in \mathfrak{B} . By 599, 9, these paths split β into two precincts β_1, β_2 . We can join v_1, v_2 by a path u_1 lying in β_1 , and by a path u_2 lying in β_2 .

Now the precinct \mathfrak{B} and its complement each contain a point of the connex u_1 . Hence by 599, 1, b contains a point of u_1 . Similarly b contains a point of u_2 . Thus u_1, u_2 cut b , and as they do not cut b_1, b_3 by hypothesis, they cut b_2 . Thus *at least one point of β_1 and one point of β_2 lie in b_2 .*

Let p be a point of β_1 lying in b_2 , let $p \sim p$ on the circle. Let b' be an arc of b_2 containing p . Let $b' \sim b'$. As the connex b' has no point in common with Front β_1 , b' must lie entirely in β_1 by 599, 1. This is independent of the choice of b' , hence the connex b_2 , except its end points, lies in β_1 . Thus β_2 can contain no point of b_2 , which contradicts the result in italics above.

Thus the supposition that $b < J$ is impossible. Hence $b = J$, and therefore $\mathfrak{F} = J$.

As a corollary we have :

2. *A Jordan curve is apantactic in \mathfrak{E} .*

3. *A closed Jordan curve J cannot determine more than two precincts.*

For suppose there were more than two precincts

$$\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3 \dots \quad (1)$$

Let us divide the circle Γ into four arcs whose images call J_1, J_2, J_3, J_4 .

Then by 1, the frontier of each of the precincts 1) is J . Thus by 599, 9, there is a path in each of the precincts $\mathfrak{A}_1, \mathfrak{A}_2 \dots$ joining J_1 and J_3 . These paths split

$$\mathfrak{k} = \text{Inter } (J_1, J_3)$$

to precincts $\mathfrak{k}_1, \mathfrak{k}_2 \dots$

Now as in 1, we show on the one hand that each \mathfrak{k}_i must contain point of J_2 or J_4 , and on the other hand neither J_2 nor J_4 can be in more than one \mathfrak{k}_i .

4. *A closed Jordan curve J must determine at least two precincts.*

Suppose that J determines but a single precinct \mathfrak{A} . From a point a of \mathfrak{A} we may draw two non-intersecting paths u_1, u_2 to points b_1, b_2 of J .

Since the point a may be regarded as a connex, a and J are two detached connexes. Hence by 599, 9, the paths u_1, u_2 split \mathfrak{A} into two precincts $\mathfrak{A}_1, \mathfrak{A}_2$. Let $j = (u_1, u_2, J)$. The points b_1, b_2 divide J into two arcs J_1, J_2 , and

$$j_1 = (u_1, u_2, J_1) \quad , \quad j_2 = (u_1, u_2, J_2) \quad .$$

Let j be a closed Jordan curve. Regarding a and J as two detached connexes, we see j determines two precincts, α_1, α_2 . By 599, 1, a path which joins a point a_1 of α_1 with a point a_2 of α_2 must cut j , and hence j . It cannot thus lie altogether in \mathfrak{A}_1 or in \mathfrak{A}_2 . Thus neither α_1, α_2 do not lie in \mathfrak{A}_1 , nor both in \mathfrak{A}_2 . Let us therefore suppose for example that \mathfrak{A}_1 lies in α_1 , and \mathfrak{A}_2 in α_2 . Hence by 2, j is pantactic in α_1 , and \mathfrak{A}_2 in α_2 . By 1, each point of j_1 is common to the frontiers of α_1 and of α_2 , and hence of \mathfrak{A}_1 and of \mathfrak{A}_2 , and these are pantactic.

Let P be a point of J_2 . It lies either in α_1 or α_2 . Suppose it lies in α_1 . Then it lies neither in α_2 nor on Front α_2 , and hence neither in \mathfrak{A}_2 nor on Front \mathfrak{A}_2 . But every point of j_2 and also every point of j_1 lies on Front \mathfrak{A}_2 . We are thus brought to a contradiction. Hence the supposition that J determines but a single precinct is untenable.

Dimensional Invariance

601. 1. In 247 we have seen that the points of a unit interval and of a unit square S may be put in one to one correspondence. This fact, due to Cantor, caused great astonishment in the mathematical world, as it seemed to contradict our intuitional views

regarding the number of dimensions necessary to define a figure. Thus it was thought that a curve required *one* variable to define it, a surface *two*, and a solid *three*.

The correspondence set up by Cantor is not continuous. On the other hand the curves invented by Peano, Hilbert, and others (cf. 573) establish a continuous correspondence between I and S , but this correspondence is not one to one. Various mathematicians have attempted to prove that a continuous one to one correspondence between spaces of m and n dimensions cannot exist. We give a very simple proof due to *Lebesgue*.*

It rests on the following theorem :

2. Let \mathfrak{A} be a point set in \mathfrak{R}_m . Let $\Omega \leq \mathfrak{A}$ be a standard cube

$$0 \leq x_i - a_i \leq 2\sigma \quad , \quad i = 1, 2 \dots m.$$

Let $\mathfrak{C}_1, \mathfrak{C}_2 \dots$ be a finite number of complete sets so small that each lies in a standard cube of edge σ . If each point of \mathfrak{A} lies in one of the \mathfrak{C} 's, there is a point of \mathfrak{A} which lies in at least $m + 1$ of them.

Suppose first that each \mathfrak{C}_i is the union of a finite number of standard cubes. Let \mathfrak{C}_1 denote those \mathfrak{C} 's containing a point of the face f_1 of Ω lying in the plane $x_1 = a_1$. The frontier \mathfrak{F}_1 of \mathfrak{C}_1 is formed of a part of the faces of the \mathfrak{C} 's. Let F_1 denote that part of \mathfrak{F}_1 which is parallel to f_1 . Let $\Omega_1 = Dv(\Omega, F_1)$. Any point of it lies in at least two \mathfrak{C} 's.

Let \mathfrak{C}_2 denote those of the \mathfrak{C} 's not lying altogether in \mathfrak{C}_1 and containing a point of the face f_2 of Ω determined by $x_2 = a_2$. Let F_2 denote that part of Front \mathfrak{C}_2 which is parallel to f_2 . Let $\Omega_2 = Dv(\Omega_1, F_2)$. Any point of it lies in at least three of the \mathfrak{C} 's.

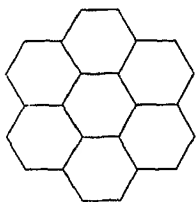
In this way we may continue, arriving finally at Ω_m , any point of which lies in at least $m + 1$ of the \mathfrak{C} 's.

Let us consider now the *general case*. We effect a cubical division of space of norm $d < \sigma$. Let C_i denote those cells of D which contain a point of \mathfrak{C}_i . Then by the foregoing, there is a point of \mathfrak{A} which lies in at least $m + 1$ of the C 's. As this is true, however small d is taken, and as the \mathfrak{C} 's are complete, there is at least one point of \mathfrak{A} which lies in $m + 1$ of the \mathfrak{C} 's.

* *Math Annalen*, vol. 70 (1911), p. 100.

. We now note that the space \mathfrak{R}_m may be divided into congruent cells so that no point is in more than $m + 1$ cells.

For $m = 1$ it is obvious. For $m = 2$ we may use a hexagonal pattern. We may also use a square division of norm δ of the plane. These squares may be grouped in horizontal bands. Let every other band be slid a distance $\delta/2$ to the right. Then no point lies in more than 3 squares. For $m = 3$ we may use a trichloric division of space, etc.



In each case no point of space is in more than $m + 1$ cells.

Let us call such a division a *reticulation* of \mathfrak{R}_m .

. Let \mathfrak{A} be a point set in \mathfrak{R}_m having an inner point a . There is a continuous unfold image \mathfrak{B} of \mathfrak{A} in \mathfrak{R}_n , $n \neq m$, such that $b \sim a$ is an inner point of \mathfrak{B} .

For let $n > m$. Let us effect a reticulation R of \mathfrak{R}_m of norm ρ . If $\rho > 0$ is taken sufficiently small $\Delta = D_{2\delta}(a)$ lies in \mathfrak{A} . Let $\mathfrak{C}_1, \mathfrak{C}_2, \dots, \mathfrak{C}_s$ be the cells of R which contain points of \mathfrak{A} , lie in Δ . Let the image of \mathfrak{C}_1 be \mathfrak{E}_1 and that of the cells 1) be $\mathfrak{E}_2, \mathfrak{E}_3, \dots, \mathfrak{E}_s$.

These are complete. Each point of \mathfrak{E} lies in one of the sets 2). Hence by 2), they contain a point β which lies in $n + 1$ of them. Then $\alpha \sim \beta$ lies in $n + 1$ of the cells 1). But these, being part of the reticulation R , are such that no point lies in more than $m + 1$ cells. Hence the contradiction.

02. 1. *Schönfliess' Theorem.* Let

$$u = f(x, y) \quad , \quad v = g(x, y) \quad (1)$$

one-valued and continuous in a unit square A whose center is the origin. These equations define a transformation T . If T is regular, we have seen in I, 742, that the domain $D_\rho(P)$ of a point $P = (x, y)$ within A goes over into a set E such that if $Q \sim P$ and $D_\sigma(Q)$ lies in E , if $\sigma > 0$ is sufficiently small.

is a continuous function of θ, μ which does not vanish for $\mu = \mu_0$, when $0 \leq \theta \leq 2\pi$. But being continuous, it is uniformly continuous. It therefore does not vanish in the rectangle

$$-\epsilon + \mu_0 \leq \mu \leq \mu_0 + \epsilon, \quad 0 \leq \theta \leq 2\pi.$$

We can now show that if $\mathfrak{B} \leq c_i$, it is identical with c_i . To this end we need only to show that any point β of c_i lies on some c_μ . In fact, as $\mu \doteq 0$, c_μ contracts to a point. Thus β is an outer point of some c_μ , and an inner point of others. Let μ_0 be the maximum of the values of μ such that β is exterior to all c_μ , if $\mu < \mu_0$. Then β lies on c_{μ_0} . For if not, β is exterior to $c_{\mu_0 + \epsilon}$, by what we have just shown, and μ_0 is not the maximum of μ .

Let us suppose that \mathfrak{B} lay without c . We show this leads to a contradiction. For let us invert with respect to a circle f , lying in c_i . Then c goes over into a curve \mathfrak{f} , and \mathfrak{A} goes over into $\mathfrak{D} = \mathfrak{E} + \mathfrak{f}$. Then \mathfrak{E} lies inside \mathfrak{f} . Let ξ, η be coördinates of a point of \mathfrak{D} . Obviously they are continuous functions of x, y in A , and

$$A \sim \mathfrak{D}, \quad c \sim \mathfrak{f}, \quad \text{uniformly.}$$

By what we have just proved, \mathfrak{E} must fill all the interior of \mathfrak{f} . This is impossible unless \mathfrak{A} is unlimited.

3. We may obviously extend the theorem 2 to the case

$$u_1 = f_1(x_1 \cdots x_m) \cdots u_m = f_m(x_1 \cdots x_m)$$

and A is a cube in m -way space \mathfrak{R}_m , provided we assume that c , the image of the boundary of A , divides space into two precincts whose frontier is c .

Area of Curved Surfaces

603. 1. *The Inner Definition.* It is natural to define the area of a curved surface in a manner analogous to that employed to define the length of a plane curve, viz. by inscribing and circumscribing the surface with a system of polyhedra, the area of whose faces converges to 0. It is natural to expect that the limits of the area of these two systems will be identical, and this common limit would then forthwith serve as the definition of the area of the surface. The consideration of the inner and the outer sys-

tems of polyhedra afford thus two types of definitions, which may be styled the inner and the outer definitions. Let us look first at the inner definition.

Let the equations of the surface S under consideration be

$$x = \phi(u, v) \quad , \quad y = \psi(u, v) \quad , \quad z = \chi(u, v), \quad (1)$$

the parameters ranging over a complete metric set \mathfrak{A} , and x, y, z being one-valued and continuous in \mathfrak{A} .

Let us effect a rectangular division D of norm d of the u, v plane. The rectangles fall into triangles t_κ on drawing the diagonals. Such a division of the plane we call *quasi rectangular*.

Let $P_0 = (u_0, v_0) \quad , \quad P_1 = (u_0 + \delta, v) \quad , \quad P_2 = (u_0, v_0 + \eta)$

be the vertices of t_κ . To these points in the u, v plane correspond three points $\mathfrak{P}_i = (x_i, y_i, z_i)$, $i=1, 2, 3$, of S which form the vertices of one of the triangular faces τ_κ of the inscribed polyhedron Π_D corresponding to the division D . Here, as in the following sections, we consider only triangles lying in \mathfrak{A} . We may do this since \mathfrak{A} is metric.

Let $X_\kappa, Y_\kappa, Z_\kappa$ be the projections of τ_κ on the coördinate planes. Then, as is shown in analytic geometry,

$$\tau_\kappa^2 = X_\kappa^2 + Y_\kappa^2 + Z_\kappa^2$$

where

$$2 X_\kappa = \begin{vmatrix} y_0 & z_0 & 1 \\ y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \end{vmatrix} = \begin{vmatrix} y_1 - y_0 & z_1 - z_0 \\ y_2 - y_0 & z_2 - z_0 \end{vmatrix} = \begin{vmatrix} \Delta' y & \Delta' z \\ \Delta'' y & \Delta'' z \end{vmatrix}$$

and similar expressions for Y_κ, Z_κ .

Thus the area of Π_D is

$$S_D = \Sigma \sqrt{X_\kappa^2 + Y_\kappa^2 + Z_\kappa^2},$$

the summation extending over all the triangles t_κ lying in the set \mathfrak{A} .

Let x, y, z have continuous first derivatives in \mathfrak{A} . Then

$$\Delta' x = x_1 - x_0 = \frac{\partial x}{\partial u} \delta + \alpha' \delta; \quad \Delta'' x = x_2 - x_0 = \frac{\partial x}{\partial v} \eta + \alpha'' \eta,$$

with similar expressions for the other increments. Let

$$A = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}, \quad B = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}, \quad C = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \quad (2)$$

Then

$$X_k = (A_k + \alpha_k)t_k, \quad Y_k = (B_k + \beta_k)t_k, \quad Z_k = (C_k + \gamma_k)t_k$$

where $\alpha_k, \beta_k, \gamma_k$ are uniformly evanescent with d in \mathfrak{A} . Thus if A, B, C do not simultaneously vanish at any point of \mathfrak{A} , we have as area of the surface S

$$\lim_{d \rightarrow 0} S_d = \int_{\mathfrak{A}} \sqrt{A^2 + B^2 + C^2} du dv. \quad (3)$$

2. An objection which at once arises to this definition lies in the fact that we have taken the faces of our inscribed polyhedra in a very restricted manner. We cannot help asking, Would we get the same area for S if we had chosen a different system of polyhedra?

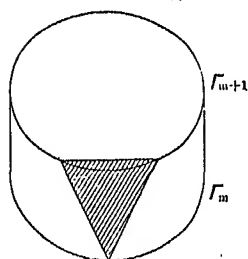
To lessen the force of this objection we observe that by replacing the parameters u, v by two new parameters u', v' we may replace the above quasi rectangular divisions which correspond to the family of right lines $u = \text{constant}$, $v = \text{constant}$ by the infinitely richer system of divisions corresponding to the family of curves $u' = \text{constant}$, $v' = \text{constant}$. In fact, by subjecting u', v' to certain very general conditions, we may transform the integral 3) to the new variables u', v' without altering its value.

But even this does not exhaust all possible ways of dividing \mathfrak{A} into a system of triangles with evanescent sides. Let us therefore take at pleasure a system of points in the u, v plane having no limiting points, and join them in such a way as to cover the plane without overlapping with a set of triangles t_k . If each triangle lies in a square of side d , we may call this a triangular division of norm d . We may now inquire if S_d still converges to the limit 3), as $d \rightarrow 0$, for this more general system of divisions. It was generally believed that such was the case, and standard treatises even contained demonstrations to this effect. These demonstrations are wrong; for Schwarz* has shown that by

* Werke, vol. 2, p. 309.

properly choosing the triangular divisions D , it is possible to make S_D converge to a value large at pleasure, for an extensive class of simple surfaces.

604. 1. Schwarz's Example. Let C be a right circular cylinder of radius 1 and height 1. A set of planes parallel to the base at a distance $\frac{1}{n}$ apart cuts out a system of circles $\Gamma_1, \Gamma_2 \dots$. Let



us divide each of these circles into m equal arcs, in such a way that the end points of the arcs on $\Gamma_1, \Gamma_3, \Gamma_5 \dots$ lie on the same vertical generators, while the end points of $\Gamma_2, \Gamma_4, \Gamma_6 \dots$ lie on generators halfway between those of the first set. We now inscribe a polyhedron so that the base of one of the triangular facets lies on one

circle while the vertex lies on the next circle above or below, as in the figure.

The area t of one of these facets is

$$t = \frac{1}{2} b h, \quad b = 2 \sin \frac{\pi}{m}, \quad h = \sqrt{\frac{1}{n^2} + \left(1 - \cos \frac{\pi}{m}\right)^2}.$$

Thus

$$t = \sin \frac{\pi}{m} \sqrt{\frac{1}{n^2} + 4 \sin^4 \frac{\pi}{2m}}.$$

There are $2m$ such triangles in each layer, and there are n layers. Hence the area of the polyhedron corresponding to this triangular division D is

$$S_D = \Sigma t_k = 2mn \sin \frac{\pi}{m} \sqrt{\frac{1}{n^2} + 4 \sin^4 \frac{\pi}{2m}}.$$

Since the integers m, n are independent of each other, let us consider various relations which may be placed on them.

Case 1°. Let $n = \lambda m$. Then

$$\begin{aligned} S_D &= 2m^2 \lambda \sin \frac{\pi}{m} \sqrt{\frac{1}{\lambda^2 m^2} + 4 \sin^4 \frac{\pi}{2m}} \\ &= 2m^2 \lambda \cdot \frac{\pi}{m} \cdot \frac{\sin \frac{\pi}{m}}{\frac{\pi}{m}} \sqrt{\frac{1}{\lambda^2 m^2} + 4 \frac{\pi^4}{2^4 m^4} \left[\frac{\sin \frac{\pi}{2m}}{\frac{\pi}{2m}} \right]^4} \\ &\doteq 2\pi, \quad \text{as } m \doteq \infty. \end{aligned}$$

Case 2°. Let $n = \lambda m^2$. Then

$$S_D = 2 \lambda m^3 \cdot \frac{\pi}{m} \left[\frac{\sin \frac{\pi}{m}}{\frac{\pi}{m}} \right] \sqrt{\frac{1}{\lambda^2 m^4} + 4 \frac{\pi^4}{2^4 m^4} \left[\frac{\sin \frac{\pi}{2m}}{\frac{\pi}{2m}} \right]^4}$$

$$\doteq 2 \pi \sqrt{1 + \frac{\pi^4}{4} \lambda^2} \quad , \quad \text{as } m \doteq \infty.$$

Case 3°. Let $n = \lambda m^3$. Then

$$S_D = 2 \pi \left[\frac{\sin \frac{\pi}{m}}{\frac{\pi}{m}} \right] \sqrt{1 + \frac{\pi^4}{2^2} m^2 \lambda^2 \left[\frac{\sin \frac{\pi}{2m}}{\frac{\pi}{2m}} \right]^4}$$

$$\doteq +\infty \quad , \quad \text{as } m \doteq \infty.$$

2. Thus only in the first case does S_D converge to 2π , which is the area of the cylinder C as universally understood. In the 1° and 3° cases the ratio $h/b \doteq 0$. As equations of C we may take

$$x = \cos u \quad , \quad y = \sin u \quad , \quad z = v.$$

Then to a triangular facet of the inscribed polyhedron will correspond a triangle in the u, v plane. In cases 2° and 3° this triangle has an angle which converges to π as $m \doteq \infty$. This is not so in case 1°. Triangular divisions of this latter type are of great importance. Let us call then a triangular division of the u, v plane such that no angle of any of its triangles is greater than $-\epsilon$, where $\epsilon > 0$ is small at pleasure but fixed, *positive triangular divisions*. We employ this term since the sine of one of the angles is $>$ some fixed positive number.

605. The Outer Definition. Having seen one of the serious difficulties which arise from the inner definition, let us consider briefly the outer definition. We begin with the simplest case in which the equation of the surface S is

$$z = f(x, y), \tag{1}$$

being one-valued and having continuous first derivatives. Let us effect a metric division Δ of the x, y plane of norm δ , and on

each cell d_κ as base, we erect a right cylinder C , which cuts out an element of surface δ_κ from S . Let \mathfrak{P}_κ be an arbitrary point of δ_κ and \mathfrak{T}_κ the tangent plane at this point. The cylinder C cuts out of \mathfrak{T}_κ an element ΔS_κ . Let ν_κ be the angle that the normal to \mathfrak{T}_κ makes with the z -axis. Then

$$\cos \nu_\kappa = \frac{1}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)_\kappa^2 + \left(\frac{\partial z}{\partial y}\right)_\kappa^2}}$$

and

$$\Delta S_\kappa = \frac{d_\kappa}{\cos \nu_\kappa}.$$

The area of S is now defined to be

$$\lim_{\delta=0} \Sigma \Delta S_\kappa \quad (2)$$

when this limit exists. The derivatives being continuous, we have at once that this limit is

$$\int_{\mathfrak{A}} dx dy \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \quad (3)$$

which agrees with the result obtained by the inner definition in 603, 3).

The advantages of this form of definition are obvious. In the first place, the nature of the divisions Δ is quite arbitrary; however they are chosen, one and the same limit exists. Secondly, the most general type of division is as easy to treat as the most narrow, viz. when the cells d_κ are squares.

Let us look at its disadvantages. In the first place, the elements ΔS_κ do not form a circumscribing polyhedron of S . On the contrary, they are little patches attached to S at the points \mathfrak{P}_κ , and having in general no contact with one another. Secondly, let us suppose that S has tangent planes parallel to the z -axis. The derivatives which enter the integral 603, 3) are no longer continuous, and the reasoning employed to establish the existence of the limit 2) breaks down. Thirdly, we have the case that z is not one-valued, or that the tangent planes to S do not turn continuously, or do not even exist at certain points.

To get rid of these disadvantages various other forms of outer definitions have been proposed. One of these is given by *Goursat* in his *Cours d'Analyse*. Instead of projecting an arbitrary element of surface on a fixed plane, the xy plane, it is projected on one of the tangent planes belonging to that element. Hereby the more general type of surfaces defined by 603, 1) instead of those defined by 1) above is considered. The restriction is, however, made that the normals to the tangent planes cut the elements of surface but once, also the first derivatives of the coördinates are assumed to be continuous in \mathfrak{M} . Under these conditions we get the same value for the area as that given in 603, 3).

When the first derivatives of x, y, z are not continuous or do not exist, this definition breaks down. To obviate this difficulty *de la Vallée-Poussin* has proposed a third form of definition in his *Cours d'Analyse*, vol. 2, p. 30 seq. Instead of projecting the element of surface on a tangent plane, let us project it on a plane for which the projection is a maximum. In case that S has a continuously turning tangent plane nowhere parallel to the z -axis, *de la Vallée-Poussin* shows that this definition leads to the same value of the area of S as before. He does not consider other cases in detail.

Before leaving this section let us note that *Jordan* in his *Cours* employs the form of outer definition first noted, using the parametric form of the equations of S . In the preface to this treatise the author avows that the notion of area is still somewhat obscure, and that he has not been able "à définir d'une manière satisfaisante l'aire d'une surface gauche que dans le cas où la surface a un plan tangent variant suivant une loi continue."

606. 1. Regular Surfaces. Let us return to the inner definition considered in 603. We have seen in 604 that not every system of triangular divisions can be employed. Let us see, however, if we cannot employ divisions much more general than the quasi rectangular. We suppose the given surface is defined by

$$x = \phi(u, v) \quad , \quad y = \psi(u, v) \quad , \quad z = \chi(u, v) \quad (1)$$

the functions ϕ, ψ, χ being one-valued, totally differentiable functions of the parameters u, v which latter range over the complete

metric set \mathfrak{A} . Surfaces characterized by these conditions we shall call *regular*. Let

$$P_0 = (u_0, v_0) \quad , \quad P_1 = (u_0 + \delta', v_0 + \eta') \quad , \quad P_2 = (u_0 + \delta'', v_0 + \eta'')$$

be the vertices of one of the triangles t_κ , of a triangular division D of norm d of \mathfrak{A} . As before let $\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2$ be the corresponding points on the surface S . Then

$$\Delta'x = x_1 - x_0 = \frac{\partial x}{\partial u} \delta' + \frac{\partial x}{\partial v} \eta' + \alpha'_x \delta' + \beta'_x \eta',$$

$$\Delta''x = x_2 - x_0 = \frac{\partial x}{\partial u} \delta'' + \frac{\partial x}{\partial v} \eta'' + \alpha''_x \delta + \beta''_x \eta'',$$

and similar expressions hold for the other increments. Also

$$2 X_\kappa = \begin{vmatrix} \frac{\partial y}{\partial u} \delta' + \frac{\partial y}{\partial v} \eta' & , & \frac{\partial z}{\partial u} \delta' + \frac{\partial z}{\partial v} \eta' \\ \frac{\partial y}{\partial u} \delta'' + \frac{\partial y}{\partial v} \eta'' & , & \frac{\partial z}{\partial u} \delta'' + \frac{\partial z}{\partial v} \eta'' \end{vmatrix} + 2 X'_\kappa,$$

where X'_κ denotes the sum of several determinants, involving the infinitesimals

$$\alpha'_y \quad , \quad \alpha''_y \quad , \quad \beta'_z \quad , \quad \beta''_z.$$

Similar expressions hold for Y_κ, Z_κ . We get thus

$$X_\kappa = A_\kappa t_\kappa + X'_\kappa \quad , \quad Y_\kappa = B_\kappa t_\kappa + Y'_\kappa \quad , \quad Z_\kappa = C_\kappa t_\kappa + Z'_\kappa$$

where A, B, C are the determinants 2) in 603. Then the area of the inscribed polyhedron corresponding to this division D is

$$S_D = \Sigma t_\kappa \sqrt{\left(A_\kappa + \frac{X'_\kappa}{t_\kappa}\right)^2 + \left(B_\kappa + \frac{Y'_\kappa}{t_\kappa}\right)^2 + \left(C_\kappa + \frac{Z'_\kappa}{t_\kappa}\right)^2}.$$

Let us suppose that

$$A^2 + B^2 + C^2 \geq q \quad , \quad q > 0 \tag{2}$$

as u, v ranges over \mathfrak{A} . Also let us assume that

$$\frac{X'_\kappa}{t_\kappa} \quad , \quad \frac{Y'_\kappa}{t_\kappa} \quad , \quad \frac{Z'_\kappa}{t_\kappa} \tag{3}$$

main numerically $< \epsilon$ for any division D of norm $d < d_0$, ϵ small pleasure, except in the vicinity of a discrete set of points, that let 3) be *in general uniformly evanescent* in \mathfrak{A} , as $d \doteq 0$. Then

$$S_D = \Sigma t_\kappa \sqrt{A_\kappa^2 + B_\kappa^2 + C_\kappa^2} + \Sigma \epsilon_\kappa t_\kappa,$$

here *in general*

$$|\epsilon_\kappa| < \frac{\epsilon}{\text{Cont } \mathfrak{A}}.$$

If now A, B, C are limited and R -integrable in \mathfrak{A} , we have at once

$$\lim_{d \doteq 0} S_D = \int_{\mathfrak{A}} du dv \sqrt{A^2 + B^2 + C^2}$$

in 603.

2. We ask now under what conditions are the expressions 3) general uniformly evanescent in \mathfrak{A} ? The answer is pretty evident from the example given by Schwarz. In fact the equation of the tangent plane \mathfrak{T} at \mathfrak{P}_0 is

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

On the other hand the equation of the plane $T = (\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2)$

$$\begin{vmatrix} x & y & z & 1 \\ x_0 & y_0 & z_0 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \end{vmatrix} = 0,$$

$$xX_\kappa + yY_\kappa + zZ_\kappa + U_\kappa = 0,$$

finally

$$x\left(A_\kappa + \frac{X'_\kappa}{t_\kappa}\right) + y\left(B_\kappa + \frac{Y'_\kappa}{t_\kappa}\right) + z\left(C_\kappa + \frac{Z'_\kappa}{t_\kappa}\right) + \frac{U_\kappa}{t_\kappa} = 0.$$

Thus for 3) to converge in general uniformly to zero, it is necessary and sufficient that the secant planes T converge in general uniformly to tangent planes. Let us call divisions such that the faces of the corresponding inscribed polyhedra converge in general uniformly to tangent planes *uniform triangular divisions*. For such divisions the expressions 3) are in general uniformly evanescent, as $d \doteq 0$. We have therefore the following theorem:

3. *Let \mathfrak{A} be a limited complete metric set. Let the coördinates y, z be one-valued totally differentiable functions of the param-*

ters u, v in \mathfrak{A} , such that $A^2 + B^2 + C^2$ is greater than some positive constant, and is limited and R -integrable in \mathfrak{A} . Then

$$S = \lim_{d \rightarrow 0} S_D = \int_{\mathfrak{A}} \sqrt{A^2 + B^2 + C^2} du dv,$$

D denoting the class of uniform triangular divisions of norms d .

This limit we shall call the *area* of S . From this definition we have at once a number of its properties. We mention only the following :

4. Let $\mathfrak{A}_1, \dots, \mathfrak{A}_m$ be unmixed metric sets whose union is \mathfrak{A} . Let S_1, \dots, S_m be the pieces of S corresponding to them. Then each S_κ has an area and their sum is S .

5. Let \mathfrak{A}_λ be a metric part of \mathfrak{A} , depending on a parameter $\lambda \doteq 0$, such that $\widehat{\mathfrak{A}}_\lambda \doteq \widehat{\mathfrak{A}}$. Then

$$\lim_{\lambda \rightarrow 0} S_\lambda = S.$$

6. The area of S remains unaltered when S is subjected to a displacement or a transformation of the parameters as in I, 744 seq.

607. 1. Irregular Surfaces. We consider now surfaces which do not have tangent planes at every point, that is, surfaces for which one or more of the first derivatives of the coördinates x, y, z do not exist, and which may be styled *irregular surfaces*. We prove now the theorem :

Let the coördinates x, y, z be one-valued functions of u, v having limited total difference quotients in the metric set \mathfrak{A} . Let D be a positive triangular division of norm $d \leq d_0$. Then

$$\text{Max } S_D$$

is finite and evanescent with \mathfrak{A} .

For let the difference quotients remain $\leq \mu$. We have

$$S_D \leq \Sigma |X_\kappa| + \Sigma |Y_\kappa| + \Sigma |Z_\kappa|.$$

But

$$\begin{aligned} |X_\kappa| &= \frac{1}{2} |\Delta' y \Delta'' z - \Delta' z \Delta'' y| \leq \frac{1}{2} \{ |\Delta' y| \cdot |\Delta'' z| + |\Delta' z| \cdot |\Delta'' y| \} \\ &\leq \mu^2 \bar{P}_0 \bar{P}_1 \cdot \bar{P}_0 \bar{P}_2 = 2 \mu^2 t_\kappa |\operatorname{cosec} \theta_\kappa| \end{aligned}$$

where θ_κ is the angle made by the sides $\overline{P_0P_1}$, $\overline{P_0P_2}$. As D is a positive division, one of the angles of t_κ is such that $\operatorname{cosec} \theta_\kappa$ is numerically less than some positive number M . Thus

$$|X_\kappa| < 2 \mu^2 M t_\kappa,$$

where μ , M are independent of κ and d . Similar relations hold for $|Y_\kappa|$, $|Z_\kappa|$. Hence

$$S_D < \Sigma 6 \mu^2 M \cdot t_\kappa = 6 \mu^2 M (\mathfrak{A} + \eta)$$

where $\eta > 0$ is small at pleasure, for d_0 sufficiently small.

2. Let \mathfrak{A} and x, y, z be as in 606, 3, except at certain points forming a discrete set α , the first partial derivatives do not exist. Let their total difference quotients be limited in \mathfrak{A} . Then

$$\lim_{d \rightarrow 0} S_D = \int \sqrt{A^2 + B^2 + C^2} du dv,$$

where D denotes a positive triangular division of norm d .

Let us first show that the limit on the left exists. We may choose a metric part \mathfrak{B} of \mathfrak{A} such that $\mathfrak{C} = \mathfrak{A} - \mathfrak{B}$ is complete and exterior to \mathfrak{A} and such that \mathfrak{B} is as small as we please. Let $S_{\mathfrak{C}}$ denote the area of the surface corresponding to \mathfrak{C} . The triangles t_κ fall into two groups: G_1 containing points of \mathfrak{B} ; G_2 containing only points of \mathfrak{C} . Then

$$S_D = \Sigma \sqrt{X_\kappa^2 + Y_\kappa^2 + Z_\kappa^2} = \Sigma_{\alpha_1} + \Sigma_{\alpha_2}.$$

But \mathfrak{B} may be chosen so small that the first sum is $< \epsilon/4$ for any $d < d_0$. Moreover by taking d_0 still smaller if necessary, we have

$$|\Sigma_{\alpha_2} - S_{\mathfrak{C}}| < \epsilon/4.$$

Hence

$$|S_D - S_{\mathfrak{C}}| < \epsilon/2, \quad d < d_0. \quad (1)$$

Similarly for any other division D' of norm d' ,

$$|S_{D'} - S_{\mathfrak{C}}| < \epsilon/2, \quad d' < d_0$$

decreasing d_0 still farther if necessary. Thus

$$|S_{D'} - S_D| < \epsilon, \quad d, d' < d_0.$$

Hence $\lim S_d$ exists, call it S . Since S exists we may take d_0 so small that

$$|S - S_d| < \epsilon/2 \quad , \quad d < d_0.$$

This with 1) gives

$$|S - S_{\mathfrak{G}}| < \epsilon,$$

that is,

$$\begin{aligned} S &= \lim S_{\mathfrak{G}} = \lim \int_{\mathfrak{G}} \sqrt{A^2 + B^2 + C^2} \, dudv \\ &= \int_{\mathfrak{A}} \sqrt{A^2 + B^2 + C^2} \, dudv \end{aligned}$$

by I, 724.

608. 1. The preceding theorem takes care of a large class of irregular surfaces whose total difference quotients are limited. In case they are not limited we may treat certain cases as follows:

Let us effect a quadrate division of the u, v plane of norm d , and take the triangles t_x so that for any triangular division D associated with d , no square contains more than n triangles, and no triangle lies in more than ν squares; n, ν being arbitrarily large constants independent of d . Such a division we call a *quasi quadrate division of norm d* . If we replace the quadrate by a rectangular division, we get a quasi rectangular division.

We shall also need to introduce a new classification of functions according to their variation in \mathfrak{A} , or along lines parallel to the u, v axes. Let D be a quadrate division of the u, v plane of norm $d < d_0$. Let

$$\omega_x = \text{Osc } f(u, v) \quad , \quad \text{in the cell } d_x.$$

Then

$$\text{Max } \Sigma \omega_x d$$

is the variation of f in \mathfrak{A} . If this is not only finite, but evanescent with $\bar{\mathfrak{A}}$, we say f has *limited fluctuation* in \mathfrak{A} . Obviously this may be extended to any limited point set in m -way space.

Let us now restrict ourselves to the plane. Let α denote the points of \mathfrak{A} on a line parallel to the u -axis. Let us effect a division D' of norm d' . Let $\omega'_x = \text{Osc } f(u, v)$ in one of the intervals of D' . Then

$$\eta_\alpha = \text{Max } \Sigma \omega'_x$$

is the variation of f in α .

Let us now consider all the sets α lying on lines parallel to the α -axis, and let

$$\bar{\alpha} \leq \sigma, \quad \sigma \neq 0.$$

If now there exists a constant G independent of α such that

$$\eta_{\alpha} < \sigma G,$$

that is, if η_{α} is uniformly evanescent with σ , we say that $f(u, v)$ is *limited fluctuation* in \mathfrak{A} with respect to u .

With the aid of these notions we may state the theorems:

2. *Let the coördinates x, y, z be one-valued limited functions in the limited complete set \mathfrak{A} . Let x, y have limited total difference quotients, while z has limited variation in \mathfrak{A} . Let D denote a quadratic division of norm $d \leq d_0$. Then*

$$\max_n S_n$$

finite.

For, as before,

$$2|X_{\kappa}| \leq |\Delta'_y| \cdot |\Delta''_z| + |\Delta''_y| \cdot |\Delta'_z|.$$

But μ denoting a sufficiently large constant,

$$|\Delta'_y|, \quad |\Delta''_y| \quad \text{are} < \mu d.$$

Let $\omega_i = (\text{osc } z \text{ in the square } s_i)$. If the triangle t_k lies in the squares s_{i_1}, \dots, s_{i_k} ,

$$|\Delta'_z|, |\Delta''_z| \leq \omega_{i_1} + \dots + \omega_{i_k}.$$

Thus, n denoting a sufficiently large constant,

$$\sum_{\kappa} |X_{\kappa}| < \mu \sum_{\kappa} d (\omega_{i_1} + \dots + \omega_{i_k})$$

$$< n \mu \sum_i \omega_i d,$$

the summation extending over those squares containing a triangle of D . But z having limited variation,

$$\sum \omega_i d < \text{some } M.$$

Hence $\sum |X_{\kappa}|, \quad \sum |Y_{\kappa}| \quad \text{are} < n \mu M.$

Finally, as in 607,

$$\sum |Z| < \text{some } M'.$$

The theorem is thus established.

3. The coördinates x, y, z , being as in 2, except that z has limited fluctuation in \mathfrak{N} , and D denoting a quasi quadrate division of norm $d \leq d_0$,

$$\text{Max}_D S_D$$

is finite and evanescent with $\bar{\mathfrak{N}}$.

The reasoning is the same as in 2 except that now M, M' are evanescent with $\bar{\mathfrak{N}}$.

4. Let the coördinates x, y, z have limited total difference quotients in \mathfrak{N} , while the variation of z along any line parallel to the u or v axis is $< M$. Let \mathfrak{N} lie in a square of side $s \doteq 0$. Then

$$\text{Max}_D S_D < sG,$$

where G is some constant independent of s , and D is a quasi rectangular division of norm $d \leq d_0$.

For here

$$2 \Sigma |X_\kappa| \leq \Sigma |\Delta' y| \cdot |\Delta'' z| + \Sigma |\Delta'' y| \cdot |\Delta' z| \\ < M' \Sigma \omega_u d_v + M' \Sigma \omega_v d_u,$$

where M' denotes a sufficiently large constant; d_u, d_v denote the length of the sides of one of the triangles t_κ parallel respectively to the u, v axes, and ω_u, ω_v the oscillation of z along these sides. Since the variation is $< M$ in both directions,

$$\Sigma \omega_u d_v = \Sigma_v d_v \Sigma \omega_u < M \Sigma d_v < Ms.$$

Similarly

$$\Sigma \omega_v d_u < Ms.$$

The rest of the proof follows as before.

5. The symbols having the same meaning as before, except that z has limited fluctuation with respect to u, v ,

$$\text{Max}_D S_D < s^2 G.$$

The demonstration is similar to the foregoing. Following the line of proof used in establishing 607, 2 and employing the theorems just given, we readily prove the following theorems:

6. Let \mathfrak{A} be a metric set containing the discrete set α . Let \mathfrak{b} be a metric part of \mathfrak{A} , containing α such that $\mathfrak{B} = \mathfrak{A} - \mathfrak{b}$ is exterior to α , and $\bar{\mathfrak{b}} \doteq 0$. Let the coördinates x, y, z be one-valued totally differentiable functions in \mathfrak{B} , such that $A^2 + B^2 + C^2$ never sinks below a positive constant in any \mathfrak{B} , is properly R -integrable in any \mathfrak{B} , and improperly integrable in \mathfrak{A} . Let x, y have limited total difference quotients, and z limited fluctuation in \mathfrak{b} . Then

$$\lim_{d=0} S_D = \int_{\mathfrak{A}} \sqrt{A^2 + B^2 + C^2} dudv$$

where A, B, C are the determinants in 603, 2), and D is any quasi quadrature division of norm d .

7. Let the symbols have the same meaning as in 6, except that

1° α reduces to a finite set.

2° z has limited variation along any line parallel to the u, v axes.

3° D denotes a uniform quasi rectangular division. Then

$$\lim_{d=0} S_D = \int_{\mathfrak{A}} \sqrt{A^2 + B^2 + C^2} dudv.$$

8. The symbols having the same meaning as in 6, except that

1° z has limited fluctuation with respect to u, v in \mathfrak{b} .

2° D denotes a uniform quasi rectangular division. Then

$$\lim_{d=0} S_D = \int_{\mathfrak{A}} \sqrt{A^2 + B^2 + C^2} dudv.$$

9. If we call the limits in theorems 6, 7, 8, area, the theorems 606, 3, 4, 5 still hold.

INDEX

(Numbers refer to pages)

- identity, 87
- ies, 87
- olutely convergent integrals, 31
- ies, 79
- oducts, 247
- tion of cardinals, 292
- inals, 312
- ies, 128
- rence, 340
- ut product, 247
- ies, 77, 139
- of intervals, 337
- egates, cardinal number, 278
- inition, 276
- tribution, 295
- umerable, 280
- ivalence, 276
- actic, 304
- ponents, 294
- ered, 302
- ver or potency, 278
- tions, 307
- ilar, 303
- nsfinite, 278
- form or 1-1 correspondence, 276
- ate series, 83
- ytical curve, 582
- tactic, 325
- of curve, 599, 602
- face, 623
- 2, 305, 555
- iated simple series, 144
- ducts, 247
- ltiple series, 145
- rgal series, 245
- arithmic series, 243
- er sets, 365
- Associated*, outer sets, 365
- non-negative functions, 41
- Baire*, 326, 452, 482, 587
- Bernouillian* numbers, 265
- Bertram's* test, 104
- Bessel* functions, 238
- Beta* functions, 267
- Binomial* series, 110
- Bôcher*, 165
- Bonnet's* test, 121
- Borel*, 324, 542
- Brouwer*, 614
- Cahen's* test, 340
- Cantor's* 1° and 2° principle, 316
- theorem, 450
- Category* of a set, 326
- Cauchy's* function, 214
- integral test, 99
- radical test, 98
- theorem, 90
- Cell* of convergence, 144
- standard rectangular, 359
- Chain*, 612
- Class* of a function, 468, 469
- Conjugate* functions, 238
- series, 147
- products, 249
- Connex*, 605
- Connected* sets, 605
- Contiguous* functions, 231
- Continuity*, 452
- infra, 487
- semi, 487
- supra, 487
- Continuous* image, 608

Contraction, 287
Convergence, infra-uniform, 502
 monotone, 176
 uniform, 156
 at a point, 157
 in segments, 556
 sub-uniform, 555
Co-product, 212
Curves, analytical, 582
 area, 599, 602
 Faber, 546
 Jordan, 595, 610
 Hilbert, 590
 length, 579
 non-intuitional, 537
 Osgood, 600
 Pompeiu, 542
 rectifiable, 583
 space-filling, 588
D'Alembert, 96
Deleted series, 139
Derivates, 404
Derivative of a set, 330
 order of, 331
Detached sets, 604
Dilation, 287
Dini, 176, 185, 438, 538
 series, 86
Discontinuity, 452
 at a point, 454
 of 1^o kind, 416
 of 2^o kind, 455
 pointwise, 457
 total, 457
Displacement, 286
Distribution, 295
Divergence of a series, 440
Division, complete, 30
 separated, 306, 371
 unmixed, 2
 of series, 196
 of products, 253
Divisor of a set, 23
 quasi, 390

Divisor, semi, 390
Du Bois Reymond, 103
 \mathfrak{C}_ϵ , $\mathfrak{C}_{\epsilon \neq 0}$ sets, 473
Elimination, 594
Enclosures, complementary ϵ -, 355
 deleted, 452
 distinct, 344
 divisor of, 344
 ϵ -, 355
 measurable, 356
 non-overlapping, 344
 null, 366
 outer, 343
 standard, 359
Enumerable, 280
Equivalent, 276
Essentially positive series, 78
 negative series, 78
Euler's constant, 269
Eutactic, 304
Exponents, 201
Exponential series, 96
Extremal sequence, 374
Faber curves, 546
Fluctuation, 634, 635
Fourier's coefficient, 416
 constants, 416
 series, 416
Function, associated non-negative func-
 tions, 41
 Bessel's, 238
 Beta, 267
 Cauchy's, 214
 class of, 468, 469
 conjugate, 233
 contiguous, 231
 continuous, 452
 infra, 487
 semi, 487
 supra, 487
 discontinuous, 452
 of 1^o kind, 416
 of 2^o kind, 455

- Function*, Gamma, 267
 Gauss' $\Pi(x)$, 238
 hyperbolic, 228
 hypergeometric, 228
 lineo-oscillating, 528
 maximal, 488
 measurable, 338
 minimal, 488
 monotone, 137
 null, 385
 oscillatory, 488
 pointwise discontinuous, 457
 residual, 561
 Riemann's, 459
 totally discontinuous, 457
 truncated, 27
 uniformly limited, 160, 567
 Volterra's, 501, 583
 Weierstrass', 498, 523, 581, 588

Gamma function, 267
Gauss' function $\Pi(x)$, 238
 test, 109
Geometric series, 81, 139

Harnack, divergence of series, 440
 sets, 354
Hermite, 300
Hilbert's curves, 590
Hobson, 389, 412, 555
Hyperbolic functions, 228
Hypercomplete sets, 472
Hypergeometric functions, 229
 series, 112

Images, simple, multiple, 606
 unfold, manifold, 606
 continuous, 606, 608
Integrals, absolutely convergent, 31
 L- or Lebesgue, proper, 372
 improper, 403, 405
 improper, author's, 32
 classical, 26
 de la Vallée-Poussin, 27
 inner, 20
 Integrals, R- or Riemannian, 372
 Integrand set, 385
 Intervals, of convergence, 90
 adjoint set of, 337
 set of, belonging to, 337
 Inversion, geometric, 287
 of a series, 204
 Iterable sets, 14
 Iterated products, 251
 series, 149

Jordan curves, 595, 610
 variation, 430
 theorem, 436

König, 527
Kummer's test, 106, 124

Lattice points, 137
 system, 137
Law of Mean, generalized, 505
Layers, 555
 deleted, 563
Lebesgue or L- integrals, 372
 theorems, 413, 424, 426, 452, 475,
 520, 619
Leibnitz's formula, 226
Length of curve, 579
Lindermann, 300, 599
Lineo-oscillating functions, 528
Link, 612
Liouville numbers, 301
Lipschitz, 438
Logarithmic series, 97
Löbrot, 448

Maclaurin's series, 206
Maximal, minimal functions, 488
Maximum, minimum, 521
 at a point, 485
Measure, 348
 lower, 348
 upper, 343
Mertens, 130
Metric sets, 1

- Monotone convergence*, 176
 functions, 137
Moore-Osgood theorem, 170
Motion, 570
Multiplication of series, 129
 cardinals, 293
 ordinals, 314
 infinite products, 253

Normal form of infinite product, 245
Null functions, 385
 sets, 348
Numbers, Bernoullian, 265
 cardinal, 278
 class of ordinal numbers, 318
 limitary, 314.
 Liouville, 301
 ordinal, 310
 rank of limitary numbers, 331

Ordered sets, 302
Order of derivative of a set, 331
Oscillation at a point, 454
Oscillatory function, 488
Osgood curves, 600
 -Moore theorem, 170
 theorems, etc., 178, 555, 622

Pantactic, 325
Path, 612
Peaks, 179
 infinite, 606
Polyant, 153
Point sets, adherence, 340
 adjoint set of intervals, 337
 apantactic, 325
 associated inner set, 365
 outer set, 365
 Baire sets, 326
 category 1° and 2° , 326
 coherence, 340
 conjugate, 51
 connected, 605
 convex, 605
 detached, 604

 Point sets, divisor, 23
 \mathfrak{C}_ϵ , $\mathfrak{C}_{\neq 0}$ sets, 473
 Harnack sets, 354
 hypercomplete, 472
 images, 605, 606
 integrant sets, 385
 iterable, 14
 measurable, 343, 348
 metric, 1
 negative component, 37
 null, 348
 pantactic, 325
 positive component, 37
 potency or power, 278
 projection, 10
 quasidivisor, 390
 reducible, 335
 reticulation, 621
 semidivisor, 390
 separated intervals, 337
 sum, 22
 transfinite derivatives, 330
 union, 27
 well-ordered, 304

 Pointwise discontinuity, 457
 Pompeiu, curves, 542
 Potency or power of a set, 278
 Power series, 89, 144, 187, 191
 Precinct, 612
 Pringsheim, theory of convergence, 113
 theorems, etc., 141, 215, 216, 217,
 220, 273
 Projection, 10
 Products, absolute convergence, 247
 adjoint, 247
 associate simple, 247
 conjugate, 249
 co-product, 242
 iterated, 251
 normal form, 245

 Quasidivisor, 390

 Raabe's test, 107
 Rank of limitary numbers, 331

- measurable sets, 335
- remainder series, 77
- Taylor's series, 200, 210
- rectifiable curves, 583
- regular points, 428
- residual function, 561
- reticulation, 621
- Richardson, 32
- Riemann's function, 459
- theorem, 444
- for Riemann integrals, 372
- rotation, 286
- Schaeffer, theorem, 516
- sinusoidal, theorems, 598, 621
- Schwarz, theorem, etc., 448, 626
- Summation of an aggregate, 307
- Summation, constant, or of invariability, 21
- Subdivisor, 390
- Separated divisions, 366, 371
- Functions, 403
- Sets, 366
 - of intervals, 337
- Sequence, extremal, 374
- Set-tuple, 137
- Series, Abel's, 87
- Absolute convergent, 79
- Adjoint, 77, 139
- Alternate, 83
- Associate logarithmic, 243
 - normal, 245
 - simple, 144
 - multiple, 144
- Bessels, 238
- Binomial, 110
- Cell of convergence, 144
- Conjugate, 147
- Deleted, 139
- Dini's, 86
- Divergence of, 440
- Essentially positive or negative, 78
- Exponential, 96

- harmonic, 82
 - general of exponent μ , 82
- hypergeometric, 112
- interval of convergence, 90
- inverse, 204
- iterated, 149
- logarithmic, 97
- Maclaurin's, 206
- power, 89, 144, 187
- rate of convergence or divergence, 102
- remainder, 77
- simple convergence, 80
- Taylor's, 206
- tests of convergence, *see* Tests
- telescopic, 85
- trigonometric, 88
- two-way, 133
- Similar sets, 303
- Similitude, 287
- Simple convergence of series, 80
- Singular points, 26
- Space-filling curves, 588
- Steady convergence, 176
- Submeasurable, 405
- Sum of sets, 22
- Surface, area, 623
 - irregular, 632
 - regular, 629
- Taylor's series, 206
- Telescopic series, 85
- Tests of convergence, Bertram, 101
 - Bonnet, 121
 - Cahen, 108
 - Cauchy, 98, 99
 - d'Alembert, 96
 - Gauss, 109
 - Kummer, 106, 124
 - Pringsheim, 123
 - Raabe, 107
 - radical, 98
 - ratio, 96

Tests of convergence, tests of 1° and 2°
kind, 120

Weierstrass, 120

Theta functions, 135, 184, 256

Total discontinuity, 457

Transfinite cardinals, 278

derivatives, 330

Translation, 286

Trigonometric series, 88

Truncated function, 27

Two-way series, 133

Undetermined coefficients, 197

Unfold image, 606

Uniform convergence, 156

at a point, 157

correspondence, 276

Uniformly limited function, 160, 567

Union of sets, 22

Vallée-Poussin (de la), 27, 594

Van Vleck sets, 361

Variation, limited or finite, 420, 530

positive and negative, 430

Volterra curves, 501, 587

Wallis formula, 260

Weierstrass' function, 498, 523, 588

test, 120

Well-ordered sets, 304

Wilson, W. A., vii, 395, 401

Young, W. H., theorems, 360, 363

Zeros of power series, 191

SYMBOLS EMPLOYED IN VOLUME II

(Numbers refer to pages)

Front \mathfrak{A} , 1. $F_{\mathfrak{A}}$, 614

$\int_{\mathfrak{A}}^*$, 20

\mathfrak{A} , 1

$U, \{ \}$, 22

Dv , 22

Adj \int , 31

$f_{\lambda, \mu}$, 31

$\mathfrak{A}_{\alpha, \beta}$, 32. $\mathfrak{A}_{\alpha, \beta}$, 34

$\mathfrak{A}_{\beta}, \mathfrak{A}_{-\alpha}$, 34

A_m, \bar{A}_m , Adj A , 77. $A_{n, p}$, 78

$A_p = A_{v_1 \dots v_n}$, 138; $A_p = A_{v_1 \dots v_n}$, 139

$R_v = v_1 \dots v_n$, 139

$\mathfrak{A} \sim \mathfrak{B}$, 270; $\mathfrak{A} \simeq \mathfrak{B}$, 303

Card \mathfrak{A} , 278

$c = \aleph_0$, 280; c , 287

\aleph_ω , 290

Sa, 307

Ord \mathfrak{A} , 311

ω , 311; Ω , 318

$\aleph_1, \aleph_2 \dots$, 318, 323

$Z_1, Z_2 \dots$, 318

$\mathfrak{A}^{(\omega)} = \mathfrak{A}^\omega$, 330; $\mathfrak{A}^{(\alpha)} = \mathfrak{A}^\alpha$, 331

$\mathfrak{A} = \text{Meas } \mathfrak{A}$, 343; $\mathfrak{A} = \text{Meas } \mathfrak{A}$, 348

$\mathfrak{A} = \text{Meas } \mathfrak{A}$, 348

\int, \int, \int , 372, 403, 405

Sdv, Qdv, 390

V_m , 420; $\text{Var } f = V_f$, 420

Osc $f = \text{oscillation in a given set.}$

Osc f , 454

$\text{Disc } f$, 454

$\mathfrak{E}_\epsilon, \mathfrak{E}_{\epsilon, n}$, 473

$f(x), f(x)$, 488

$f'(x), f'(x)$, 403

$\bar{R}f', Rf', Lf', \underline{L}f', Uf', \underline{U}f', R(a)$,

$R(a)$, 404

$\Delta(a, \beta)$, 404

The following symbols are defined in Volume I and are repeated here for the convenience of the reader.

$\text{Dist}(a, x)$ is the distance between a and x

$D_\delta(a)$, called the *domain* of the point a of norm δ is the set of points x , such that $\text{Dist}(a, x) \leq \delta$

$V_\delta(a)$, called the *vicinity* of the point a of norm δ , refers to some set \mathfrak{A} , and is the set of points in $D_\delta(a)$ which lie in \mathfrak{A}

$D_\delta^*(a)$, $V_\delta^*(a)$ are the same as the above sets, omitting a . They are called *deleted domains*, *deleted vicinities*

$a_n \doteq \alpha$ means a_n converges to α

$f(x) \doteq \alpha$, means $f(x)$ converges to α
A line of symbols as:

$$\epsilon < 0, m, | \alpha - a_n | < \epsilon, n > m$$

is of constant occurrence, and is to be read: for each $\epsilon > 0$, there exists an index m , such that $| \alpha - a_n | < \epsilon$, for every $n > m$

Similarly a line of symbols as:

$$\epsilon > 0, \delta > 0, | f(x) - \alpha | < \epsilon, x \text{ in } V_\delta^*(a)$$

is to be read: for each $\epsilon > 0$, there exists a $\delta > 0$, such that

$$| f(x) - \alpha | < \epsilon,$$

for every x in $V_\delta^*(a)$